Proceedings of the Edinburgh Mathematical Society (2007) **50**, 459–475 ©
DOI:10.1017/S0013091505000891 Printed in the United Kingdom

SPECIALIZATIONS OF DIRECT LIMITS AND OF LOCAL COHOMOLOGY MODULES

DAM VAN NHI

Department of Mathematics, Pedagogical University Ha Noi, 136 Xuan Thuy Road, Hanoi, Vietnam (damvannhi@yahoo.com)

(Received 21 June 2005)

Abstract In this paper we introduce the specialization of an S-module which is a direct limit of a direct system of finitely generated S-modules indexed by $\mathbb{N}$. This specialization preserves the Buchsbaum property, the generalized Cohen–Macaulay property and the Castelnuovo–Mumford regularity of a module.

Keywords: specialization; direct limit; local cohomolgy module

2000 *Mathematics subject classification:* Primary 13A30
Secondary 13D45

1. Introduction

Specialization is a technique to prove the existence of algebraic structures over a field which has prescribed properties. More concretely, one starts with a module M_u with certain properties over a polynomial ring $R_u = k(u)[x]$, where $k(u)$ is an extension of a base field k by a finite set $u = (u_1, \dots, u_m)$ of parameters, and substitute u by a set $\alpha = (\alpha_1, \dots, \alpha_m)$ of elements of k to obtain ideally a module M_α over $R = k[x]$ with the same properties.

The theory of specialization of ideals was introduced by Krull [**4**], (see also [**11**]). Following [**4**], the specialization of an ideal I of a polynomial ring R_u with respect to the substitution $u \to \alpha$ was defined as the ideal I_α generated by elements of the set $\{f(\alpha, x) \mid f(u, x) \in I \cap k[u, x]\}$. For almost all substitutions $u \to \alpha$, that is for all α lying outside a proper algebraic subvariety of k^m, specializations preserve basic properties and operations on ideals, and the ideal I_α inherits most of the basic properties of I. In [**6**–**8**], the authors developed the theory of specializations of finitely generated modules, and we showed that basic properties and operations on modules are preserved by specializations. An outstanding problem is how to specialize non-finitely generated modules. In this paper, we propose a way to specialize modules that are direct limits of a direct system of finitely generated modules in the case where the base field is uncountable. Our approach can be applied to study specializations of local cohomology modules. We show that the Buchsbaum property, the generalized Cohen–Macaulay property and the Castelnuovo–Mumford regularity of a module are preserved by specializations.

2. Specializations of finitely generated modules

In this section we recall basic facts on specializations of finitely generated modules over a local ring.

Let P be an arbitrary separable prime ideal of R_u. By [**4**, Satz 14], P_α is a radical ideal of R_α for almost all α. Assume that $\mathfrak{p}$ is an arbitrary associated prime ideal of P_α. In brief, we will set $S = R_{u_P}$ and $S_\alpha = R_{\alpha_\mathfrak{p}}$. We denote PS and $\mathfrak{p}S_\alpha$ by $\mathfrak{m}$ and $\mathfrak{m}_\alpha$. An arbitrary element $f \in R_u$ may be written in the form $f = p(u,x)/q(u)$ with $p(u,x) \in k[u,x]$, $q(u) \in k[u] \setminus \{0\}$. For any α such that $q(\alpha) \neq 0$, we define $f_\alpha := p(\alpha,x)/q(\alpha)$. For every element $a = f/g \in S$ with $f, g \in R$, $g \notin P$, we define $a_\alpha := f_\alpha/g_\alpha$ if $g_\alpha \notin P_\alpha$. This definition is valid for almost all α, i.e. it holds for all α except perhaps those lying on a proper algebraic subvariety of k^m.

Let L be a finitely generated S-module. Let $S^r \xrightarrow{\phi} S^t \to L \to 0$ be a finite free presentation of L, where ϕ is represented by a matrix $A = (a_{ij}(u,x))$ with $a_{ij}(u,x) \in S$. Set $A_\alpha = (a_{ij}(\alpha,x))$, and let $\phi_\alpha : S^s_\alpha \to S^t_\alpha$ be the homomorphism represented by the matrix A_α. The specialization of L with respect to the substitution $u \to \alpha$ is defined as $L_\alpha := \operatorname{Coker} \phi_\alpha$ (see [**8**]).

This definition of L_α clearly depends on the chosen presentations of L. If we choose a different finite free presentation we may get a different specialization L'_α of L, but L_α and L'_α are canonically isomorphic for almost all α. For this we need to introduce the specialization of a homomorphism of finitely generated S-modules.

Let $v : L \to M$ be a homomorphism of finitely generated S-modules. Consider a commutative diagram

$$\begin{array}{ccccccc}
S^m & \xrightarrow{\phi} & S^n & \longrightarrow & L & \longrightarrow & 0 \\
\downarrow{\scriptstyle v_1} & & \downarrow{\scriptstyle v_0} & & \downarrow{\scriptstyle v} & & \\
S^r & \xrightarrow{\psi} & S^s & \longrightarrow & M & \longrightarrow & 0,
\end{array}$$

where the rows are finite free presentations of L and M, respectively. It is obvious that $\psi_\alpha(v_1)_\alpha = (v_0)_\alpha\phi_\alpha$, and that the diagram

$$\begin{array}{ccccccc}
S^m_\alpha & \xrightarrow{\phi_\alpha} & S^n_\alpha & \longrightarrow & L_\alpha & \longrightarrow & 0 \\
\downarrow{\scriptstyle (v_1)_\alpha} & & \downarrow{\scriptstyle (v_0)_\alpha} & & & & \\
S^r_\alpha & \xrightarrow{\psi_\alpha} & S^s_\alpha & \longrightarrow & M_\alpha & \longrightarrow & 0
\end{array}$$

is commutative for almost all α. Hence, there is an induced homomorphism $v_\alpha : L_\alpha \to M_\alpha$, which makes the diagram

$$\begin{array}{ccccccc}
S^m_\alpha & \xrightarrow{\phi_\alpha} & S^n_\alpha & \longrightarrow & L_\alpha & \longrightarrow & 0 \\
\downarrow{\scriptstyle (v_1)_\alpha} & & \downarrow{\scriptstyle (v_0)_\alpha} & & \downarrow{\scriptstyle v_\alpha} & & \\
S^r_\alpha & \xrightarrow{\psi_\alpha} & S^s_\alpha & \longrightarrow & M_\alpha & \longrightarrow & 0
\end{array}$$

https://doi.org/10.1017/S0013091505000891 Published online by Cambridge University Press

commutative for almost all α. The induced homomorphism v_α is called a *specialization* of $v : L \to M$ with respect to (ϕ, ψ) (see [**8**]). For almost all α, this definition does not depend on the choice of v_0 and v_1. Indeed, if we are given two other maps $w_0 : S^n \to S^s$ and $w_1 : S^m \to S^r$ which lift the same homomorphism $v : L \to M$, then the maps $(w_i)_\alpha$ induce the same map $v_\alpha : L_\alpha \to M_\alpha$ for almost all α. The specialization of id_L with respect to (ϕ, ϕ) is the identity map id_{L_α}.

We shall need the following basic properties of specializations of finitely generated modules over the local ring S.

Lemma 2.1 (Nhi and Trung [8, Lemma 1.4]). *Let $v, w : L \to M$ and $u : M \to N$ be homomorphisms of finitely generated S-modules. Then, for almost all α,*

$$(v + w)_\alpha = v_\alpha + w_\alpha,$$
$$(uv)_\alpha = u_\alpha v_\alpha.$$

Lemma 2.2 (Nhi and Trung [8, Theorem 2.2]). *Let $0 \to L \to M \to N \to 0$ be an exact sequence of finitely generated S-modules. Then the sequence $0 \to L_\alpha \to M_\alpha \to N_\alpha \to 0$ is exact for almost all α.*

By virtue of this lemma, one can identify each module L_α with its canonical image in M_α when L is a submodule of M, and deduce the following results.

Lemma 2.3 (Nhi and Trung [8, Lemma 2.3]). *Let M and N be submodules of a finitely generated S-module L. For almost all α, there are canonical isomorphisms*

$$(L/M)_\alpha \cong L_\alpha / M_\alpha,$$
$$(M \cap N)_\alpha \cong M_\alpha \cap N_\alpha,$$
$$(M + N)_\alpha \cong M_\alpha + N_\alpha.$$

Lemma 2.4 (Nhi and Trung [8, Theorem 2.6]). *Let L be a finitely generated S-module. Then, for almost all α, we have $\operatorname{Ann} L_\alpha = (\operatorname{Ann} L)_\alpha$ and $\dim L_\alpha = \dim L$.*

Lemma 2.5 (Nhi and Trung [8, Proposition 3.3]). *Let L and M be finitely generated S-modules. For almost all α, we have*

$$\operatorname{Ext}^i_{S_\alpha}(L_\alpha, M_\alpha) \cong \operatorname{Ext}^i_S(L, M)_\alpha, \quad i \geqslant 0.$$

3. Specializations of direct limits

Modifying the well-known definition of specialization of a finitely generated S-module, we will give the definition of specialization of an S-module which is a direct limit of a direct system of finitely generated S-modules indexed by $\mathbb{N}$.

