ON AN INTEGRAL TRANSFORM

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1. Introduction. In this paper the author continues the search for a suitable integral transform that can be applied to certain boundary value problems involving the Helmholtz equation and the condition of radiation. The transform in question must be capable of eliminating the r-dependence appearing in the partial differential equation

$$r^2 w_{rr} + r w_r + k^2 r^2 w + w_{\psi\psi} = 0. \tag{1}$$

Here $k > 0$, (r, ψ) are polar coordinates and the coordinate r varies over some infinite interval $a \leq r < \infty$, where $a > 0$. In the type of problem considered the function w is $O(r^{-1/2})$ as $r \to \infty$ and satisfies the radiation condition

$$\lim_{r \to \infty} r^{1/2}(w_r - ikw) = 0. \tag{2}$$

The transform that is adapted to treat such a problem is that defined by the equation

$$F_1(v) = \int_a^\infty H_v^{(1)}(kr) f(r) \frac{dr}{r}, \tag{3}$$

where the kernel is a Hankel function of the first kind. The formal inverse of (3) is an expansion involving the eigenfunctions $H_{v_n}^{(1)}(kr)$ where $v_1, v_2, \ldots$ are the zeros of the function $H_v^{(1)}(ka)$ regarded as a function of the order v. However these zeros are complex and the expansion itself is sometimes convergent and sometimes divergent, a phenomenon that has not as yet been satisfactorily explained [**2**, **6**]. In an attempt to resolve this problem the author [**3**, **4**] has suggested two alternative formulas of inversion for the transform (3), one as a series and the other as an integral. However both of these formulas involve a summability factor of one kind or another, which renders their application somewhat cumbersome. The author [**5**] has also derived a formula of inversion for the related transform

$$F_2(v) = \int_a^\infty [J_v(kr) H_v^{(1)}(ka) - J_v(ka) H_v^{(1)}(kr)] f(r) \frac{dr}{r}.$$

This transform, first proposed by Chakrabarti [**1**], can, like (3), be applied to problems similar to that outlined above for the function $w(r, \psi)$ but its formula of inversion also requires a summability factor.

With a view to avoiding the complications associated with summability factors the author proposes an alternative transform in which the kernel involves the Neumann function $Y_u(kr)$ rather than a Hankel function. The resulting transform, which does not appear to have been used previously, is that defined by the equation

$$F(u) = \int_a^\infty Y_u(kr) f(r) \frac{dr}{r}. \tag{4}$$

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Despite the singular non-self-adjoint nature of the underlying expansion problem, the transform (4) does possess a formula of inversion which expresses $f(r)$ as the sum of a series and an integral both of which are convergent in the classical sense. The formula of inversion in question appears as equation (5) of the theorem that follows.

THEOREM. *Suppose that $f(r)$ is twice continuously differentiable for $r \geqslant a$, $f(r)$ and $f'(r)$ are $O(r^{-1/2})$ as $r \to \infty$, $r^{-1/2}(rf_{rr}+f_r+k^2rf) \in L(a,\infty)$ and $r^{1/2}(f_r - ikf) \to 0$ as $r \to \infty$, where k is real and positive. Let $F(u)$ be defined by the equation* (4); *then if $r > a$,*

$$f(r) = -\frac{1}{2i}\int_{-\infty}^{\infty} \frac{[J_{is}(kr)Y_{is}(ka) - J_{is}(ka)Y_{is}(kr)]F(is)s\,ds}{Y_{is}(ka)} + \pi \sum_{u=u_n} \frac{uJ_u(ka)Y_u(kr)F(u)}{(\partial/\partial u)Y_u(ka)}, \tag{5}$$

where the summation includes all the real negative zeros u_n of the function $Y_u(ka)$ regarded as a function of the order u.

It should be noticed that the underlying expansion problem is not self-adjoint so that the formula of inversion cannot be constructed by following a procedure such as that described in Titchmarsh's treatise [**7**] on this subject. Instead the method developed by the author to invert the transform (3) has been applied. This method is described in [**4**].

Before proceeding to the proof of the above theorem it is necessary to investigate the distribution of the zeros of the function $Y_u(x)$ regarded as a function of the order u, when the argument x is supposed to be prescribed and positive. This is carried out in the next section of this paper where it is proved that the function $Y_u(x)$ possesses three infinite sets of zeros as follows:

