FIXED POINTS AND CHARACTERS IN GROUPS WITH NON-COPRIME OPERATOR GROUPS

I. M. ISAACS

1. Let A and H be finite groups with A acting on H, i.e., there is a given, fixed homomorphism $A \to \mathrm{Aut}(H)$. In this situation, A acts on the set of conjugacy classes of H and also on the set of irreducible characters of H. If A is cyclic, it follows from a lemma of Brauer (see, for instance, **1**, 12.1) that the number of fixed points in these two actions are equal and therefore one can conclude that for any A, the number of orbits in the two actions are equal. In general, little more can be said; however, if $(|A|, |H|) = 1$, Glauberman and Dade have shown (in as yet unpublished papers) that the number of fixed points is the same in both actions. It is my aim in this paper to obtain the following partial result in this direction in the non-coprime situation.

THEOREM. *Let A act on a solvable group H and suppose that A has a normal p-complement for every prime $p \mid |H|$. Then the following are equivalent:*

(1) *A fixes an irreducible character $\neq 1$ of H,*

(2) *A fixes an element $\neq 1$ of H,*

(3) *A fixes a class $\neq \{1\}$ of H.*

The example given below shows that some hypothesis on A of the nature of the one given in the theorem is necessary. It may, however, be possible to remove the solvability condition on H.

Example. Let p and q be primes with $q \mid (p-1)$ and let $a \neq 1$, $a \in \mathrm{GF}(p)$ with $a^q = 1$. Let $A \subseteq \mathrm{GL}(2, p)$ be the group of order pq generated by

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

and let H be the elementary abelian group of order p^2 on which A naturally acts. Now A has no fixed points on H but $[A, H]$ has order p and A fixes the $p - 1$ linear characters of H which have kernel $= [A, H]$.

2. We work toward a proof of the theorem by deriving some character-theoretic facts in this section. Here, $[\ ,\ ]_G$ denotes a character inner product.

LEMMA 1. *If $H \subseteq G$ and χ is a character of G, then*

$$[\chi|H, \chi|H]_H \leqq [G : H][\chi, \chi]_G$$

with equality if and only if χ vanishes on $G - H$.

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Proof. This is a routine calculation using the definition of inner product.

LEMMA 2. *Let $K \triangle H$ with H/K an elementary abelian p-group, and let $K \subseteq U \subseteq H$. If χ is an irreducible character of H and*

$$[\chi|U, \chi|U]_U < [\chi|K, \chi|K]_K,$$

then there exists M, $K \subseteq M < U$, $[U : M] = p$ such that χ vanishes on $U - M$.

Proof. Since $U \triangle H$, we have that $\chi|U = a\sum_1^t \theta_i$, where the θ_i are distinct irreducible characters of U. Then

$$a^2t = [\chi|U, \chi|U]_U < [\chi|K, \chi|K]_K = a^2\sum_{ij}[\theta_i|K, \theta_j|K]_K.$$

Therefore for some i and j, $[\theta_i|K, \theta_j|K]_K > \delta_{ij}$. If $i = j$ in the above, then $\theta_i|K$ is reducible and we may choose L, minimal satisfying $K < L \subseteq U$, such that $\theta_i|L$ is irreducible. If $L_0 \supseteq K$ has index p in L, then $\theta_i|L_0$ is reducible and thus is a sum of p distinct irreducible constituents (see, e.g., **1**, 9.12). Let M be the inertia group in U of one of these constituents. Then $[U : M] = p$ and $\theta_i|M$ is reducible. Since $M \triangle H$, each of the θ_j reduces on M and thus $[\theta_j|M, \theta_j|M]_M = p$ and by Lemma 1, θ_j vanishes on $U - M$. The same is therefore true for χ.

The remaining possibility is that each $\theta_i|K$ is irreducible, and thus some $\theta_i|K = \theta_j|K$ for $i \neq j$. It follows then that $\theta_j = \lambda\theta_i$ for some linear character λ of U/K (see, e.g., **3**, Lemma 3.1). Now, $M = \operatorname{Ker} \lambda$ has index p in U and $M \triangle H$. We then have that $\chi|M = a' \sum_1^{t'} \phi_i$, where the ϕ_i are distinct irreducible characters of M and are restrictions of the θ's. Since $\theta_i|M = \theta_j|M$, we have that $t' < t$, and since $at = a't'$, we have that $a' > a$. Therefore

$$[\chi|M, \chi|M]_M = a'^2t' > a^2t = [\chi|U, \chi|U]_U.$$

Since t and t' are powers of p, we have that $[\chi|M, \chi|M]_M \geqq p[\chi|U, \chi|U]_U$ and Lemma 1 yields the result.