Let $\{L_i\}_{i \in \mathbb{N}}$ be a family of finitely generated S-modules indexed by $\mathbb{N}$. For each pair $i, j \in \mathbb{N}$ such that $i \leqslant j$, let $f_{ij} : L_i \to L_j$ be an S-homomorphism, and suppose that the following conditions are satisfied:

(1) f_{ii} is the identity mapping on L_i for all $i \in \mathbb{N}$;

(2) $f_{ih} = f_{jh} f_{ij}$ whenever $i \leqslant j \leqslant h$.

https://doi.org/10.1017/S0013091505000891 Published online by Cambridge University Press

Then we have a direct system $\{L_i, f_{ij}\}$ over $\mathbb{N}$. Let C be the direct sum of the L_i, and identify each module L_i with its canonical image in C. Let D be the submodule of C generated by all elements of the form $\ell_i - f_{ij}(\ell_i)$, where $i \leqslant j$, $\ell_i \in L_i$. Denote by f the projection $C \to C/D$ and by f_i the projection $f|_{L_i} : L_i \to C/D$ for all $i \in \mathbb{N}$. The S-module C/D or the system $\{C/D, f_i\}$ is called the *direct limit* of the direct system $\{L_i, f_{ij}\}$ and is written $\varinjlim L_i$ (see [**2**, **5**]). Let $\{L_i, f_{ij}\}$ be a direct system of finitely generated S-modules over $\mathbb{N}$. By the definition of specialization of a finitely generated S-module, there is a polynomial $t_i(u) \in k[u]$ such that if $t_i(\alpha) \neq 0$, then we have a specialization $(L_i)_\alpha$. Moreover, there are polynomials $t_{ij}(u) \in k[u]$ such that if $t_{ij}(\alpha) \neq 0$, then we have a homomorphism $(f_{ij})_\alpha : (L_i)_\alpha \to (L_j)_\alpha$ for each pair $i, j \in \mathbb{N}$ with $j \geqslant i$. Therefore, there exists a countable family of non-zero polynomials such that if α is not a zero of all these polynomials, then $\{(L_i)_\alpha, (f_{ij})_\alpha\}$ is again a direct system of finitely generated S_α-modules. In general, the set of all such α may be empty. However, if the base field is uncountable, such a set is always non-empty due to the following observation.

Lemma 3.1. *Let k be an uncountable field. Let $\{t_i(u) \mid i \in \mathbb{N}\}$ be a family of non-zero polynomials in indeterminates $u_1, \dots, u_m$. Set $B_i = \{\alpha \mid t_i(\alpha) \neq 0\}$ for all $i \in \mathbb{N}$ and $B = \bigcap_{i \in \mathbb{N}} B_i$. Then B is an uncountable set.*

Proof. The claim will be proved by induction on m. We begin with the case $m = 1$. The sets $A_i = \{\alpha = (\alpha_1) \mid t_i(\alpha) = 0\}$ are countable sets. Then the set $A^1 = \bigcup_{i \in \mathbb{N}} A_i$ is a countable set. Therefore, $B^1 = k \setminus A^1$ is an uncountable set, because k is uncountable by assumption. In the case $m = 2$, the polynomials $t_i(u_1, u_2)$ are presented as polynomials of u_2:

$$t_i(u_1, u_2) = a_{i0}(u_1)u_2^{h_i} + a_{i1}(u_1)u_2^{h_i - 1} + \cdots, \quad a_{i0} \neq 0, \ i \in \mathbb{N}.$$

Since B^1 is an uncountable set, there exists $\lambda_1 \in B^1$ such that $a_{i0}(\lambda_1) \neq 0$ for all $i \in \mathbb{N}$. Then $t_i(\lambda_1, u_2) \neq 0$ for all $i \in \mathbb{N}$. By applying the result of the case $m = 1$, it is easy to show that the set of all $\lambda_2 \in k$ such that $t_i(\lambda_1, \lambda_2) \neq 0$ for all $i \in \mathbb{N}$ is an uncountable set. Thus, the set $B^2 = \{(\lambda_1, \lambda_2) \mid t_i(\lambda_1, \lambda_2) \neq 0 \text{ for all } i \in \mathbb{N}\}$ is an uncountable set. Now it will be assumed that B^{m-1} is an uncountable set. For the case $m > 2$, note that the polynomials $t_i(u_1, \dots, u_m)$ are presented as polynomials of u_m:

$$t_i(u_1, \dots, u_m) = a_{i0}(u_1, \dots, u_{m-1})u_m^{h_i} + a_{i1}(u_1, \dots, u_{m-1})u_m^{h_i - 1} + \cdots, \quad a_{i0} \neq 0, \ i \in \mathbb{N}.$$

Since B^{m-1} is uncountable by the inductive assumption, there exists $(\lambda_1, \dots, \lambda_{m-1}) \in B^{m-1}$ such that $a_{i0}(\lambda_1, \dots, \lambda_{m-1}) \neq 0$ for all $i \in \mathbb{N}$. Then $t_i(\lambda_1, \dots, \lambda_{m-1}, u_m) \neq 0$ for all $i \in \mathbb{N}$. By applying the result of the case $m = 1$, it is easy to show that the set of all $\lambda_m \in k$ such that $t_i(\lambda_1, \dots, \lambda_{m-1}, \lambda_m) \neq 0$ for all $i \in \mathbb{N}$ is an uncountable set. Hence, $B^m = \{(\lambda_1, \dots, \lambda_{m-1}, \lambda_m) \mid t_i(\lambda_1, \dots, \lambda_{m-1}, \lambda_m) \neq 0 \text{ for all } i \in \mathbb{N}\}$ is uncountable. Because $B^m \subset B$, B is therefore an uncountable set. □

Let $\{L_i, f_{ij}\}$ be a direct system of finitely generated S-modules over $\mathbb{N}$. This lemma will be needed to show the existence of a direct system $\{(L_i)_\alpha, (f_{ij})_\alpha\}$ of finitely generated S_α-modules over $\mathbb{N}$. Now we can modify the notion 'almost all' as follows.

Definition 3.2. A subset $V \subseteq k^m$ is called a *quasi-closed set* of k^m if V can be represented as a union $\bigcup_{i=0}^{\infty} V(\mathfrak{a}_i)$, where each $\mathfrak{a}_i$ is an ideal of $k[u]$ and $V(\mathfrak{a}) = \{P \in k^m \mid f(P) = 0, \text{ for all } f \in \mathfrak{a}\}$.

Lemma 3.3. *The union of two quasi-closed sets is a quasi-closed set. The intersection of a family of quasi-closed sets is again a quasi-closed set. The empty set $\emptyset$ and the whole space k^m are the quasi-closed sets.*

Proof. Assume that $V_1 = \bigcup_{i=0}^{\infty} V(\mathfrak{a}_i)$ and $V_2 = \bigcup_{j=0}^{\infty} V(\mathfrak{b}_j)$ are two quasi-closed sets. Then

$$V = V_1 \cup V_2 = \left(\bigcup_{i=0}^{\infty} V(\mathfrak{a}_i)\right) \cup \left(\bigcup_{j=0}^{\infty} V(\mathfrak{b}_j)\right) = \bigcup_{i,j=0}^{\infty} V(\mathfrak{a}_i \cap \mathfrak{b}_j)$$

is a quasi-closed set.

Assume that $V_\lambda = \bigcup_{i=0}^{\infty} V(\mathfrak{a}_{\lambda i})$, $\lambda \in \Lambda$, is a family of quasi-closed sets. Then

$$W = \bigcap_{\lambda \in \Lambda} V_\lambda = \bigcap_{\lambda \in \Lambda} \left(\bigcup_{i=0}^{\infty} V(\mathfrak{a}_{\lambda i})\right) = \bigcup_{i=0}^{\infty} \left(\bigcap_{\lambda \in \Lambda} V(\mathfrak{a}_{\lambda i})\right) = \bigcup_{i=0}^{\infty} V\left(\sum_{\lambda \in \Lambda} \mathfrak{a}_{\lambda i}\right).$$

Hence, W is a quasi-closed set. The empty set $\emptyset = V(1)$ and the whole space $k^m = V(0)$ are the quasi-closed sets. □

Definition 3.4. We define the *Zariski topology* on k^m by taking the *quasi-open* subsets to be the complements of the quasi-closed sets.

