NOTE ON AN APPLICATION OF THE δ-FUNCTION IN THE REPRESENTATION OF SOLUTIONS OF ALGEBRAIC EQUATIONS

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The "function" $\delta(x - x_o)$ is known as the Dirac Delta function and may be defined as zero everywhere except at x_o, where it is infinite in such a way that

$$(1) \qquad \int_a^b \delta(x-x_o)dx = \begin{cases} 0 & x_o \notin (a, b) \\ 1 & x_o \in (a, b) \end{cases}$$

having property that for every continuous function $\varphi(x)$ on (a, b)

$$(2) \qquad \int_a^b \varphi(x)\delta(x-x_0)dx = \begin{cases} 0 & x_o \notin (a, b) \\ \varphi(x_o) & x_o \in (a, b) \end{cases}$$

It is well known [2] $\delta(x-x_o)$ can be approximated as a limit of a sequence of piecewise continuous functions, and there is an abundance of such sequences.

The subject matter of this note is to point out an application of the δ-function in the representation of solutions of algebraic equations.

Given an algebraic equation

$$(3) \qquad \varphi(x) = 0$$

and it is known that (3) has a single root x_o in the interval (a, b). Further if $\varphi'(x)$ exist on (a, b) and $\varphi'(x) \neq 0$ on (a, b), then there exists an inverse function $x = x(\varphi(x))$ on (a, b).

Hence φ is monotone on (a, b), and from [1] we have

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(4) $$\int_a^b x\delta(\varphi(x))\,|\varphi'(x)|\,dx = \int_a^b x\delta(x-x_o)\,dx = x_o$$

which is the roof of $\varphi(x) = 0$ on (a, b).

Thus

(5) $$x_o = \int_a^b x\delta(\varphi(x))\,|\varphi'(x)|\,dx$$

The fact that (5) presupposes the knowledge of the root of (3) makes it useless for practical computation, and one must resort in (5) to the use of δ-function as a limit of an appropriate sequence.

For instance, for the sequence of functions

$$s_n = (n/\sqrt{\pi})\,e^{-n^2x^2}$$

one can easily show that $\lim_{n\to\infty} s_n(x) = \delta(x)$ and that

(6.1) $$x_o = \lim_{n\to\infty} (n/\sqrt{\pi}) \int_a^b x\,e^{-n^2(\varphi(x))^2}\,|\varphi'(x)|\,dx$$

Some of other possible choices would be to take

(6.2) $$s_n(x) = \sin nx/\pi x$$

(6.3) $$s_n(x) = \frac{1}{\pi}\,\frac{n}{1+n^2x^2},$$

or

(6.4) $$s_a(x) = \frac{1}{b}\,\frac{a^{2k-1}}{(a^{2k}+x^{2k})},\quad b = \int_{-\infty}^{\infty} \frac{dx}{1+x^{2k}}$$

Using (6.4) the root of the algebraic equation (3) is given by

(7) $$x_o = \lim_{a\to o} \frac{a^{2k-1}}{b} \int_a^b x\,\frac{|\varphi'(x)|}{a^{2k}+\varphi^{2k}(x)}\,dx$$

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As a simple illustration of method in (7) consider the equation $\cos x = 0$ on $(0, \pi)$. The single root is given by

$$x_o = \lim_{a \to o} \frac{a}{\pi} \int_o^\pi x \frac{\sin x}{a^2 + \cos^2 x} dx$$

with $x = \pi - t$

We have

$$\int_o^\pi x \frac{\sin x}{a^2 + \cos^2 x} dx = \frac{\pi}{2} \int_o^\pi \frac{\sin t}{a^2 + \cos^2 t} dt = \frac{\pi}{2a} [\tan^{-1}(\frac{1}{a}) - \tan^{-1}(-\frac{1}{a})]$$

and thus

$$x_o = \lim_{a \to o} \frac{a}{\pi} , \frac{\pi}{2a} [\tan^{-1}(\frac{1}{a}) - \tan^{-1}(\frac{-1}{a})] = \frac{\pi}{2}$$

Moreover, methods above can be easily extended to systems of algebraic equations.

For instance, given $\varphi_i(x_1, x_2, \ldots, x_n) = 0$, $i = 1, 2, \ldots, n$, suppose that in a region R there exists a single root for the system in (8) and that the Jacobian of the system is different from zero in R. Then it can be easily shown that (7) takes the form

$$x_i^o = \lim_{a \to o} \frac{a^{n(2k-1)}}{b^n} \int \ldots \int_R \frac{x_i}{\prod_{j=1}^n (a^{2k} + \varphi_j^{2k})} |J| \, dx_1 dx_2 \ldots dx_n$$

One can generalise the main results of this note as follows: Let φ be a continuous, monotone function in $[a, b]$, possessing a continuous derivative on (a, b). Assume

$$\varphi(a) = c < 0, \quad \varphi(b) = d > 0 \text{ and } \varphi(x_o) = 0.$$

Denote by ψ the inverse function of φ defined in $[c, d]$ by $\psi(y) = x$ if and only if $y = \varphi(x)$. We have $< \delta , \psi > = \psi(0) = x_o$.

Let s_n be a sequence of integrable functions, having their supports in $[c, d]$, tending weakly to δ.

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Then $$x_o = \langle \delta, \psi \rangle = \lim_{n\to\infty} \int_c^d s_n(y)\psi(y)dy$$

$$= \lim_{n\to\infty} \int_a^b x\, s_n(x)|\varphi'(x)|dx.$$

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