PROPOSITION 3. *Let $K \triangle H$ be such that H/K is an elementary abelian p-group and let A act on H and normalize K with the induced act on on H/K being non-trivial and irreducible. Then we have the following.*

(1) *If $\chi \neq 1$ is an irreducible character of H which is fixed by A, we have that either*

(a) *$\chi|K$ is irreducible and $\neq 1$,*

(b) *$\chi|K = a\theta$, θ is irreducible, $\neq 1$ and $a^2 = [H : K]$, or*

(c) *$\chi|K = \sum_1^t \theta_i$, where $t = [H : K]$ and the θ_i are distinct irreducible characters of K, transitively permuted by H.*

(2) *If $\theta \neq 1$ is an irreducible character of K which is fixed by A and θ^H is the induced character, we have that either:*

(a) *θ^H is irreducible,*

(b) *$\theta^H = a\chi$, χ is irreducible and $a^2 = [H : K]$, or*

(c) *$\theta^H = \sum_1^t \chi_i$, where $t = [H : K]$ and the χ_i are distinct irreducible characters* *of H which are transitively permuted by multiplication by the linear characters of H/K.*

Proof. (1) We have that $\chi|K = a\sum_1^t \theta_i$, where the θ_i are distinct irreducible characters of K which are transitively permuted by H. Let T be the inertia group of θ_1 in H, $[H : T] = t$. Since H/K is abelian, $T \triangle H$ and thus is the inertia group of each of the θ_i. Since T is determined by χ which is fixed by A, T admits A and by the irreducibility of H/K, either (*) $T = H$ or (**) $T = K$. In the latter situation we have that $\theta_i^H(1) = [H : K]\theta_i(1) = \chi(1)/a$. Since χ is a constituent of θ_i^H we have that $a = 1$ and this is case (c).

In situation (*) we have that $t = 1$ and $\chi|K = a\theta$. If $\theta = 1$, then $K \subseteq \operatorname{Ker} \chi$ and χ is an irreducible character of H/K and thus is linear. $\operatorname{Ker} \chi \neq H$ and admits A and therefore is K. Thus, H/K is cyclic and χ has a different value on each of its elements. Since A fixes χ, this implies that A centralizes H/K, which is not the case, and therefore $\theta \neq 1$. If $a = 1$, we have case (a) of the proposition. If $a > 1$, we must show that $a^2 = [H : K]$. This will follow immediately once we have shown part (2). Since θ is fixed by A and θ^H has the constituent χ with multiplicity $a > 1$, we have case (2) (b).

(2) We have $\theta^H = \sum_1^r a_i\chi_i$, where each χ_i is an irreducible character of H. Let T be the inertia group of θ. Then T admits A and thus either (*) $T = H$ or (**) $T = K$. In situation (**), $\chi_i|K = a_i\sum_1^t \theta_j$, where $t = [H : K]$. However, $\chi_i(1) \leqq t\theta(1)$, and thus $a_i = 1$, $\chi_i(1) = t\theta(1)$, and therefore $\theta^H = \chi_1$ and we have case (a). In situation (*) we have $\chi_i|K = a_i\theta$. If $a_1 = 1$, then $\chi_1|K$ is irreducible and $\lambda\chi_1 = \chi_1$ implies that χ_1 vanishes on $H - \operatorname{Ker} \lambda$, and thus $\chi_1|(\operatorname{Ker} \lambda)$ is reducible. We may thus conclude that all $\lambda\chi_1$ are distinct for linear characters λ of H/K. Each $\lambda\chi_1$ is one of the χ_i, and thus $r \geqq [H : K]$. Since $\sum_1^r a_i \leqq [H : K]$ we must have each $a_i = 1$ and $r = [H : K]$, and thus each χ_i is some $\lambda\chi_1$. This is case (c).

For the remaining possibility, where $a_1 > 1$, let $U_i = \bigcap\{M \subseteq H|\ M \supseteq K$ and χ_i vanishes on $H - M\}$. Let V/K be a complement for U_1/K in H/K. Let $\lambda \neq 1$ be a linear character of H/V. Thus, if $\lambda\chi_1 = \chi_1$, then χ_1 vanishes on $H - \operatorname{Ker} \lambda$ and $\operatorname{Ker} \lambda \supseteq U_1$. Since $\operatorname{Ker} \lambda \supseteq V$, this is not possible and it follows that all $\lambda\chi_1$ are distinct for linear characters λ of H/V. Each $\lambda\chi_1$ is one of the χ_i and we may suppose that they are $\chi_1, \ldots, \chi_{[H:V]}$. Now, χ_1 vanishes on $H - U_1$, and thus $[\chi_1|U_1, \chi_1|U_1]_{U_1} = [H : U_1][\chi_1, \chi_1]_H = [H : U_1]$. By the definition of U_1, there exists no $M \supseteq K$, $[U_1 : M] = p$ with χ_1 vanishing on $U_1 - M$, and thus by Lemma 2,

$$[\chi_1|U_1, \chi_1|U_1]_{U_1} = [\chi_1|K, \chi_1|K]_K = a_1^2[\theta, \theta]_K.$$