Proposition 3.5. *Let k be an uncountable field. If $\{B_h, h = 1, \dots, s\}$ is a finite family of quasi-open sets of the form $B_h = k^m \setminus \bigcup_{i=0}^{\infty} V(t_{hi}(u))$, $t_{hi}(u) \in k[u]$, then $\bigcap_{h=1}^{s} B_h$ is an uncountable quasi-open subset of k^m.*

Proof. Without loss of generality, we need only prove the proposition for $s = 2$. The intersection $B_1 \cap B_2$ is a quasi-open subset of k^m by Lemma 3.3. Suppose that $B_1 = k^m \setminus \bigcup_{i=0}^{\infty} V(t_i(u))$ and $B_2 = k^m \setminus \bigcup_{j=0}^{\infty} V(p_j(u))$. Then

$$B_1 \cap B_2 = k^m \setminus \bigcup_{i,j=0}^{\infty} V(t_i(u) p_j(u)).$$

Thus, the quasi-open set $B_1 \cap B_2$ is an uncountable set by Lemma 3.1. □

Let T be a property which can be asserted or denied for each $\alpha \in k^m$. We shall say that T holds for almost all $\alpha \in k^m$ if it holds for all α lying outside all the zeros sets of a countable collection of polynomials. If k is assumed to be an uncountable field, then the set of points P for which the property T holds should contain an uncountable quasi-open subset of k^m. From now on, the base field k is assumed to be the uncountable and perfect field. For the sake of simplicity, the phrase 'for almost all α' will be deleted in the proofs of all results.

Let $\{L_i, f_{ij}\}$ be a direct system of finitely generated S-modules over $\mathbb{N}$. One can raise a question about the existence of $\varinjlim (L_i)_\alpha$. The following result is an answer to this question.

https://doi.org/10.1017/S0013091505000891 Published online by Cambridge University Press

Proposition 3.6. *Let $\{L_i, f_{ij}\}$ be a direct system of finitely generated S-modules over $\mathbb{N}$. There exists $\varinjlim (L_i)_\alpha$ for almost all α.*

Proof. From the definitions of specializations of a finitely generated S-module and of a homomorphism, it was well-known that there are polynomials $t_i(u), t_{ij}(u) \in k[u]$ such that if $t_i(\alpha)t_j(\alpha)t_{ij}(\alpha) \neq 0$, then we have the specializations $(L_i)_\alpha$, $(L_j)_\alpha$ and homomorphisms $(f_{ij})_\alpha : (L_i)_\alpha \to (L_j)_\alpha$. Since the set $A = \{t_i(u), t_{ij}(u) \mid i, j \in \mathbb{N}\}$ is a countable collection of polynomials, the set $B = \{\alpha \in k^m \mid t(\alpha) \neq 0 \text{ for all } t(u) \in A\}$ is countable by Lemma 2.1. Thus, the family $\{(L_i)_\alpha, (f_{ij})_\alpha\}$ exists for almost all α. Since $(f_{ii})_\alpha$ is the identity mapping on $(L_i)_\alpha$ for all $i \in \mathbb{N}$ and $(f_{ih})_\alpha = (f_{jh} f_{ij})_\alpha = (f_{jh})_\alpha \cdot (f_{ij})_\alpha$ for all $i \leqslant j \leqslant h$ by Lemma 2.1, therefore $\{(L_i)_\alpha, (f_{ij})_\alpha\}$ is a direct system of finitely generated S_α-modules over $\mathbb{N}$. Therefore, $\varinjlim (L_i)_\alpha$ exists for almost all α. $\square$

Definition 3.7. Let $\{L_i, f_{ij}\}$ be a direct system of finitely generated S-modules over $\mathbb{N}$. We call the direct system $\{(L_i)_\alpha, (f_{ij})_\alpha\}$ of finitely generated S_α-modules over $\mathbb{N}$ a *specialization* of $\{L_i, f_{ij}\}$.

Definition 3.8. If L is an S-module and $L = \varinjlim L_i$, then we define $L_\alpha = \varinjlim (L_i)_\alpha$ for almost all α and call such a *specialization* of L (with respect to $\{L_i, f_{i,j}\}$).

We shall see that, up to an isomorphism, this definition does not depend on the chosen direct system of finitely generated S-modules. First of all, we need to define the specialization of a map of direct systems.

Let $\{L_i, f_{ij}\}$ and $\{M_i, g_{ij}\}$ be direct systems of finitely generated S-modules over $\mathbb{N}$. Then a homomorphism $\phi : \{L_i, f_{ij}\} \to \{M_i, g_{ij}\}$ consists of a family of homomorphisms $\{\phi_i : L_i \to M_i\}$ such that all diagrams

$$\begin{array}{ccc} L_i & \xrightarrow{\phi_i} & M_i \\ {\scriptstyle f_{ij}}\downarrow & & \downarrow{\scriptstyle g_{ij}} \\ L_j & \xrightarrow[\phi_j]{} & M_j \end{array}$$

are commutative for all pairs $i, j \in \mathbb{N}$ with $i \leqslant j$. Suppose that $L = \varinjlim L_i$, $M = \varinjlim M_i$. There is a unique homomorphism $\phi = \varinjlim \phi_i : L \to M$ (see [**2, 5**]). One can define specializations of homomorphisms between direct systems of finitely generated S-modules over $\mathbb{N}$.

Lemma 3.9. *Let $\{L_i, f_{ij}\}$ and $\{M_i, g_{ij}\}$ be direct systems of finitely generated S-modules over $\mathbb{N}$ and consider a homomorphism $\phi = \{\phi_i\} : \{L_i, f_{ij}\} \to \{M_i, g_{ij}\}$. Then there are homomorphisms $\{(\phi_i)_\alpha\} : \{(L_i)_\alpha, (f_{ij})_\alpha\} \to \{(M_i)_\alpha, (g_{ij})_\alpha\}$ and $\varinjlim (\phi_i)_\alpha : \varinjlim (L_i)_\alpha \to \varinjlim (M_i)_\alpha$ for almost all α.*

Proof. The homomorphism $\phi : \{L_i, f_{ij}\} \to \{M_i, g_{ij}\}$ consists of a family of homomorphisms $\{\phi_i : L_i \to M_i\}$. By Proposition 3.5, there are specializations $\{(L_i)_\alpha, (f_{ij})_\alpha\}$ and

https://doi.org/10.1017/S0013091505000891 Published online by Cambridge University Press

$\{(M_i)_\alpha, (g_{ij})_\alpha\}$ of $\{L_i, f_{ij}\}$ and $\{M_i, g_{ij}\}$, respectively. By Lemma 3.1 and by Proposition 3.5, there are homomorphisms $(\phi_i)_\alpha : (L_i)_\alpha \to (M_i)_\alpha$ such that all diagrams

$$\begin{array}{ccc} (L_i)_\alpha & \xrightarrow{(\phi_i)_\alpha} & (M_i)_\alpha \\ {\scriptstyle (f_{ij})_\alpha}\downarrow & & \downarrow{\scriptstyle (g_{ij})_\alpha} \\ (L_j)_\alpha & \xrightarrow[(\phi_j)_\alpha]{} & (M_j)_\alpha, \end{array}$$

are commutative for all pairs $i, j \in \mathbb{N}$ with $i \leqslant j$. Hence, we obtain a homomorphism $\{(L_i)_\alpha, (f_{ij})_\alpha\} \to \{(M_i)_\alpha, (g_{ij})_\alpha\}$ consisting of a family of homomorphisms $(\phi_i)_\alpha : (L_i)_\alpha \to (M_i)_\alpha$, and a homomorphism $\varinjlim(\phi_i)_\alpha : \varinjlim(L_i)_\alpha \to \varinjlim(M_i)_\alpha$ for almost all α. □

Definition 3.10. Let $\{L_i, f_{ij}\}$ and $\{M_i, g_{ij}\}$ be direct systems of finitely generated S-modules over $\mathbb{N}$, respectively, and let $\{\phi_i : L_i \to M_i\}$ be a family of homomorphisms determining a homomorphism $\phi : \{L_i, f_{ij}\} \to \{M_i, g_{ij}\}$. We call the homomorphism

$$\{(L_i)_\alpha, (f_{ij})_\alpha\} \to \{(M_i)_\alpha, (g_{ij})_\alpha\}$$

defined by a family of S_α-module homomorphisms $(\phi_i)_\alpha$ for all $i \in \mathbb{N}$ a *specialization* of ϕ, and it will be denoted by ϕ_α. The homomorphism $\varinjlim(\phi_i)_\alpha : \varinjlim(L_i)_\alpha \to \varinjlim(M_i)_\alpha$ is called a *specialization* of $\varphi = \varinjlim \phi_i : \varinjlim L_i \to \varinjlim M_i$ with respect to the system $\{\phi_i\}$ and is denoted by φ_α.

Next, we want to prove the following proposition about specializations of direct systems of finitely generated S-modules over the directed set $\mathbb{N}$.