(i) an infinite set of real zeros u_n where $u_{n+1} < u_n < \ldots < u_1$ such that $u_n \to -\infty$ as $n \to \infty$; for n large the zero u_n is given by the asymptotic formula

$$u_n = -(n+\tfrac{1}{2}) + \frac{1}{2\pi}(ex/2n+1)^{2n+1}: \tag{6}$$

(ii) an infinite set of complex zeros u'_n located in the first quadrant of the complex u-plane together with the corresponding set of conjugate complex zeros $\bar{u}'_n$ in the fourth quadrant, the zeros $u'_n = R_n e^{i\theta_n}$ are such that $\mathrm{Re}(u'_n)$ and $|\mathrm{Im}(u'_n)|$ tend to $+\infty$ as $n \to \infty$ and for large n are given by the asymptotic equations

$$\theta_n = \frac{\pi}{2}\left[1 - \frac{1}{\log(2R_n/ex)}\right], \tag{7}$$

$$R_n \log(2R_n/ex) = (n - \tfrac{1}{4})\pi. \tag{8}$$

2. The distribution of the eigenvalues. The following arguments can be used to investigate the properties of the zeros of the function $Y_u(x)$. We write $y = Y_u(kr)$ so that

$$r^2 y_{rr} + ry_r + (k^2r^2 - u^2)y = 0. \tag{9}$$

This equation is multiplied by $r^{-1}\bar{y}$, where $\bar{y}$ denotes the complex conjugate of y. The resulting equation is subtracted from its conjugate and integrated for $a \leqslant r < \infty$. Upon

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setting $x = ka$, we obtain the equation

$$(u^2-\bar{u}^2)\int_a^\infty |y|^2\frac{dr}{r} = \lim_{r\to\infty} r(\bar{y}y_r - y\bar{y}_r) + \pi x \operatorname{Im} Y_u(x)Y'_{\bar{u}}(x)$$

The value of the limit appearing in the above equation can be obtained after substituting the Hankel asymptotic formulas for the Bessel functions of large argument. Upon setting $u = t + is$ we obtain the equation

$$2\pi st\int_a^\infty |y|^2\frac{dr}{r} = \sinh(s\pi) + \pi x \operatorname{Im} Y_u(x)Y'_{\bar{u}}(x), \tag{10}$$

where $x = ka$. It follows immediately from this equation that complex zeros of $Y_u(x)$ must be positioned in the half plane $\operatorname{Re}(u) > 0$, since $t > 0$ whenever $s \neq 0$. In particular there are no purely imaginary zeros. In addition since the complex conjugate of $Y_u(x)$ is $Y_{\bar{u}}(x)$ it also follows that complex zeros occur at pairs of conjugate points $(u'_n, \bar{u}'_n)$. Further information on the distribution of the zeros can be obtained from the equation [**8**, p. 178]

$$\pi Y_u(x) = \int_0^\pi \sin(x\sin\theta - u\theta)\,d\theta - \int_0^\infty \exp(u\theta - x\sinh\theta)\,d\theta$$
$$-\cos u\pi\int_0^\infty \exp(-u\theta - x\sinh\theta)\,d\theta. \tag{11}$$

It can be shown from this equation that for given positive x the function $Y_u(x)$ cannot vanish as $\operatorname{Re}(u)\to+\infty$ if $\operatorname{Im}(u)$ remains bounded. For upon expanding the trigonometric function and integrating by parts it is seen that the first integral on the right hand side of (11) is $O(u^{-1})$ as $u\to\infty$ with $\operatorname{Im}(u)$ bounded. The third integral in (11) is also $O(u^{-1})$, since its modulus cannot exceed

$$\int_0^\infty \exp(-t\theta)\,d\theta = t^{-1}.$$

It follows that

$$\pi Y_u(x) = O(u^{-1}) - \int_0^\infty \exp(u\theta - x\sinh\theta)\,d\theta \tag{12}$$

as $\operatorname{Re}(u)\to+\infty$ with $\operatorname{Im}(u)$ bounded. Since in these conditions the integral surviving in (12) tends to infinity it follows that $Y_u(x)$ cannot vanish as $\operatorname{Re}(u)\to+\infty$ in any strip in which $\operatorname{Im}(u)$ is bounded. In particular $Y_u(x)$ does not vanish whenever u is sufficiently large and positive.

A formula to determine the large complex zeros can be obtained by using the formula

$$J_u(x) = \frac{(x/2)^u}{\Gamma(u+1)}[1 + O(u^{-1})]. \tag{13}$$

This formula holds for fixed x and u large enough and bounded away from the negative

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integers. If (13) is substituted in the equation defining the function $Y_u(x)$, that is

$$Y_u(x)=\frac{J_u(x)\cos u\pi - J_{-u}(x)}{\sin u\pi}, \tag{14}$$

then, after using the identity $\Gamma(u)\Gamma(1-u)\sin u\pi=\pi$, it is found that

$$Y_u(x)=\left[\frac{(x/2)^u \cot u\pi}{\Gamma(u+1)}-\frac{1}{\pi}(x/2)^{-u}\Gamma(u)\right]\left[1+O(u^{-1})\right] \tag{15}$$

for large u bounded away from the integers. The Γ-functions appearing in the preceding formula may be estimated with the aid of Stirling's formula

$$\Gamma(u)=(2\pi/u)^{1/2}\exp(u\log u-u)[1+O(u^{-1})], \tag{16}$$

which holds for $u\to\infty$ in $|\arg u|\leqslant\pi-\delta$. Substitution of (16) into (15) leads to the equation

$$Y_u(x)=2(\pi u)^{-1/2}[\cot u\pi\exp\{-u\log(2u/ex)\}-2\exp\{u\log(2u/ex)\}]\left[1+O\left(\frac{1}{u}\right)\right]. \tag{17}$$