Therefore $[H : U_1] = a_1^2$. Now, if $\chi_i = \lambda\chi_1$, then $U_i = U_1$ and thus

$$[H : K] = \sum_1^r a_i^2 \geqq [H : V]a_1^2 = [H : V][H : U_1] = [H : K].$$

Therefore, we have equality and $r = [H : V]$ and each $U_i = U_1$. It follows that U_1 admits A, and thus $U_1 = K$. Therefore $[H : K] = a_1^2$ and $r = 1$. This yields $\theta^H = a_1\chi_1$ and we have case (b). The proof is complete.

3. We now offer a lemma concerning coprime action which is essentially due to Glauberman (**2**). This lemma is the key to the proof of the theorem. The solvability hypothesis is superfluous because of the solvability of groups of odd order; however, we do not need that fact here.

LEMMA 4. *Let A act on H with $(|A|, |H|) = 1$ and one of A and H solvable. Suppose that both A and H act on a set S so that $(xh)^a = x^a h^a$ for all $x \in S$, $h \in H$, and $a \in A$. If the action of H is transitive on S, then A fixes some $x \in S$. If, in addition, H acts regularly and $\mathfrak{C}_H(A) = 1$, then A fixes a unique element of S.*

Proof. For the existence of a fixed point see (**2**, Theorem 4). If H is regular on S, $\mathfrak{C}_H(A) = 1$, and A fixes both x and y in S, then since H is transitive, $y = xh$ for some $h \in H$ and $xh = (xh)^a = x^a h^a = xh^a$ and $x = xh^a h^{-1}$. Because of regularity we conclude that $h^a h^{-1} = 1$ and a centralizes h. Since $a \in A$ is arbitrary, we have a contradiction.

PROPOSITION 5. *Let A act on H where H is solvable and A has a normal p-complement for each prime p which divides $|H|$. If $[A, H] < H$, then $\mathfrak{C}_H(A) > 1$.*

Proof. Since $[A, H] \triangle H$ we may choose $K \triangle H$, $[A, H] \subseteq K < H$ with $[H : K] = p$, a prime. Since A acts trivially on $H/[A, H]$, K admits A. Clearly, we may assume that $K > 1$ and choose $L < K$, $L \triangle H$ maximal such that L admits A. If $\mathfrak{C}_{H/L}(A) = C/L > 1$, we may choose C_0, $L < C_0 < H$ with $C_0 \subseteq C$. Then C_0 admits A and $[A, C_0] \subseteq L < C_0$ and the result will follow by induction applied to C_0. We may therefore assume that $\mathfrak{C}_{H/L}(A) = 1$.

Now, K/L is clearly an elementary abelian q-group for some prime q and we consider first the case $q = p$. Let F/L be the Frattini subgroup of the p-group H/L and let B be the normal p-complement of A. Now, $L \subseteq F \subseteq K$, $F \triangle H$, and F admits A, and thus either $F = K$ or $F = L$. If $F = K$, then since B centralizes H/K we may conclude that B centralizes H/L. If $F = L$, then H/L is elementary abelian and B normalizes K/L, and thus by Maschke's theorem (**4**, 12.1.2), there is a complement M/L for K/L which admits B. Thus M admits B and $[B, M] \subseteq [A, H] \cap M \subseteq K \cap M \subseteq L$ and B centralizes M/L. Therefore, if $F = K$ or $F = L$, $\mathfrak{C}_{H/L}(B) > 1$. This is a p-group admitting A and thus it is acted on by the p-group A/B. This implies that $\mathfrak{C}_{H/L}(A) > 1$ and this contradiction completes the proof of the case where $p = q$.

Assuming that $p \neq q$, let U/L be an S_p subgroup of H/L. If U admits A, as it will if $U \triangle H$ for instance, then $[A, U] \subseteq U \cap K \subseteq L$, and thus we may assume that no S_p of H/L admits A and that U is not normal. Now, $(\mathfrak{N}(U) \cap K)/L \subseteq \mathfrak{Z}(H/L) < K/L$ since K/L is abelian. If $\mathfrak{Z}(H/L) = Z/L$, then $Z \triangle H$ and Z admits A, and thus by the choice of L, $Z = L$, and hence $\mathfrak{N}(U) \cap K = L$ and K/L acts regularly on the S_p subgroups of H/L. If B is the normal q-complement of A, then B also permutes the set of S_p's of H/L and these two actions are compatible in the sense of Lemma 4. Therefore, some S_p (say U/L) is fixed by B. The set of fixed points of B is permuted by

A and thus, if there is only one, U admits A and our proof is complete. By Lemma 4 it is sufficient to show that $\mathfrak{C}_{K/L}(B) = 1$. Suppose then that $\mathfrak{C}_{K/L}(B) = C/L > 1$. Then C/L is a q-group acted on by the q-group A/B and thus $T/L = \mathfrak{C}_{C/L}(A) > 1$. However, since $\mathfrak{C}_{H/L}(A) = 1$, this is a contradiction and the proposition is proved.