Proposition 3.11. *Let $\{L_i, f_{ij}\}$, $\{M_i, g_{ij}\}$, $\{N_i, h_{ij}\}$ be direct systems of finitely generated S-modules over the directed set $\mathbb{N}$. If*

$$0 \to \{L_i, f_{ij}\} \xrightarrow{\phi} \{M_i, g_{ij}\} \xrightarrow{\psi} \{N_i, h_{ij}\} \to 0$$

is an exact sequence, then the sequence

$$0 \to \{(L_i)_\alpha, (f_{ij})_\alpha\} \xrightarrow{\phi_\alpha} \{(M_i)_\alpha, (g_{ij})_\alpha\} \xrightarrow{\psi_\alpha} \{(N_i)_\alpha, (h_{ij})_\alpha\} \to 0$$

is exact for almost all α.

Proof. Assume that the sequence $0 \to \{L_i, f_{ij}\} \xrightarrow{\phi} \{M_i, g_{ij}\} \xrightarrow{\psi}\!\to \{N_i, h_{ij}\} \to 0$ is an exact sequence of direct systems of finitely generated S-modules over the same directed set $\mathbb{N}$. Then the sequences

$$0 \to L_i \xrightarrow{\phi_i} M_i \xrightarrow{\psi_i} N_i \to 0$$

are exact for all $i \in \mathbb{N}$. By Lemma 2.2, the sequences

$$0 \to (L_i)_\alpha \xrightarrow{(\phi_i)_\alpha} (M_i)_\alpha \xrightarrow{(\psi_i)_\alpha} (N_i)_\alpha \to 0$$

https://doi.org/10.1017/S0013091505000891 Published online by Cambridge University Press

are exact sequences and all diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (L_i)_\alpha & \xrightarrow{(\phi_i)_\alpha} & (M_i)_\alpha & \xrightarrow{(\psi_i)_\alpha} & (N_i)_\alpha & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (f_{ij})_\alpha} & & \downarrow{\scriptstyle (g_{ij})_\alpha} & & \downarrow{\scriptstyle (h_{ij})_\alpha} & & \\ 0 & \longrightarrow & (L_j)_\alpha & \xrightarrow{(\phi_j)_\alpha} & (M_j)_\alpha & \xrightarrow{(\psi_j)_\alpha} & (N_j)_\alpha & \longrightarrow & 0 \end{array}$$

exist and are commutative for all $i, j \in \mathbb{N}$ with $i \leqslant j$ by Lemma 2.1 and Proposition 3.5. Therefore, the sequence

$$0 \to \{(L_i)_\alpha, (f_{ij})_\alpha\} \xrightarrow{\phi_\alpha} \{(M_i)_\alpha, (g_{ij})_\alpha\} \xrightarrow{\psi_\alpha} \{(N_i)_\alpha, (h_{ij})_\alpha\} \to 0$$

is exact for almost all α. □

Corollary 3.12. *Let $\{L_i, f_{ij}\}$ and $\{M_i, g_{ij}\}$ be direct systems of finitely generated S-modules over the same directed set $\mathbb{N}$. Let $\phi : \{L_i, f_{ij}\} \to \{M_i, g_{ij}\}$ be a homomorphism. Then, for almost all α, ϕ_α is injective (surjective) if ϕ is injective (surjective).*

Corollary 3.13. *Let $\{L_i, f_{ij}\}$, $\{M_i, g_{ij}\}$, $\{N_i, h_{ij}\}$ be direct systems of finitely generated S-modules over the directed set $\mathbb{N}$. If*

$$0 \to \{L_i, f_{ij}\} \xrightarrow{\phi} \{M_i, g_{ij}\} \xrightarrow{\psi} \{N_i, h_{ij}\} \to 0$$

is an exact sequence, then

$$0 \to \varinjlim (L_i)_\alpha \xrightarrow{\varphi_\alpha} \varinjlim (M_i)_\alpha \xrightarrow{\psi_\alpha} \varinjlim (N_i)_\alpha \to 0$$

is exact for almost all α, too.

Proof. The sequence

$$0 \to \{(L_i)_\alpha, (f_{ij})_\alpha\} \xrightarrow{\phi_\alpha} \{(M_i)_\alpha, (g_{ij})_\alpha\} \xrightarrow{\psi_\alpha} \{(N_i)_\alpha, (h_{ij})_\alpha\} \to 0$$

is exact by Proposition 3.11, and by Lemma 3.9 there are homomorphisms

$$\varphi_\alpha : \varinjlim (L_i)_\alpha \to \varinjlim (M_i)_\alpha,$$
$$\psi_\alpha : \varinjlim (M_i)_\alpha \to \varinjlim (N_i)_\alpha.$$

The exactness of the sequence $0 \to \varinjlim (L_i)_\alpha \xrightarrow{\varphi_\alpha} \varinjlim (M_i)_\alpha \xrightarrow{\psi_\alpha} \varinjlim (N_i)_\alpha \to 0$ follows from [**5**, Theorem A2]. □

The following theorem shows that we may speak about a unique specialization of a direct limit of a direct system of finitely generated S-modules.

Theorem 3.14. *Let L be an S-module. Let $\{L_i, f_{ij}\}$ and $\{M_i, g_{ij}\}$ be two different direct systems of finitely generated S-modules over the same directed set $\mathbb{N}$ such that $\varinjlim L_i = L = \varinjlim M_i$. Denote by f_i and g_i the projections $L_i \to \varinjlim L_i$ and $M_i \to \varinjlim M_i$, respectively, for all $i \in \mathbb{N}$. Then, for almost all α, we have $\varinjlim \operatorname{Im}(f_i)_\alpha = \varinjlim \operatorname{Im}(g_i)_\alpha$.*

https://doi.org/10.1017/S0013091505000891 Published online by Cambridge University Press

Proof. By Proposition 3.5, there exist the S_α-modules $(L_i)_\alpha$ and $(M_i)_\alpha$ for all $i \in \mathbb{N}$. By Proposition 3.6, there exist $L_\alpha = \varinjlim (L_i)_\alpha$ and $M_\alpha = \varinjlim (M_i)_\alpha$. Denote by g_i the projection $M_i \to L$ for all $i \in \mathbb{N}$. For each $r \in \mathbb{N}$ there exist $i, j \in \mathbb{N}$ with $i < j$ such that $\operatorname{Im} f_i \subseteq \operatorname{Im} g_r \subseteq \operatorname{Im} f_j$ and $s > r$ such that $\operatorname{Im} g_r \subseteq \operatorname{Im} f_j \subseteq \operatorname{Im} g_s$. Consider two S_α-modules $P = \varinjlim (\operatorname{Im} f_i)_\alpha$, $Q = \varinjlim (\operatorname{Im} g_r)_\alpha$. From Lemma 2.2, the module $(\operatorname{Im} g_r)_\alpha$ is considered as a submodule of $(\operatorname{Im} f_j)_\alpha$. Because the modules $(\operatorname{Im} f_i)_\alpha$ are submodules of P, there are the inclusions

$$(\operatorname{Im} f_i)_\alpha \subseteq (\operatorname{Im} g_r)_\alpha \subseteq (\operatorname{Im} f_j)_\alpha$$

in the module P. By an analogous argument, we have the inclusions

$$(\operatorname{Im} g_r)_\alpha \subseteq (\operatorname{Im} f_j)_\alpha \subseteq (\operatorname{Im} g_s)_\alpha$$

in module Q. Hence, $P = Q$. Consider the surjective maps $f_i : L_i \to \operatorname{Im} f_i$ and $g_r : M_r \to \operatorname{Im} g_r$. We also have the surjective maps $(f_i)_\alpha : (L_i)_\alpha \to (\operatorname{Im} f_i)_\alpha$ and $(g_r)_\alpha : (M_r)_\alpha \to (\operatorname{Im} g_r)_\alpha$ by Lemma 2.2. Since $\operatorname{Im}(f_i)_\alpha = (\operatorname{Im} f_i)_\alpha$ for each $i \in \mathbb{N}$, we therefore obtain $\varinjlim \operatorname{Im}(f_i)_\alpha = P$ by [**2**, Corollary 4.14]. Analogously, $\varinjlim \operatorname{Im}(g_r)_\alpha = P$. Hence, $\varinjlim \operatorname{Im}(f_i)_\alpha = \varinjlim \operatorname{Im}(g_i)_\alpha$. The claim is proved. □

Notice that our definition of L_α depends on the chosen direct system $\{L_i, f_{ij}\}$. If we choose a different specialization of $\{L'_i, f'_{ij}\}$ we may get a different specialization L'_α of L. However, L_α and L'_α are canonically isomorphic for almost all α. To see this we need to introduce the specialization of a homomorphism between two direct limits.

We turn to the definition of specializations of homomorphisms between direct limits.