To determine the complex zeros situated in the first quadrant we may suppose, as proved above, that $\mathrm{Im}(u)\to+\infty$ as $u\to\infty$ so that $\cot u\pi\to-i$. On simplifying equation (17) by means of this result we find, after slight rearrangement, that

$$Y_u(x)=-(2\pi u)^{-1/2}e^{-i\pi/4}\sinh[u\log(2u/ex)+\log\sqrt{2}+\tfrac{1}{4}i\pi][1+O(u^{-1})] \tag{18}$$

as $u\to\infty$ in $\delta\leqslant\arg u\leqslant\pi-\delta$. The corresponding formula valid as $u\to\infty$ in the region $-(\pi-\delta)\leqslant\arg u\leqslant-\delta$ can be obtained from (17) by noting that now $\cot u\pi\to+i$ giving the formula

$$Y_u(x)=-2(\pi u)^{-1/2}e^{-i\pi/4}\sinh[u\log(2u/ex)+\log\sqrt{2}-\tfrac{1}{4}i\pi][1+O(u^{-1})]. \tag{19}$$

Complex zeros occur only in the half plane $\mathrm{Re}(u)>0$ and so by (18) those located in the first quadrant are given by the equation

$$u\log(2u/ex)+\log\sqrt{2}=(n-\tfrac{1}{4})i\pi, \tag{20}$$

where n is a large positive integer. Upon setting $u=Re^{i\theta}$ in (20) and equating real and imaginary parts, we find the equations

$$R\cos\theta\log(2R/ex)-R\theta\sin\theta=-\log 2, \tag{21}$$

$$R\sin\theta\log(2R/ex)+R\theta\cos\theta=(n-\tfrac{1}{4})\pi. \tag{22}$$

It follows from equation (21) after division by the factor $R\log(2R/ex)$ that $\cos\theta\to 0$ and therefore $\theta\to\pi/2$ as $R\to\infty$. We therefore set $\theta=\pi/2-\varepsilon$, where ε is small, and determine ε. This leads, after slight calculation, to the formulas (7), (8) already stated.

It remains to discuss the large negative zeros of $Y_u(x)$. The existence of such zeros is well known; here we wish only to obtain the formula (6). If $u\to-\infty$ it is now the third integral in (11) that is the dominant one, the remaining integrals being each $O(u^{-1})$ so that

$$\pi Y_u(x) = O(u^{-1}) - \cos u\pi \int_0^\infty \exp(-u\theta - x \sinh \theta)\, d\theta$$

as $u \to -\infty$. As $u \to -\infty$ the function $Y_u(x)$ alternates in sign since the integral in the preceding equation increases without bound. It is evident that the large negative zeros of $Y_u(x)$ must be such that $\cos u\pi \to 0$ so that u_n must tend to half an odd negative integer. To construct the formula (6) the equation (15) must be rewritten in terms of Γ-functions of positive argument to permit the use of Stirling's formula. This can be carried out with the aid of the identity $\Gamma(u)\Gamma(1-u)\sin u\pi = \pi$ which enables (15) to be written in the form

$$Y_u(x) = \left[-\frac{1}{\pi}(x/2)^u\Gamma(-u)\cos u\pi + \frac{1}{u}(x/2)^{-u}\Gamma(-u)\operatorname{cosec} u\pi\right]\left[1 + O\left(\frac{1}{u}\right)\right]. \tag{23}$$

Since u is now large and negative we may write, by (13),

$$\Gamma(-u) = (2\pi/|u|)^{1/2} \exp[|u| \log |u| - |u|][1 + O(u^{-1})]$$

as $u \to -\infty$. After inserting this formula into the right hand side of (23) we find that the large real zeros of $Y_u(x)$ satisfy the equation

$$\sin 2u\pi = -\exp[-2|u| \log |2u/ex|][1 + O(u^{-1})].$$

Since $u_n \to -(n+\frac{1}{2})$ an approximate solution of this equation can be obtained by setting $u_n = -(n+\frac{1}{2}) + \varepsilon$ and solving for ε. This procedure leads after some calculation to the formula (6).

3. The Green's function. To construct the expansion formula (5) let $f(r)$, the function to be expanded, satisfy the conditions of the theorem and let $g(r)$ be defined by the equation

$$r^2 f_{rr} + r f_r + (k^2 r^2 - v^2) f = g(r), \tag{24}$$

where $r \geq a$ and v is some negative number which is not a zero of $Y_u(ka)$. The equation (24) is now inverted in terms of a suitable Green's function. The Green's function that generates the expansion in question is defined by the equations

$$G(r,\rho) = \begin{cases} \pi[J_v(kr)Y_v(ka) - J_v(ka)Y_v(kr)]Y_v(k\rho)/2Y_v(ka) & (a \leq r \leq \rho), \\ \pi[J_v(k\rho)Y_v(ka) - J_v(ka)Y_v(k\rho)]Y_v(kr)/2Y_v(ka) & (a \leq \rho \leq r). \end{cases} \tag{25}$$

