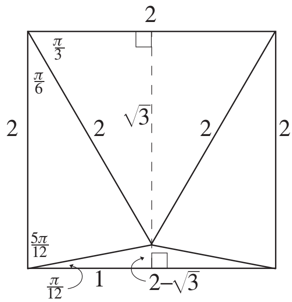
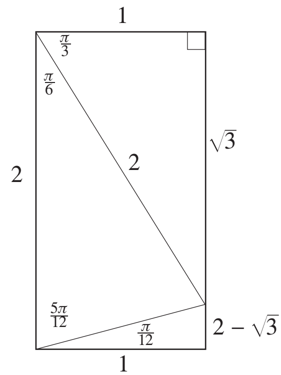
Feedback

On Feedback for July 2022: Nick Lord writes: Martin Lukarevski, in his response to 'What makes a good Proof without Words', challenges the reader to come up with a visual demonstration of the identity $\tan\frac{\pi}{12} = 2 - \sqrt{3}$. Here are some of my thoughts.

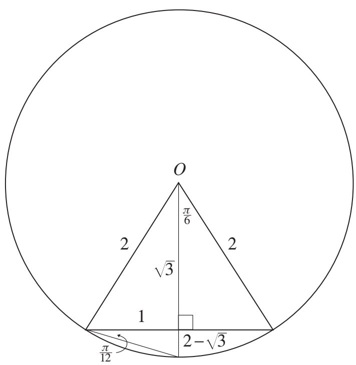
Proof 1:



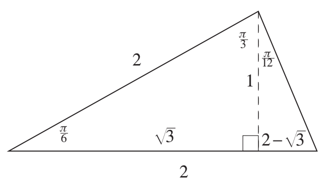
Proof 2 (variation on Proof 1):



Proof 3:



Proof 4 (variation on Proof 3):



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On 106.34 Owen Toller writes: The author, Sabu Sebastian, gives an algorithm for testing divisibility by two-digit numbers. This is a topic on the OCR Further Mathematics (A) Additional Pure option, for which the only currently available textbook on the topic [1] presents an algorithm, similar but not identical to that in the Note, but without justification. The use of modular arithmetic in the Note seems rather obviously at odds with the spirit of the task—after all, if you are having to do arithmetic modulo 19 in the course of the algorithm, why not just find the remainder of the original number to that modulus?—which is avoided by that in [1]. Here we present the latter's algorithm together with a proof more elementary than that given in the Note. The two algorithms are in essence identical apart from the use or otherwise of modular arithmetic, and in particular it will be noted that both algorithms involve the finding of an integer k such that $kp \equiv -1 \pmod{10}$. I admit that neither proof is particularly transparent!

The task is to test whether an integer N is divisible by a prime p, where $p > 5$ (no algorithm is needed for $p = 2$, 3 or 5). The need for this restriction will be seen in due course.

Description of the algorithm

Suppose N can be written as $10a + b$, where $a, b \in \mathbb{N}$. For example, for $N = 92796$ (the example given in the Note), $a = 9279$ and $b = 6$. We construct a smaller integer M such that $p \mid N \Leftrightarrow p \mid M$.

We seek to write $M = a + mb$ for some suitable integer m.

To find m, find a multiple k of p such that the last digit of kp is 9, in other words $kp \equiv -1 \pmod{10}$; then choose $m = \frac{1}{10}(kp + 1) \pmod{p}$. Because $p > 5$, p is coprime to 10, so it is always possible to find such a k. Then reiterate the algorithm to obtain a decreasing sequence, until it is obvious whether or not the value of M is divisible by p.

Example

Determine whether 19 divides 92796. (This is the example used by Sebastian.)

Solution

Here $p = 19$. Take $k = 1$, so $kp = 19$ and $m = \frac{1}{10}(19 + 1) = 2$. So $M = a + 2b$. The implementation of the algorithm is shown in the following table.

N	a	b	$M = a + 2b$
92796	9279	6	9291
9291	929	1	931
931	93	1	95
95	9	5	19

As 19 is divisible by 19, it follows that 92796 is divisible by 19.

The choice of k depends only on the last digit of p. For final digits of 1, 3, 7 or 9, we can take k to be 9, 3, 7 or 1 respectively. (Sebastian does not spell out that the algorithm is usable for any two-digit prime, and indeed for primes of any magnitude.) Had p been, for example, 37, then take $k = 7$, when $m = \frac{1}{10}(259 + 1) = 26$, although in that case it might make the numbers smaller to use $m = -11$, noting that $-11 \equiv 26 \pmod{37}$.

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Why the algorithm works

We choose m such that, for all a, b there exists c such that

$$p \mid (10a + b) + c(a + mb).$$

There are, of course, multiple possible choices for c and m. For instance, for division by 7, you could take $c = 4$ and $m = 5$:

$$(10a + b) + 4(a + 5b) = 14a + 21b$$

which is clearly divisible by 7. We could replace 5 by –2, as $5 \equiv -2 \pmod 7$; this would give smaller, and hence more convenient, numbers.

Now

$$p \mid (10 + c)a + (1 + mc)b \text{ for all } a, b \Rightarrow p \mid 10 + c \text{ and } p \mid 1 + mc.$$

As $p > 5$, p is coprime to 10, so the congruency equation $kp \equiv 9 \pmod{10}$ can always be solved for any prime p. Also, for this value of k, $\frac{1}{10}(kp + 1)$ is an integer; let this be m.

So the integer m satisfies $10m = 1 + kp$. Then

$$1 + mc = \frac{10 + c(1 + kp)}{10} = \frac{(10 + c) + kcp}{10}.$$

As 10 and p have no common factor,

$$p \mid \frac{(10 + c) + kcp}{10} \Leftrightarrow p \mid (10 + c) + kcp \Leftrightarrow p \mid 10 + c.$$

Hence $p \mid 1 + mc \Leftrightarrow p \mid 10 + c$,

So we can choose any c for which $c \equiv -10 \pmod p$; and with this choice of c and m, p divides $(10a + b) + c(a + mb)$. Thus $10a + b$ is divisible by p if, and only if, $a + mb$ is divisible by p. It is nice to observe that you don't need to use the value of c, or even to find it.

This proof will appear in my book written for the OCR option, forthcoming if I can find a publisher!

Reference

1. John Sykes, *A Level Further Mathematics for OCR A Additional Pure*, Cambridge University Press, 2021.

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On 107.03: Zoltan Retkes writes: The author gives an argument for the volume of an ungula (including a typo, where $z = 2y$ should be $z = 2x$). However, the argument can be replaced by this one-liner. If S is half the base circle lying in $x \geqslant 0$, then

$$\iint_S 2x\,dx\,dy = \int_{-a}^{a} dy \int_0^{\sqrt{a^2 - y^2}} 2x\,dx = \int_{-a}^{a} (a^2 - y^2)\,dy = \left[a^2 y - \frac{y^3}{3}\right]_{y=-a}^{y=a} = \frac{4a^3}{3}.$$

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