Let L and M be the S-modules with $L = \varinjlim L_i$ and $M = \varinjlim M_i$ and a homomorphism $\phi : L \to M$. The projections $L_i \to \varinjlim L_i$ and $M_i \to \varinjlim M_i$ will be denoted by f_i and g_i, respectively. Then $L = \varinjlim L_i = \varinjlim \operatorname{Im} f_i$, $M = \varinjlim M_i = \varinjlim \operatorname{Im} g_i$, and ϕ can be considered as a homomorphism

$$\phi : \varinjlim \operatorname{Im} f_i \to \varinjlim \operatorname{Im} g_i.$$

Now $\operatorname{Im} f_i \subseteq \operatorname{Im} f_j$ and $\operatorname{Im} g_i \subseteq \operatorname{Im} g_j$ for $j \geqslant i$. We write

$$\pi_{ij} : \operatorname{Im} f_i \to \operatorname{Im} f_j \quad \text{and} \quad \delta_{ij} : \operatorname{Im} g_i \to \operatorname{Im} g_j$$

for the natural inclusions. For each $i \in \mathbb{N}$ there exists $p(i)$ such that $\phi(\operatorname{Im} f_i) \subseteq \operatorname{Im} g_{p(i)}$, $\phi(\operatorname{Im} f_i) \nsubseteq \operatorname{Im} g_{p(i)-1}$. Set $\phi_i := \phi|_{\operatorname{Im} f_i}$. Then $\{\phi_i(\operatorname{Im} f_i), \delta_{p(i)p(j)}|_{\phi_i(\operatorname{Im} f_i)}\}$ is a direct subsystem of submodules of $\{\operatorname{Im} g_i, \delta_{ij}\}$. We have a homomorphism $\phi = \varinjlim \phi_i : \varinjlim \operatorname{Im} f_i \to \varinjlim \phi_i(\operatorname{Im} f_i)$. By the definition of specialization of a limit-homomorphism, there is a homomorphism

$$\varinjlim (\phi_i)_\alpha : \varinjlim (\operatorname{Im} f_i)_\alpha \to \varinjlim \phi_i(\operatorname{Im} f_i)_\alpha.$$

Because

$$\{\phi_i(\operatorname{Im} f_i), \delta_{p(i)p(j)}|_{\phi_i(\operatorname{Im} f_i)}\}$$

is a direct system of submodules of $\{\operatorname{Im} g_{p(i)}, \delta_{p(i)p(j)j}\}$, therefore $\varinjlim \phi_i(\operatorname{Im} f_i)_\alpha$ is a submodule of $\varinjlim (\operatorname{Im} g_i)_\alpha$. Hence, there is a homomorphism

$$\varinjlim (\phi_i)_\alpha : \varinjlim (\operatorname{Im} f_i)_\alpha \to \varinjlim (\operatorname{Im} g_i)_\alpha.$$

Definition 3.15. Let L, M be S-modules and consider a homomorphism $\phi : L \to M$, where $L = \varinjlim L_i$, $M = \varinjlim M_i$. Let the projections $L_i \to \varinjlim L_i$ and $M_i \to \varinjlim M_i$ be denoted by f_i and g_i, respectively, and set $\phi_i := \phi|_{\operatorname{Im} f_i}$. The homomorphism $\phi_\alpha := \varinjlim(\phi_i)_\alpha : \varinjlim(\operatorname{Im} f_i)_\alpha \to \varinjlim(\operatorname{Im} g_i)_\alpha$ is called *a specialization* of ϕ for almost all α.

As a consequence of the above definition we have the following corollary.

Corollary 3.16. *Let L, M and N be S-modules and consider homomorphisms $\phi : L \to M$ and $\psi : M \to N$. Then, for almost all α, $(\psi \cdot \phi)_\alpha = \psi_\alpha \cdot \phi_\alpha$.*

Lemma 3.17. *Let L, M be S-modules and $\phi : L \to M$ a homomorphism, where $L = \varinjlim L_i$, $M = \varinjlim M_i$. Let the projections $L_i \to \varinjlim L_i$, $M_i \to \varinjlim M_i$ be denoted by f_i and g_i, respectively. Then, for almost all α,*

(i) *each homomorphism $(f_i)_\alpha$ is a projection $(L_i)_\alpha \to \varinjlim(L_i)_\alpha$,*

(ii) $\varinjlim(L_i)_\alpha = \varinjlim \operatorname{Im}(f_i)_\alpha$,

(iii) $(\phi \cdot f_i)_\alpha = \phi_\alpha \cdot (f_i)_\alpha$.

Proof. (i) Assume that $\{L_i, f_{ij}\}$ and $\{M_i, g_{ij}\}$ are direct systems of finitely generated S-modules over the directed set $\mathbb{N}$, where $L = \varinjlim L_i$, $M = \varinjlim M_i$. Denote the homomorphism $\phi \cdot f_i : L_i \to \varinjlim M_i$ by d_i for each $i \in \mathbb{N}$. Since L_i is a finitely generated S-module, $d_i(L_i)$ is also a finitely generated S-module. Therefore, $d_i(L_i)$ may be specialized. Since $d_i(L_i) = \phi(\operatorname{Im} f_i) \subset \operatorname{Im} g_{p(i)}$, in the notation introduced after the proof of Theorem 3.14, $d_i(L_i)_\alpha$ can be considered as a submodule of $\varinjlim(\operatorname{Im} g_i)_\alpha$ by Lemma 3.3. On the other hand, since $f_i = f_j \cdot f_{ij}$ for all pairs i, j with $j \geqslant i$, we have $d_i = d_j \cdot f_{ij}$, and $(d_i)_\alpha = (d_j)_\alpha \cdot (f_{ij})_\alpha$ by Lemma 2.1. Thus, by the universal property of $\varinjlim(L_i)_\alpha$, there exists a unique homomorphism $\varinjlim(L_i)_\alpha \to \varinjlim(\operatorname{Im} g_i)_\alpha$ which will be denoted by φ_α and which satisfies $(d_i)_\alpha = \varphi_\alpha \cdot \pi_i$ or $(\phi \cdot f_i)_\alpha = \varphi_\alpha \cdot \pi_i$ for all $i \in \mathbb{N}$, where π_i is the projection $(L_i)_\alpha \to \varinjlim(L_i)_\alpha$. It is easy to show that if $\phi = \mathrm{id}$, then $\varphi_\alpha = \mathrm{id}$ by Theorem 3.14. Hence, $(f_i)_\alpha = \pi_i$, and the homomorphism $(f_i)_\alpha$ is a projection $(L_i)_\alpha \to \varinjlim(L_i)_\alpha$.

(ii) Consider the epimorphism $f_i : L_i \to \operatorname{Im} f_i$. The homomorphism $(f_i)_\alpha : (L_i)_\alpha \to (\operatorname{Im} f_i)_\alpha$ is an epimorphism by Lemma 3.3. Then $\operatorname{Im}(f_i)_\alpha = (\operatorname{Im} f_i)_\alpha$ for each $i \in \mathbb{N}$. Because $(f_i)_\alpha$ is the projection $(L_i)_\alpha \to \varinjlim(L_i)_\alpha$ by (i), we therefore obtain

$$\varinjlim(L_i)_\alpha = \varinjlim \operatorname{Im}(f_i)_\alpha.$$

(iii) Since $(\phi \cdot f_i)_\alpha = (\phi \cdot f_i)_\alpha = (\phi_i)_\alpha \cdot (f_i)_\alpha$ for each $i \in \mathbb{N}$, by Lemma 2.1, $(\phi_i)_\alpha \cdot (f_i)_\alpha = \varphi_\alpha \cdot (f_i)_\alpha$. Thus, $\phi_\alpha = \varinjlim(\phi_i)_\alpha = \varphi_\alpha$, and then $(\phi \cdot f_i)_\alpha = \phi_\alpha \cdot (f_i)_\alpha$ for almost all α. □

Corollary 3.18. *Let L be an S-module. If $\{L_i, f_{ij}\}$ and $\{M_i, g_{ij}\}$ are direct systems of finitely generated S-modules over the directed set $\mathbb{N}$ and $\varinjlim L_i = L = \varinjlim M_i$, then $\varinjlim(L_i)_\alpha = \varinjlim(M_i)_\alpha$ for almost all α.*

Proof. The projections

$$L_i \to \varinjlim L_i \quad \text{and} \quad M_i \to \varinjlim M_i$$

will be denoted by f_i and g_i, respectively. Then $\varinjlim(L_i)_\alpha = \varinjlim(\operatorname{Im} f_i)_\alpha$ and $\varinjlim(M_i)_\alpha = \varinjlim(\operatorname{Im} g_i)_\alpha$ by Lemma 3.17 (ii). Since $\varinjlim(\operatorname{Im} f_i)_\alpha = \varinjlim(\operatorname{Im} g_i)_\alpha$ by Theorem 3.14, $\varinjlim(L_i)_\alpha = \varinjlim(M_i)_\alpha$ for almost all α. □

Corollary 3.19. *If L is a finitely generated S-module, then L_α is also a finitely generated S_α-module for almost all α.*

Proof. Assume that L is a finitely generated S-module. By Theorem 3.14, we need only construct a direct system of the form $\{L, f_{ij} : L \to L \text{ is the identity map}\}$ such that $L = \varinjlim L$. The projections $L \to \varinjlim L$ are given by $f_i = \mathrm{id}$. By Lemma 3.17 (ii), we have $L_\alpha = \varinjlim L_\alpha$. Hence, L_α is also a finitely generated S_α-module for almost all α. □

From this corollary we see that the above definition of L_α is of course an extension of the definition of specialization of finitely generated modules in [**8**].