Upon inverting the equation (24) in terms of the above Green's function it is found that

$$f(r) = f(a)Y_v(kr)/Y_v(ka) + \lim_{\rho\to\infty}[\rho f(\rho)G_\rho(r,\rho) - \rho f'(\rho)G(r,\rho)]$$
$$+ \int_a^\infty G(r,\rho)g(\rho)\frac{d\rho}{\rho}. \tag{26}$$

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The limit appearing in (26) can be evaluated with the aid of the asymptotic formula

$$Y_v(k\rho)=(2/\pi k\rho)^{1/2}\sin\left[k\rho-(v+\tfrac{1}{2})\frac{\pi}{2}\right][1+O(\rho^{-1})].$$

This applies as $\rho\to\infty$ when v is fixed. It follows from this result together with the fact that the function $f(\rho)$ satisfies the radiation condition that

$$\lim_{\rho\to\infty}[k\rho f(\rho)Y_v'(k\rho)-\rho f'(\rho)Y_v(k\rho)]=c\exp(iv\pi/2), \tag{27}$$

where

$$c=(2k/\pi)^{1/2}\exp(i\pi/4)\lim_{\rho\to\infty}[\rho^{1/2}f(\rho)e^{-ik\rho}].$$

The limit in (26) can now be determined with the aid of (27) on using the first of the expressions given in equation (25) for $G(r,\rho)$. This leads to the formula

$$f(r)=f(a)\frac{Y_v(kr)}{Y_v(ka)}+\frac{\pi c[J_v(kr)Y_v(ka)-J_v(ka)Y_v(kr)]e^{1/2iv\pi}}{2Y_v(ka)}+\int_a^\infty G(r,\rho)g(\rho)\frac{d\rho}{\rho}. \tag{28}$$

The Green's function defined by the composite formula (25) is now represented by means of the following single expression, which will be inserted into (28),

$$\begin{aligned}G(r,\rho)=&\frac{1}{2i}\int_L\frac{[J_u(kr)Y_u(ka)-J_u(ka)Y_u(kr)]Y_u(k\rho)u\,du}{(u^2-v^2)Y_u(ka)}\\&+\pi\sum_{u=u_n}\frac{uJ_u(ka)Y_u(kr)Y_u(k\rho)}{(u^2-v^2)(\partial/\partial u)Y_u(ka)}.\end{aligned} \tag{29}$$

In this formula, which holds for all $r,\rho\geqslant a$, the path L denotes the imaginary axis of the complex u-plane and the summation extends over all the real negative zeros of the function $Y_u(ku)$. To establish the formula (29) we consider first the case $a\leqslant r\leqslant\rho$. The integral appearing in (29) may then be evaluated by closing the contour on the left by means of a sequence of large semicircles which avoid the poles and which recede to infinity. In the half plane $\mathrm{Re}(u)\leqslant 0$ the integrand is singular at the point $u=v$ and at the negative zeros u_n of $Y_u(ka)$. Upon calculating the residues at these points it is seen that the series in (29) cancels with the series of corresponding residues whilst the residue at the point v reproduces the first of the expressions appearing in equation (25).

To justify this procedure it is necessary to prove that the integral over the semicircle tends to zero as the radius tends to infinity and that the L-integral itself is convergent. With this in view we first determine the asymptotic behaviour of the cross product of Bessel functions occurring in (29). By means of the definition (14) we may write, as $u\to\infty$,

$$J_u(kr)Y_u(ka)-J_u(ka)Y_u(kr)=-\frac{[J_u(kr)J_{-u}(ka)-J_u(ka)J_{-u}(kr)]}{\sin u\pi} \tag{30}$$

$$=-\frac{1}{u\pi}[(r/a)^u-(a/r)^u][1+O(u^{-1})] \tag{31}$$

after estimating the Bessel functions by means of equation (13) and using the identity $\Gamma(1+u)\Gamma(1-u)\sin u\pi = u\pi$. The behaviour of the quotient of Y-type functions appearing in the integral in (29) can be deduced from the formula (18) if it is noted that

$$\mathrm{Re}[u\log(2u/ex)+\log\sqrt{2}+\tfrac{1}{4}i\pi] = R\cos\theta\log(2R/ex) - R\theta\sin\theta + \log\sqrt{2}, \tag{32}$$

where $u=Re^{i\theta}$. The expression on the right hand side of (32) tends to $-\infty$ as $R=|u|\to\infty$ in the left hand half plane $\mathrm{Re}(u)\leqslant 0$ since $\cos\theta\leqslant 0$ there. Therefore

$$|\sinh[u\log(2u/ex)+\log\sqrt{2}+\tfrac{1}{4}i\pi]| \sim 2^{-3/2}\exp[R\theta\sin\theta - R\cos\theta\log(2R/ex)].$$