Proof of the theorem. We shall show that (1) implies (2) and that (3) implies (1), and since (2) implies (3) is obvious, the result will follow.

Suppose that $\chi \neq 1$ is an irreducible character of H which is fixed by A. If $[A, H] < H$, then (2) follows from Proposition 5 and we may thus assume that $[A, H] = H$. Let $K < H$, $K \triangle H$ be maximal admitting A. Then H/K is an elementary abelian p-group and A acts irreducibly and non-trivially on it. By Proposition 3 we have that either (a) $\chi|K \neq 1$ is irreducible on K, (b) $\chi|K = a\theta$, where $\theta \neq 1$ is irreducible on K, or (c) $\chi|K = \sum_1^t \theta_i$, where the θ_i are distinct irreducible characters of K which are transitively permuted by H and where $t = [H:K]$. In cases (a) and (b), χ determines a unique non-principal irreducible character of K which is necessarily fixed by A and, by induction applied to K, we conclude that A fixes some element $\neq 1$ of K. In case (c), the group H/K regularly permutes the θ_i which are also permuted by A. If B is the normal p-complement of A and $\mathfrak{C}_{H/K}(B) > 1$, then $\mathfrak{C}_{H/K}(A) = C/K > 1$. Then C admits A and $[A, C] < C$ and our proof is complete by Proposition 5. We therefore may assume that $\mathfrak{C}_{H/K}(B) = 1$, and thus B fixes a unique one of the θ_i which is thus fixed by A. Thus (2) follows by induction.

Now suppose that $C \neq \{1\}$ is a class of H which is fixed by A. If $[A, H] < H$, then any irreducible non-principal character of $H/[A, H]$ is fixed by A and there is nothing further to show. We assume then that $H = [A, H]$ and we let $K < H$ be a maximal normal subgroup of H which admits A. If $C \not\subseteq K$ and $x \in C$, then $C \subseteq Kx$ since H/K is abelian. Thus, $\langle K, x\rangle$ admits A and is normal in H, and thus $\langle K, x\rangle = H$. Since $[x, A] \subseteq K$, we have that $[A, H] \subseteq K$ and this is a contradiction. We conclude that $C = C_1 \cup C_2 \cup \ldots \cup C_s$, where the C_i are classes of K. The C_i are transitively permuted by H and thus by H/K. Since C is fixed by A, A permutes the C_i and the kernel of the permutation representation of H on $\{C_i\}$ admits A. If this kernel is less than H, it must be K and H/K acts faithfully and transitively on $\{C_i\}$. Since H/K is abelian, the action is regular. Now, H/K is a p-group and we let B be the normal p-complement of A. If $\mathfrak{C}_{H/K}(B) > 1$, then by the usual argument $\mathfrak{C}_{H/K}(A) > 1$. This admits A, and thus A acts trivially on H/K and $[A, H] < A$, a contradiction. Thus $\mathfrak{C}_{H/K}(B) = 1$ and in the situation that H/K acts non-trivially on $\{C_i\}$ we conclude that B fixes a unique C_i which is therefore fixed by A. If H/K acts trivially on $\{C_i\}$, then C is a class of K, and thus in any case, A fixes a non-trivial class of K and by induction applied to K, there exists a non-principal irreducible character θ, of K, which is fixed by A.

By Proposition 3 we have that either (a) θ^H is irreducible, (b) $\theta^H = a\chi$,

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where χ is irreducible, or (c) $\theta^H = \sum_1^t \chi_i$, where the χ_i are irreducible characters of H which are transitively permuted by multiplication by linear characters of H/K and $t = [H : K]$. If either (a) or (b) holds, then θ determines a unique, non-principal irreducible character of H which is necessarily fixed by A. In case (c), we note that A permutes the χ_i and that the character group L of H/K regularly permutes them. If B is the normal p-complement of A, then B acts on L in a manner compatible with its action on $\{\chi_i\}$, and thus fixes some χ_i. If $\mathfrak{C}_L(B) > 1$, then since L is a p-group, $\mathfrak{C}_L(A) > 1$, i.e., there exists a non-principal linear character of H which is fixed by A. If $\mathfrak{C}_L(B) = 1$, then B fixes a unique χ_i which is thus fixed by A, completing the proof of the theorem.

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University of Chicago,
Chicago, Illinois