The following theorem states one of the most important properties of specializations.

Theorem 3.20. *Let L, M and N be S-modules with $L = \varinjlim L_i$, $M = \varinjlim M_i$ and $N = \varinjlim N_i$. If the sequence $0 \to L \xrightarrow{\phi} M \xrightarrow{\psi} N \to 0$ is exact, then the sequence $0 \to L_\alpha \xrightarrow{\phi_\alpha} M_\alpha \xrightarrow{\psi_\alpha} N_\alpha \to 0$ is also exact for almost all α.*

Proof. The projections $L_i \to \varinjlim L_i$, $M_i \to \varinjlim M_i$ and $N_i \to \varinjlim N_i$ will be denoted by f_i, g_i and h_i, respectively. The exact sequence

$$0 \to L \xrightarrow{\phi} M \xrightarrow{\psi} N \to 0$$

can be replaced by the exact sequence

$$0 \to \varinjlim \operatorname{Im} f_i \xrightarrow{\phi} \varinjlim \operatorname{Im} g_i \xrightarrow{\psi} \varinjlim \operatorname{Im} h_i \to 0.$$

Because ϕ is injective, $\phi_i = \phi|_{\operatorname{Im} f_i}$ is injective and then $(\phi_i)_\alpha$ is also injective for all $i \in \mathbb{N}$, by Lemma 3.3. Thus, the homomorphism $\phi_\alpha = \varinjlim(\phi_i)_\alpha : \varinjlim(\operatorname{Im} f_i)_\alpha \to \varinjlim(\operatorname{Im} g_i)_\alpha$ is an injective map by Corollary 3.13.

Set $\psi_i := \psi|_{\operatorname{Im} g_i}$, for all $i \in \mathbb{N}$. We have a homomorphism

$$\psi = \varinjlim \psi_i : \varinjlim \operatorname{Im} g_i \to \varinjlim \operatorname{Im} h_i.$$

Since each $\psi_i : \operatorname{Im} g_i \to \psi(\operatorname{Im} g_i)$ is a surjective map and ψ is surjective,

$$\varinjlim \operatorname{Im} h_i = \psi(\varinjlim \operatorname{Im} g_i) = \varinjlim \psi(\operatorname{Im} g_i) = \varinjlim \psi_i(\operatorname{Im} g_i).$$

By virtue of Theorem 3.14 one can obtain $\varinjlim(\operatorname{Im} h_i)_\alpha = \varinjlim \psi_i(\operatorname{Im} g_i)_\alpha$. By the definition of specialization of a limit-homomorphism, there is a homomorphism $\psi_\alpha = \varinjlim(\psi_i)_\alpha : \varinjlim(\operatorname{Im} g_i)_\alpha \to \varinjlim \psi_i(\operatorname{Im} g_i)_\alpha$. Similarly, we have the homomorphism $(\psi \cdot \phi)_\alpha$.

https://doi.org/10.1017/S0013091505000891 Published online by Cambridge University Press

Since each $\psi_i : \operatorname{Im} g_i \to \psi(\operatorname{Im} g_i)$ is a surjective map, the homomorphism $(\psi_i)_\alpha$ is an epimorphism by Lemma 3.3 and $\psi_\alpha = \varinjlim(\psi_i)_\alpha$ is also an epimorphism by Corollary 3.13. Since $(\psi_i)_\alpha((\operatorname{Im} g_i)_\alpha) = \psi_i(\operatorname{Im} g_i)_\alpha$, $\psi_\alpha : \varinjlim(\operatorname{Im} g_i)_\alpha \to \varinjlim(\psi_i)_\alpha((\operatorname{Im} g_i)_\alpha)$ is an epimorphism.

From Corollary 3.13 we have

$$\varinjlim(\operatorname{Im} h_i)_\alpha = \varinjlim\{(\psi_i)_\alpha((\operatorname{Im} g_i)_\alpha)\} = \varinjlim(\psi_i)_\alpha(\varinjlim(\operatorname{Im} g_i)_\alpha) = \psi_\alpha(\varinjlim(\operatorname{Im} g_i)_\alpha).$$

Hence, the homomorphism $\psi_\alpha : \varinjlim(\operatorname{Im} g_i)_\alpha \to \varinjlim(\operatorname{Im} h_i)_\alpha$ is also an epimorphism. Since $\phi = \varinjlim\phi_i$ and $\psi = \varinjlim\psi_i$, we have $\operatorname{Im}\phi = \varinjlim\phi_i(\operatorname{Im} f_i)$ and $\operatorname{Ker}\psi = \varinjlim \operatorname{Ker}\psi_i$. By the universal property for direct limits there is $\psi\cdot\phi = \varinjlim\psi_i\cdot\phi_i$. Now $(\psi\cdot\phi)_\alpha = \psi_\alpha\cdot\phi_\alpha$ by Corollary 3.16. By Definition 3.15 we have

$$\begin{aligned}(\operatorname{Im}\phi)_\alpha &= \varinjlim\phi_i(\operatorname{Im} f_i)_\alpha = \phi_\alpha(\varinjlim(\operatorname{Im} f_i)_\alpha) = \operatorname{Im}\phi_\alpha,\\ (\operatorname{Ker}\psi)_\alpha &= \varinjlim(\operatorname{Ker}\psi_i)_\alpha = \varinjlim \operatorname{Ker}((\psi_i)_\alpha) \quad \text{by [7, Corollary 2.5]}\\ &= \operatorname{Ker}(\varinjlim(\psi_i)_\alpha) = \operatorname{Ker}\psi_\alpha.\end{aligned}$$

Since $\operatorname{Im}\phi = \operatorname{Ker}\psi$, $(\operatorname{Im}\phi)_\alpha = (\operatorname{Ker}\psi)_\alpha$ by Theorem 3.14 so $\operatorname{Im}\phi_\alpha = \operatorname{Ker}\psi_\alpha$. Hence, the sequence

$$0 \to \varinjlim \operatorname{Im}(f_i)_\alpha \xrightarrow{\phi_\alpha} \varinjlim \operatorname{Im}(g_i)_\alpha \xrightarrow{\psi_\alpha} \varinjlim \operatorname{Im}(h_i)_\alpha \to 0$$

is an exact sequence, so the sequence $0 \to L_\alpha \xrightarrow{\phi_\alpha} M_\alpha \xrightarrow{\psi_\alpha} N_\alpha \to 0$ is exact for almost all α. □

4. Specialization of local cohomology

We will study first the relation between Ext^j_S and direct limits under specializations.

Proposition 4.1. *Let L be the direct limit of a direct system of finitely generated S-modules over the directed set $\mathbb{N}$. Let M be a finitely generated S-module. Then, for almost all α, we have $\operatorname{Ext}^j_S(M,L)_\alpha \cong \operatorname{Ext}^j_{S_\alpha}(M_\alpha, L_\alpha)$, $j \geqslant 0$.*

Proof. Let $L = \varinjlim L_i$. Since

$$\operatorname{Ext}^j_S(M,L) = \operatorname{Ext}^j_S(M,\varinjlim L_i) = \varinjlim \operatorname{Ext}^j_S(M,L_i), \quad j \geqslant 0,$$

by [9, Lemma 3.3.7], for almost all α, we have

$$\operatorname{Ext}^j_S(M,L)_\alpha = \varinjlim \operatorname{Ext}^j_S(M,L_i)_\alpha.$$

Since $\operatorname{Ext}^j_S(M,L_i)_\alpha \cong \operatorname{Ext}^j_{S_\alpha}(M_\alpha,(L_i)_\alpha)$ by Lemma 2.5, for almost all α, there are isomorphisms

$$\operatorname{Ext}^j_S(M,L)_\alpha \cong \varinjlim \operatorname{Ext}^j_{S_\alpha}(M_\alpha,(L_i)_\alpha) = \operatorname{Ext}^j_{S_\alpha}(M_\alpha, L_\alpha), \quad j \geqslant 0,$$

by Theorem 3.20. □

https://doi.org/10.1017/S0013091505000891 Published online by Cambridge University Press

Since $H^j_{\mathfrak{a}}(M) = \varinjlim \operatorname{Ext}^j_S(S/\mathfrak{a}^i, M)$ and $\operatorname{Ext}^j_S(S/\mathfrak{a}^i, M)$ is a finitely generated S-module, $H^j_{\mathfrak{a}}(M)$ may be specialized and $H^j_{\mathfrak{a}}(M)_\alpha = \varinjlim \operatorname{Ext}^j_S(S/\mathfrak{a}^i, M)_\alpha$ by the above definition. Thus, we can specialize local cohomology modules of finitely generated S-modules, even when they themselves are not finitely generated modules.