It follows from (18), (19) that

$$\left|\frac{Y_u(k\rho)}{Y_u(ka)}\right| = \exp[R\cos\theta\log(\rho/a)][1+O(u^{-1})] \tag{33}$$

as $u\to\infty$ in the sectors $\pi/2\leqslant|\theta|\leqslant\pi-\delta$. On combining (31) and (33) it follows that the integrand occurring in (29) is

$$O\{R^{-2}\exp[-|R\cos\theta|\log(\rho/r)]\} \tag{34}$$

as $u\to\infty$ in the sectors $\pi/2\leqslant|\theta|\leqslant\pi-\delta$. This bound reduces when $\theta=\pm\pi/2$ to $O(R^{-2})$ so that the integral in (29) is convergent. The bound (33) does not apply in the sector $|\pi\pm\theta|\leqslant\delta$ containing the negative real axis since (18), (19) cease to hold there. The function $Y_u(ka)$ vanishes at the points u_n in this region and it is required to close the contour by means of a path C avoiding these points. A suitable path can be made up of the two circular arcs $u=Re^{i\theta}$, $\pi/2\leqslant|\theta|\leqslant\pi-\delta$, connected by the part of the straight line $u=-(n+\frac{1}{4})+is$ located in the wedge $\pi-\delta\leqslant|\theta|\leqslant\pi$. The radius R of the arcs is chosen so that $R\cos\delta=n+\frac{1}{4}$ to ensure that a continuous curve is formed. It is evident from formula (6) which gives the positions of the large real zeros that the curve C constructed in this way will avoid the zeros when n is large enough. In the vicinity of the negative real axis the asymptotic behaviour of the Y-type Bessel functions can be obtained from formula (23) in which the function $\Gamma(-u)$ is estimated by means of Stirling's formula (16). In order to apply (16) in the wedge $|\pi\pm\theta|\leqslant\delta$ we ensure that $|\arg(-u)|\leqslant\pi-\delta$ therein by writing $u=Re^{i\theta}$ and $-u=Re^{i\psi}$, where $\psi=\theta-\pi$ for $\pi-\delta\leqslant\theta\leqslant\pi$ and $\psi=\theta+\pi$ for $-\pi\leqslant\theta\leqslant-(\pi-\delta)$. With this choice $\arg(-u)=\psi$ and $|\psi|\leqslant\delta$, and (16) gives the formula

$$|\Gamma(-u)| = (2\pi/R)^{1/2}\exp[R\cos\psi\log(R/e) - R\psi\sin\psi][1+O(R^{-1})]. \tag{35}$$

On the line $u=-(n+\frac{1}{4})+is$, $|\sin u\pi|=|\cos u\pi|=(\frac{1}{2}\cosh 2s\pi)^{1/2}$ so that on taking the modulus of (23) and using (35) we find that

$$|Y_u(x)| = \left[\frac{1}{\pi R}\cosh(2\pi R\sin\psi)\right]^{1/2}\exp[R\cos\psi\log(2R/ex) - R\psi\sin\psi]\left[1+O\left(\frac{1}{R}\right)\right] \tag{36}$$

as $u\to\infty$ on the sequence of segments $\mathrm{Re}(u)=-(n+\frac{1}{4})$ inside the wedge $\pi-\delta\leqslant|\arg u|\leqslant\pi$.

Since $\cos\psi = -\cos\theta$ it follows on applying (36) to estimate the Bessel functions that the formula (33) and hence the bound (34) apply on the stated line segments as well as on the circular arcs.

If $\rho \geq r$ the expression (34) tends to zero sufficiently rapidly as $R \to \infty$ to ensure that the integral over the curve C vanishes in the limit so that the expression for the Green's function is established for such values of r, ρ.

If $\rho < r$ the integral appearing in (29) cannot be evaluated by closing the contour since the integrand no longer tends to zero as $u \to \infty$ except on L itself. In this case however a direct evaluation of the integral can be avoided by the following method which in effect establishes the symmetry of the expression (29) claimed to represent the Green's function. Upon interchanging the variables r and ρ in the proposed expression (29) it follows that

$$G(r, \rho) - G(\rho, r) = \frac{1}{2i} \int_L \frac{[J_u(kr)Y_u(k\rho) - J_u(k\rho)Y_u(kr)]u\,du}{u^2 - v^2}.$$

By (30) the integrand appearing in the preceding equation is an odd function of u so that the integral taken along the entire imaginary axis is zero. Hence $G(r, \rho) = G(\rho, r)$ for all values of r, ρ so that when $\rho < r$ the value of the integral appearing in (29) can be obtained by interchanging r and ρ and evaluating the integral as before by taking the residues at the poles. This procedure evidently leads to the second expression given on the right hand side of equation (25).

4. The expansion theorem. The expression (29) for the Green's function is now inserted into the formula (28) for $f(r)$. If the order of integration in the resulting repeated integral be changed and the order of integration and summation in the series be changed we find the formula

$$\begin{aligned} f(r) = f(a)\frac{Y_v(kr)}{Y_v(ka)} &+ \frac{\pi c[J_v(kr)Y_v(ka) - J_v(ka)Y_v(kr)]e^{(1/2)iv\pi}}{2Y_v(ka)} \\ &+ \frac{1}{2i}\int_L \frac{[J_u(kr)Y_u(ka) - J_u(ka)Y_u(kr)]G(u)u\,du}{(u^2 - v^2)Y_u(ka)} \\ &+ \pi \sum_{u = u_n} \frac{uJ_u(ka)Y_u(kr)G(u)}{(u^2 - v^2)(\partial/\partial u)Y_u(ka)}, \end{aligned} \tag{37}$$

where

$$G(u) = \int_a^\infty Y_u(k\rho)g(\rho)\frac{d\rho}{\rho}. \tag{38}$$

To justify the above procedure it is sufficient to verify that the integral and the series appearing in (37) are absolutely convergent.