Theorem 4.2. *Let L be a finitely generated S-module and let $\mathfrak{a}$ be an ideal of S. Then, for almost all α, there is*

$$H^j_{\mathfrak{a}}(L)_\alpha \cong H^j_{\mathfrak{a}_\alpha}(L_\alpha), \quad j \geqslant 0.$$

Proof. Since $H^j_{\mathfrak{a}}(L) = \varinjlim \operatorname{Ext}^j_S(S/\mathfrak{a}^i, L)$, $j \geqslant 0$, we have

$$H^j_{\mathfrak{a}}(L)_\alpha = \varinjlim \operatorname{Ext}^j_S(S/\mathfrak{a}^i, L)_\alpha.$$

Since $\operatorname{Ext}^j_S(S/\mathfrak{a}^i, L)_\alpha \cong \operatorname{Ext}^j_{S_\alpha}(S_\alpha/\mathfrak{a}^i_\alpha, L_\alpha)$ by Lemma 2.5, we obtain

$$H^j_{\mathfrak{a}}(L)_\alpha \cong \varinjlim \operatorname{Ext}^j_{S_\alpha}(S_\alpha/\mathfrak{a}^i_\alpha, L_\alpha) = H^j_{\mathfrak{a}_\alpha}(L_\alpha)$$

for almost all α. □

We use $E(S/\mathfrak{m})$ to denote the injective envelope of $S/\mathfrak{m}$. Since $E(S/\mathfrak{m}) = H^t_{\mathfrak{m}}(S)$, $E(S/\mathfrak{m})$ can be specialized. We shall see that forming the injective envelope commutes with a specialization.

Corollary 4.3. *For almost all α, we have $E(S_\alpha/\mathfrak{m}_\alpha) \cong E(S/\mathfrak{m})_\alpha$.*

Proof. Set $\dim S = t$. Then $\dim S_\alpha = t$ by Lemma 2.4. Since S and S_α are Gorenstein rings, $E(S/\mathfrak{m}) = H^t_{\mathfrak{m}}(S)$, and $E(S_\alpha/\mathfrak{m}_\alpha) = H^t_{\mathfrak{m}_\alpha}(S_\alpha)$ by [**9**, Corollary 10.1.10]. By Theorem 4.2, $E(S/\mathfrak{m})_\alpha = H^t_{\mathfrak{m}}(S)_\alpha = H^t_{\mathfrak{m}_\alpha}(S_\alpha) = E(S_\alpha/\mathfrak{m}_\alpha)$ for almost all α. □

Now k will be assumed to be a perfect field. The preservation of the Buchsbaum property of modules by specialization was proved in [**8**, Corollary 3.8] by using its characterization in terms of systems of parameters. This result will be now reproved by using Theorem 4.2.

Theorem 4.4. *Let L be a Buchsbaum S-module. Then L_α is a Buchsbaum S_α-module for almost all α.*

Proof. By [**10**, Corollary 2.16] the Buchsbaum property of L means that the canonical map $\phi^i_L : \operatorname{Ext}^i_S(S/\mathfrak{m}, L) \to H^i_{\mathfrak{m}}(L)$ is surjective for all $i < \dim L$. We have $\dim L_\alpha = \dim L$ by Lemma 2.4. Since $\operatorname{Ext}^i_S(S/\mathfrak{m}, L)_\alpha \cong \operatorname{Ext}^j_{S_\alpha}(S_\alpha/\mathfrak{m}_\alpha, L_\alpha)$ by Lemma 2.5 and $H^i_{\mathfrak{m}}(L)_\alpha \cong H^j_{\mathfrak{m}_\alpha}(L_\alpha)$ by Theorem 4.2, $\phi^i_{L_\alpha}$ are surjective for all $i < \dim L_\alpha$ by Theorem 3.20. Thus, L_α is a Buchsbaum module. □

Recall that, more generally, a non-zero finitely generated module L of dimension $d > 0$ is said to be a *generalized Cohen–Macaulay module* if $H^i_{\mathfrak{m}}(L)$ is finitely generated for all $i = 0, \dots, d-1$ (see, for example, [**12**]).

https://doi.org/10.1017/S0013091505000891 Published online by Cambridge University Press

Theorem 4.5. *Let L be a generalized Cohen–Macaulay S-module. Then L_α is also a generalized Cohen–Macaulay S_α-module for almost all α.*

Proof. Assume that L is a generalized Cohen–Macaulay S-module of dimension $d > 0$. We have $\dim L_\alpha = d$ by Lemma 2.4. Since $H^i_{\mathfrak{m}}(L)$ is finitely generated for all $i = 0, \dots, d-1$, $H^i_{\mathfrak{m}}(L)_\alpha$ is also finitely generated for all $i = 0, \dots, d-1$ by Corollary 3.19. But $H^j_{\mathfrak{m}_\alpha}(M_\alpha) \cong H^j_{\mathfrak{m}}(L)_\alpha$ by Theorem 4.2, so $H^j_{\mathfrak{m}_\alpha}(M_\alpha)$ is finitely generated for all $i = 0, \dots, d-1$. Thus, L_α is also a generalized Cohen–Macaulay S_α-module for almost all α. □

In [**1**], the ith *pseudo-support* $\operatorname{Psupp}^i(L)$ and ith *pseudo-dimension* $\operatorname{psd}^i(L)$ of L are defined as follows:

$$\operatorname{Psupp}^i(L) = \{\mathfrak{p} \in \operatorname{Spec}(S) \mid H^{i-\dim S/\mathfrak{p}}_{\mathfrak{p}S_{\mathfrak{p}}}(L_{\mathfrak{p}}) \neq 0\},$$
$$\operatorname{psd}^i(L) = \sup\{\dim S/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Psupp}^i(L)\}.$$

Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal of S. The multiplicity of local cohomology modules is defined by

$$e'(\mathfrak{q}, H^i_{\mathfrak{m}}(L)) = \sum_{\substack{\mathfrak{p}\in\operatorname{Psupp}^i(L),\\ \dim S/\mathfrak{p}=\operatorname{psd}^i(L)}} \ell_{S_{\mathfrak{p}}}(H^{i-\dim S/\mathfrak{p}}_{\mathfrak{p}S_{\mathfrak{p}}}(L_{\mathfrak{p}}))e(\mathfrak{q}, S/\mathfrak{p}).$$

We shall see that the pseudo-dimension and the multiplicity of local cohomology modules of L are preserved under specializations.

First recall that a sequence of elements $\underline{a} = a_1, \dots, a_p$ in $\mathfrak{m}$ is a multiplicity system of L if $\lambda(L/(\underline{a})L)$ is finite, where λ is the length of modules. Let $\mathfrak{q} = (a_1, \dots, a_p)S$, an ideal of S. Let d denote the dimension of L. We know that the *multiplicity* of L with respect to $\mathfrak{q}$ is defined as the number

$$e(\mathfrak{q}, L) = \lim_{h\to\infty} \frac{\lambda(L/\mathfrak{q}^h L)\cdot d!}{h^d}.$$

Note that $e(\mathfrak{q}, S)$ will be denoted by $e(\mathfrak{q})$. The following lemma shows that the multiplicity of L with respect to $\mathfrak{q}$ is unchanged by a specialization.

Lemma 4.6. *Let L be a finitely generated S-module of dimension d and let*

$$\mathfrak{q} = (y_1, \dots, y_d)S$$

be a parameter ideal on L. Then $e(\mathfrak{q}_\alpha; L_\alpha) = e(\mathfrak{q}; L)$ for almost all α.

Proof. Since $y_1, \dots, y_d \in PS$, for almost all α there are $(y_1)_\alpha, \dots, (y_d)_\alpha \in P_\alpha S_\alpha$. By Lemmas 2.3 and 2.4, $\dim L_\alpha/((y_1)_\alpha, \dots, (y_d)_\alpha)L_\alpha = \dim L/(y_1, \dots, y_d)L = 0$. Then $(y_1)_\alpha, \dots, (y_d)_\alpha$ is a system of parameters on L_α. Since

$$\lambda(L/\mathfrak{q}^h L) = \lambda(L_\alpha/\mathfrak{q}^h_\alpha L_\alpha), \quad \text{for all } h \in \mathbb{N},$$

by [**8**, Proposition 2.8] and by Lemma 3.1, $e(\mathfrak{q}_\alpha; L_\alpha) = e(\mathfrak{q}; L)$ for almost all α. □

https://doi.org/10.1017/S0013091505000891 Published online by Cambridge University Press

Set $K_L^i = \operatorname{Hom}_S(H_{\mathfrak{m}}^i(L), E(S/\mathfrak{m}))$, $i = 0, \dots, d$. The modules K_L^i are again finitely generated S-modules. Clearly, $K_L^i = 0$ for all $i < 0$ and $i > \dim L$. We often write K_L instead of K_L^d. This module K_L is called the *canonical module* of L (see [**1**, **10**]).

Lemma 4.7. *Let L be a finitely generated S-module. Then $(K_L^i)_\alpha \cong K_{L_\alpha}^i$, $i = 0, \dots, d$, for almost all α.*

Proof. By Lemma 2.4, $\dim L_\alpha = \dim L = d$ for almost all α. Since $H_{\mathfrak{m}_\alpha}^j(L_\alpha) \cong H_{\mathfrak{m}}^j(L)_\alpha$ by Theorem 4.2 and $E(S_\alpha/\mathfrak{m}_\alpha) \cong E(S/\mathfrak{m})_\alpha$ by Corollary 4.3,

$$\operatorname{Hom}_S(H_{\mathfrak{m}}^i(L), E(S/\mathfrak{m}))_\alpha \cong \operatorname{Hom}_{S_\alpha}(H_{\mathfrak{m}_\alpha}^i(L), E(S_\alpha/\mathfrak{m}_\alpha))$$

by Proposition 4.1 and so $(K_L^i)_\alpha \cong K_{L_\alpha}^i$ for almost all α. □

Proposition 4.8. *Let L be a finitely generated S-module and $\mathfrak{q}$ an $\mathfrak{m}$-primary ideal of S. Set $\mathfrak{n} = \mathfrak{m}_\alpha$. Then, for almost α, we have*

(i) *an ideal $\mathfrak{b}$ of S with $\operatorname{Psupp}^i(L) = V(\mathfrak{b})$ such that $\operatorname{Psupp}^i(L_\alpha) = V(\mathfrak{b}_\alpha)$,*

(ii) $e'(\mathfrak{q}_\alpha, H_{\mathfrak{n}}^i(L_\alpha)) = e'(\mathfrak{q}, H_{\mathfrak{m}}^i(L))$,

(iii) $\operatorname{psd}^i(L_\alpha) = \operatorname{psd}^i(L)$.

Proof. (i) By [**1**, Proposition 1.2 (iii)], $\operatorname{Psupp}^i(L) = \operatorname{Supp}(K_L^i) = V(\mathfrak{b})$, where $\mathfrak{b} = \operatorname{Ann} K_L^i$. Since $(K_L^i)_\alpha \cong K_{L_\alpha}^i$ by Lemma 4.7 and $\mathfrak{b}_\alpha = (\operatorname{Ann} K_L^i)_\alpha = \operatorname{Ann} K_{L_\alpha}^i$ by Lemma 2.4, $\operatorname{Psupp}^i(L_\alpha) = \operatorname{Supp}(K_{L_\alpha}^i) = V(\mathfrak{b}_\alpha)$.

(ii) By [**1**, Proposition 1.2 (ii)], we know that

$$e'(\mathfrak{q}, H_{\mathfrak{m}}^i(L)) = e(\mathfrak{q}, K_L^i) \quad \text{and} \quad e'(\mathfrak{q}_\alpha, H_{\mathfrak{n}}^i(L_\alpha)) = e(\mathfrak{q}_\alpha, K_{L_\alpha}^i).$$

Since $(K_L^i)_\alpha \cong K_{L_\alpha}^i$ by Lemma 4.7 and $e(\mathfrak{q}_\alpha, (K_L^i)_\alpha) = e(\mathfrak{q}, K_L^i)$ by Lemma 4.6,

$$e(\mathfrak{q}_\alpha, K_{L_\alpha}^i) = e(\mathfrak{q}, K_L^i).$$

Hence, $e'(\mathfrak{q}_\alpha, H_{\mathfrak{n}}^i(L_\alpha)) = e'(\mathfrak{q}, H_{\mathfrak{m}}^i(L))$.

(iii) By [**1**, Theorem 2.4], we know that $\operatorname{psd}^i(M)$ is equal to the dimension of $H_{\mathfrak{m}}^i(L)$. Since $\operatorname{Ann} H_{\mathfrak{n}}^i(L_\alpha) = \operatorname{Ann} H_{\mathfrak{m}}^i(L)_\alpha$ by [**8**, Lemma 3.5], upon simple computation, we get $\dim H_{\mathfrak{n}}^i(L_\alpha) = \dim H_{\mathfrak{m}}^i(L)$. Hence, $\operatorname{psd}^i(L_\alpha) = \operatorname{psd}^i(L)$ for almost all α. □

We want to prove again that specializations of graded modules preserve the *Castelnuovo–Mumford regularity* (see [**3**, **6**]).

For a finitely generated graded R-module $L = \bigoplus_{t\in\mathbb{Z}} L_t$, we set

$$a(L) = \begin{cases} \max\{t \mid L_t \neq 0\} & \text{if } L \neq 0, \\ -\infty & \text{if } L = 0, \end{cases}$$

$$\operatorname{reg}(L) = \max\{a(H_{\mathfrak{m}}^i(L)) + i \mid i \geqslant 0\}.$$

The number $\operatorname{reg}(L)$ is called the *Castelnuovo–Mumford regularity* of L.

In [**6**] it is known that if $F = \bigoplus_{j=1}^{s} R(-h_j)$ is a free graded R-module, then its specialization $F_\alpha = \bigoplus_{j=1}^{s} R_\alpha(-h_j)$ is again a free graded R-module and if

$$\phi : \bigoplus_{j=1}^{s_1} R(-h_{1j}) \to \bigoplus_{j=1}^{s_0} R(-h_{0j})$$

is a graded homomorphism of degree 0, then the homomorphism

$$\phi_\alpha : \bigoplus_{j=1}^{s_1} R_\alpha(-h_{1j}) \to \bigoplus_{j=1}^{s_0} R_\alpha(-h_{0j})$$

is also a graded homomorphism of degree 0. Therefore, L_α is a graded R_α-module for almost all α if L is a finitely generated graded R-module, and if

$$\boldsymbol{F}_\bullet : 0 \to F_\ell \xrightarrow{\phi_\ell} F_{\ell-1} \to \cdots \to F_1 \xrightarrow{\phi_1} F_0$$

is a minimal graded free resolution of L, then the complex

$$(\boldsymbol{F}_\bullet)_\alpha : 0 \to (F_\ell)_\alpha \xrightarrow{(\phi_\ell)_\alpha} (F_{\ell-1})_\alpha \to \cdots \to (F_1)_\alpha \xrightarrow{(\phi_1)_\alpha} (F_0)_\alpha \to L_\alpha \to 0$$

is a minimal graded free resolution of L_α with the same graded Betti numbers for almost all α. Now we shall see that the Castelnuovo–Mumford regularity of a graded R-module is preserved under specializations.

Proposition 4.9. *Let L be a finitely generated graded R-module. For almost all α, we have* $\operatorname{reg}(L_\alpha) = \operatorname{reg}(L)$.

Proof. The equality $a(H^i_{\mathfrak{m}}(L)_\alpha) = a(H^i_{\mathfrak{m}}(L))$ follows from the definition of $H^i_{\mathfrak{m}}(L)_\alpha$. Since $H^i_{\mathfrak{a}}(L)_\alpha \cong H^i_{\mathfrak{a}_\alpha}(L_\alpha)$ by Theorem 4.2, $a(H^i_{\mathfrak{m}_\alpha}(L_\alpha)) = a(H^i_{\mathfrak{m}}(L)_\alpha)$ for most all α. Hence, $\operatorname{reg}(L_\alpha) = \operatorname{reg}(L)$ for almost all α. □

Acknowledgements. The author is grateful to N. V. Trung for his guidance and encouragement during the preparation of this paper and to Iain Gordon (an editor of the journal) and the referee for suggestions.

References

1. M. Brodmann and R. Y. Sharp, On the dimension and multiplicity of local cohomology modules, *Nagoya Math. J.* **167** (2002), 217–233.
2. S. Eilenberg and N. Steenrod, *Foundations of algebraic topology* (Princeton University Press, 1952).
3. D. Eisenbud and S. Goto, Linear free resolutions and minimal multiplicity, *J. Alg.* **88** (1984), 89–133.
4. W. Krull, Parameterspezialisierung in Polynomringen, *Arch. Math.* **1** (1948), 56–64.
5. H. Matsumura, *Commutative ring theory* (Cambridge University Press, 1986).
6. D. V. Nhi, Specialization of graded modules, *Proc. Edinb. Math. Soc.* **45** (2002), 491–506.
7. D. V. Nhi and N. V. Trung, Specialization of modules, *Commun. Alg.* **27** (1999), 2959–2978.

8. D. V. Nhi and N. V. Trung, Specialization of modules over a local ring, *J. Pure Appl. Alg.* **152** (2000), 275–288.
9. J. R. Strooker, *Homological questions in local algebra* (Cambridge University Press, 1990).
10. J. Stückrad and W. Vogel, *Buchsbaum rings and applications* (Springer, 1986).
11. N. V. Trung, Spezialisierungen allgemeiner Hyperflächenschnitte und Anwendungen, in *Seminar D. Eisenbud/B. Singh/W. Vogel, Volume 1*, Teubner-Texte zur Mathematik, Band 29, pp. 4–43 (Teubner, 1980).
12. N. V. Trung, Toward a theory of generalized Cohen–Macaulay modules, *Nagoya Math. J.* **102** (1986), 4–43.