To establish the absolute convergence of the integral in (37) it is necessary to obtain a suitable bound on the values of the function $G(u)$ on the imaginary axis and in view of

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(38) this requires a bound on the function $Y_u(k\rho)$ valid on L uniformly for all $\rho \geq a$. Such a bound is obtained in the appendix to this paper where it is proved that

$$|Y_{is}(k\rho)| \leq (2/\pi k\rho)^{1/2} \cosh(\tfrac{1}{2}s\pi) \tag{39}$$

for s real. On inserting this bound into the equation (38) defining $G(u)$ it follows that

$$|G(is)| \leq (2/\pi k)^{1/2} \cosh(\tfrac{1}{2}s\pi) \int_a^\infty \rho^{-3/2} |g(\rho)| \, d\rho. \tag{40}$$

Since $\rho^{-3/2} g(\rho) \in L(a, \infty)$ by hypothesis it follows that $|G(is)| \leq C \cosh(\tfrac{1}{2}s\pi)$, where C is a constant. The asymptotic behaviour of the function $Y_{is}(ka)$ can be obtained from the equation (15) if it is noted that $|\Gamma(is)| = [\pi/(s \sinh s\pi)]^{1/2}$, which tends to zero as $s \to \infty$. The dominant term in (15) is then the first so that

$$\begin{aligned} |Y_{is}(ka)| &= \left| \frac{\coth s\pi}{\Gamma(1+is)} \right| [1 + O(s^{-1})] \\ &= \frac{\cosh s\pi}{(s\pi \sinh s\pi)^{1/2}} [1 + O(s^{-1})]. \end{aligned} \tag{41}$$

Finally it follows from (31) that the cross product of Bessel functions appearing in the integral in (37) is $O(s^{-1})$ and so on using this result together with (40) and (41) it is seen that the integrand in (37) is $O(s^{-3/2})$ as $s \to \infty$ and this is absolutely convergent.

To discuss the convergence of the series appearing in (37) a suitable bound must be found on the values of the function $G(u)$ at the negative zeros of $Y_u(ka)$. A bound will first be obtained on the corresponding values of the function $F(u)$ and it will be shown that $F(u_n) = O(u_n^{-1/2})$ as $n \to \infty$. The required bound on the function $G(u)$ can then be deduced from the equation

$$G(u) = (u^2 - v^2) F(u) + kaf(a) Y_u'(ka) - af'(a) Y_u(ka) - ce^{(1/2)iu\pi}. \tag{42}$$

This equation follows on multiplying the equation (24) by $r^{-1} Y_u(kr)$, integrating twice by parts and applying the result (27). The stated bound on $F(u)$ can be found on applying the Schwarz inequality to the definition (4), which leads to the inequality

$$|F(u)| \leq \left\{ \int_a^\infty |Y_u(kr)|^2 \frac{dr}{r} \int_a^\infty |f(r)|^2 \frac{dr}{r} \right\}^{1/2}. \tag{43}$$

The value of the Bessel function integral occurring here can be obtained from Watson [**8**, p. 135, equation (14)]. The formula given there reduces, when u is a zero of $Y_u(ka)$, to the equation

$$2u \int_a^\infty Y_u(kr)^2 \frac{dr}{r} = 1 + \frac{2(\partial/\partial u) Y_u(ka)}{\pi J_u(ka)}. \tag{44}$$

The preceding equation is now transformed with the aid of the following formulae,

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also given by Watson [8, p. 444]:

$$J_u(x)^2+Y_u(x)^2=\frac{8}{\pi^2}\int_0^\infty K_0(2x\sinh\theta)\cosh 2u\theta\,d\theta,$$

$$J_u(x)\frac{\partial}{\partial u}Y_u(x)-Y_u(x)\frac{\partial}{\partial u}J_u(x)=-\frac{4}{\pi}\int_0^\infty K_0(2x\sinh\theta)e^{-2u\theta}\,d\theta. \tag{45}$$

If $Y_u(ka)=0$ it follows from these formulae that

$$\frac{\partial}{\partial u}Y_u(ka)=-\frac{\pi J_u(ka)\displaystyle\int_0^\infty K_0(2ka\sinh\theta)e^{-2u\theta}\,d\theta}{2\displaystyle\int_0^\infty K_0(2ka\sinh\theta)\cosh 2u\theta\,d\theta}, \tag{46}$$

so that (44) can be rewritten as the equation

$$2u\int_a^\infty Y_u(kr)^2\frac{dr}{r}=\frac{\displaystyle\int_0^\infty K_0(2ka\sinh\theta)\sinh 2u\theta\,d\theta}{\displaystyle\int_0^\infty K_0(2ka\sinh\theta)\cosh 2u\theta\,d\theta}.$$

Since $|\sinh 2u\theta|<\cosh 2u\theta$ if u is real it follows that

$$\int_a^\infty Y_u(kr)^2\frac{dr}{r}\leqslant\frac{1}{|2u|} \tag{47}$$

whenever u is a real zero of $Y_u(ka)$. It is then seen from (43) and (47) that $F(u)=O(u^{-1/2})$ for such values of u.

Finally we note that at a zero of $Y_u(ka)$ the Wronskian identity

$$J_u(ka)Y_u'(ka)-Y_u(ka)J_u'(ka)=2/(\pi ka) \tag{48}$$

reduces to $J_u(ka)Y_u'(ka)=2/(\pi ka)$. If we now let $u\to-\infty$ through the sequence u_n of negative zeros of $Y_u(ka)$, then (45) shows that $J_u(ka)\to\infty$ and the Wronskian condition then reveals that $Y_u'(ka)\to 0$. The relation (42) then shows that $G(u_n)=O(u_n^{3/2})$ as $n\to\infty$.

To obtain suitable bounds on the Bessel functions appearing in the series (37) we first note that $\cosh 2u\theta\leqslant e^{2|u|\theta}$ so that (46) implies, if u is a negative zero of $Y_u(ka)$, that

$$\left|\frac{\partial}{\partial u}Y_u(ka)\right|\geqslant\frac{\pi}{2}|J_u(ka)|\geqslant 2^{-3/2}(2/ka)^{|u|}\Gamma(|u|), \tag{49}$$

the final inequality following from the bound (59) derived in the appendix.

An estimate of the function $Y_u(kr)$ appearing in the series in (37) can be deduced from (31) which reduces when u is a large negative zero of $Y_u(ka)$ to the equation

$$|J_u(ka)Y_u(kr)|=\frac{1}{|\pi u|}(r/a)^{|u|}[1+O(u^{-1})]. \tag{50}$$

Upon combining the bounds (49), (50) and using the fact that $G(u_n)=O(u_n^{3/2})$, it follows that the typical term in the series (37) is

$$O\left[\frac{(kr/2)^{|u_n|}}{|u_n|^{1/2}\Gamma(|u_n|)}\right].$$

Since $|u_n|\sim n+\frac{1}{2}$ as $n\to\infty$ it follows that the series in question is absolutely convergent.

The final step necessary to construct the formula (5) consists of substituting the expression (42) for $G(u)$ into the formula (37), which leads to the equation

$$f(r)=\frac{1}{2i}\int_L \frac{[J_u(kr)Y_u(ka)-J_u(ka)Y_u(kr)]F(u)u\,du}{Y_u(ka)}+\pi\sum_{u=u_n}\frac{uJ_u(ka)Y_u(kr)F(u)}{(\partial/\partial u)Y_u(ka)}$$
$$+A_1f(a)+aA_2f'(a)+A_3c. \tag{51}$$

The definitions of the quantities A_1, A_2, A_3 appear below as equations (52)–(54) and it will be proved that $A_1=A_2=A_3=0$ so that the formula (51) reduces to the formula (5) of the theorem.

The quantity A_2 is defined by the equation

$$A_2=-\frac{1}{2i}\int_L \frac{[J_u(kr)Y_u(ka)-J_u(ka)Y_u(kr)]u\,du}{u^2-v^2}. \tag{52}$$

By (30) the integrand here is an odd function of u and the integral itself is convergent, by (31), so that its value taken along the entire imaginary axis is zero.

We consider now the quantity A_3 which is defined by the equation

$$A_3=\frac{\pi[J_v(kr)Y_v(ka)-J_v(ka)Y_v(kr)]e^{(1/2)iv\pi}}{2Y_v(ka)}$$
$$-\pi\sum_{u=u_n}\frac{uJ_u(ka)Y_u(kr)e^{(1/2)iu\pi}}{(u^2-v^2)(\partial/\partial u)Y_u(ka)}$$
$$-\frac{1}{2i}\int_L \frac{[J_u(kr)Y_u(ka)-J_u(ka)Y_u(kr)]e^{(1/2)iu\pi}u\,du}{(u^2-v^2)Y_u(ka)}. \tag{53}$$

If the integral appearing in the preceding expression is evaluated by closing the contour on the left and taking the residues at the poles it is seen that $A_3=0$, as required. This procedure can be justified by following the method adopted in §3 to close the contour and using the formulas (18), (31), (36) to show that on C the integrand is

$$O\left[(2R/ker)^{-|R\cos\theta|}\exp\left\{-\left(\frac{\pi}{2}+\theta\right)R\sin\theta\right\}\right].$$

This bound tends to zero fast enough to ensure that the contribution from the integral around the curve C vanishes as $R\to\infty$.

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The quantity A_1 is defined by the equation

$$A_1=\frac{Y_v(kr)}{Y_v(ka)}+\frac{ka}{2i}\int_L\frac{[J_u(kr)Y_u(ka)-J_u(ka)Y_u(kr)]Y_u'(ka)u\,du}{(u^2-v^2)Y_u(ka)}$$
$$+2\sum_{u=u_n}\frac{uY_u(kr)}{(u^2-v^2)(\partial/\partial u)Y_u(ka)}. \tag{54}$$

To show that $A_1=0$ the integral appearing in (54) is first decomposed into two parts by means of the following identity:

$$[J_u(kr)Y_u(ka)-J_u(ka)Y_u(kr)]Y_u'(ka)=-(2/\pi ka)Y_u(kr)$$
$$+[J_u(kr)Y_u'(ka)-Y_u(kr)J_u'(ka)]Y_u(ka).$$

This result is seen on simplification to reduce to the Wronskian identity (48). The integral term in (54) can now be expressed as the difference of two integrals, as given by the following equation

$$\frac{ka}{2i}\int_L\frac{[J_u(kr)Y_u'(ka)-J_u'(ka)Y_u(kr)]u\,du}{u^2-v^2}-\frac{1}{i\pi}\int_L\frac{Y_u(kr)u\,du}{(u^2-v^2)Y_u(ka)}. \tag{55}$$

The value of the first integral appearing in the preceding expression is zero since, by analogy with (30) and (31), the integrand is an odd function of u and behaves when $u=is$ is large like $s^{-1}\sin[s\log(r/a)]$, which is integrable. The integrand appearing in the second integral in (55) has similar asymptotic behaviour since when $u=is$ is large it can be shown from (18) that

$$Y_{is}(x)=\frac{-i}{(2\pi s)^{1/2}}\exp[-is\log(2s/ex)+\tfrac{1}{2}s\pi-\tfrac{1}{4}i\pi]$$

as $s\to+\infty$. The second integral in (55) is therefore also convergent and may be evaluated by closing the contour on the left by means of the sequence of paths defined in §3, and taking the residues at the poles. As noted in §3 the estimate (33) applies on the sequence of chosen paths so that the procedure is permissible. On evaluating the residues at the poles situated in the half plane $\mathrm{Re}(u)<0$ it is seen that the first and last terms in (54) cancel so that $A_1=0$, as required.

Appendix. It remains to derive the bounds (39) and (49) used in the paper. Both of these bounds can be obtained from the formula (45), which is first transformed into one involving only Y_u and Y_{-u} by means of the formula

$$J_u(x)=-Y_u(x)\cot u\pi+Y_{-u}(x)\operatorname{cosec} u\pi.$$

This result is a consequence of the definition (14) of the function $Y_u(x)$. Upon substituting the preceding expression into (45) it is found that

$$Y_u(x)^2+Y_{-u}(x)^2-2Y_u(x)Y_{-u}(x)\cos u\pi=\frac{8}{\pi^2}\sin^2 u\pi\int_0^\infty K_0(2x\sinh\theta)\cosh 2u\theta\,d\theta.$$

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When $u = is$ is purely imaginary the preceding equation shows that

$$|Y_{is}(x)|^2 \cosh(s\pi) = \mathrm{Re}[Y_{is}(x)^2] + \frac{4}{\pi^2} \sinh^2 s\pi \int_0^\infty K_0(2x \sinh \theta)\cos 2s\theta \, d\theta.$$

Since $\mathrm{Re}[Y_{ix}(x)^2] \leq |Y_{is}(x)|^2$, it follows that

$$\begin{aligned} |Y_{is}(x)|^2(\cosh s\pi - 1) &\leq \frac{4}{\pi^2} \sinh^2 s\pi \int_0^\infty K_0(2x \sinh \theta)\cos 2s\theta \, d\theta \\ &= (\pi x)^{-1} \sinh^2 s\pi, \end{aligned}$$

by Watson [**8**, p. 388]. It is seen that the bound (39) follows from the preceding result, after slight reduction, on replacing x by $k\rho$.

The bound (49) applies whenever u is a real zero of $Y_u(ka)$ such that $|u| > \frac{1}{2}$. For such values of u the identity (45) reduces to the equation

$$J_u(ka)^2 = \frac{8}{\pi^2} \int_0^\infty K_0(2ka \sinh \theta)\cosh 2u\theta \, d\theta. \tag{57}$$

Now if θ is positive $e^\theta \geq \cosh \theta$ and $e^\theta \geq 2 \sinh \theta$ so that

$$2 \cosh 2u\theta \geq e^{|2u|\theta} \geq (2 \sinh \theta)^{|2u|-1} \cosh \theta.$$

The integral (57) is not less than the expression

$$\frac{1}{2} \int_0^\infty K_0(2ka \sinh \theta)(2 \sinh \theta)^{|2u|-1} \cosh \theta \, d\theta = \frac{1}{16} (2/ka)^{2|u|}(\Gamma(|u|))^2 \tag{58}$$

by Watson [**8**, p. 388]. On combining (57), (58) it follows that

$$|J_u(ka)| \geq \frac{1}{\pi\sqrt{2}} (2/ka)^{|u|}\Gamma(|u|). \tag{59}$$

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