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OSCILLATION OF ELLIPTIC EQUATIONS IN GENERAL DOMAINS

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1. Introduction. Oscillation criteria will be obtained for the linear elliptic partial differential equation

$$Lu = (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha(a_{\alpha\beta}(x)D^\beta u) - c(x)u = 0,$$

$$x = (x_1, x_2, \ldots, x_n),$$

in an unbounded domain G of general type in n-dimensional Euclidean space E^n. The differential operator D is defined as usual by

$$D^\alpha u = D_1^{\alpha(1)} \ldots D_n^{\alpha(n)}; \quad \alpha = (\alpha(1), \alpha(2), \ldots, \alpha(n)),$$

$|\alpha| = \sum_{i=1}^n \alpha(i)$, where each $\alpha(i)$, $i = 1, \ldots, n$, is a non-negative integer. It will be assumed throughout that the coefficients $a_{\alpha\beta}$, are symmetric, i.e., $a_{\alpha\beta} = a_{\beta\alpha}$, and the operator L is uniformly strongly elliptic in G, i.e., there exists a positive constant d_0 such that

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta} \geqq d_0|\xi|^{2m}$$

for all $x \in G$ and for every $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$. The purpose of the present work is to extend some recent results by Swanson [**6**], and K. Kreith to elliptic operators of arbitrary even order.

2. Definitions and notation. A bounded domain $N \subset G$ is said to be a *nodal domain* for L if there exists a nontrivial function $w \in C^{2m}(N) \cap C^m(\bar{N})$ such that $Lw = 0$ in N, $D^\alpha w = 0$ on ∂N for all α with $|\alpha| \leqq m - 1$.

The operator L is said to be *oscillatory* in G if it has a nodal domain outside of every sphere centred at the origin.

Let the set of multi-indices α be ordered, in an arbitrary manner, in a sequence $S = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, where $\alpha_i = (\alpha_i(1), \alpha_i(2), \ldots, \alpha_i(n))$, and k is the number of multi-indices α. Let M be the $k \times k$ matrix defined by

$$M = (a_{\alpha_i\alpha_j}), \quad i, j = 1, 2, \ldots, k.$$

Let $\lambda(x)$ be the largest eigenvalue of the coefficient matrix M. An elementary argument [**5**] shows that $\lambda(x)$ does not depend on the multi-indices.

In the case the domain G is the whole of E^n, oscillation criteria were obtained by the author [**5**]. For example (1) is oscillatory in E^n if $\lambda(x)$ is bounded below

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in E^n by some number λ_1; $n \leqq m + 1$, and

$$\int_{|x|>0} c(x)dx = +\infty.$$

The example given by Swanson in [**6**] can be used to show that the above condition is not enough for (1) to be oscillatory if G is too "small" at ∞.

In this paper we only require that the interior of G is unbounded. i.e. For any $R > 0$ the set $G_R = \{x \in G : ||x|| > R\}$ has interior points. In particular, the domain G could be quasi-conical or quasi-cylindrical.

3. Basic lemmas. It is well known that the eigenvectors of the operator L, as defined by (1), on a bounded domain Ω of E^n which has sufficiently smooth boundary, lie in the Sobolev space H_0^m (the closure in the norm $||\cdot||_m$ defined by

$$||u||_m^2 = \int_\Omega \sum_{|\alpha|=m} (D^\alpha u)^2 dx$$

of the class $C_0^\infty(\Omega)$ of infinitely differentiable functions with compact support in Ω). The following lemma can be proved by using Garding's inequality [**5**].

LEMMA 1. *For $0 < t < \infty$, let Ω_t be a domain contained within a domain Ω of bounded width $\leqq t$. If $0 < r < s < \infty$ implies $\Omega_r \subset \Omega_s$, $\Omega_r \neq \Omega_s$, then the smallest eigenvalue $\mu_0(t)$ of the problem*

$$Lu = \mu(t)u \text{ in } \Omega_t,\ D^\alpha u = 0 \text{ on } \partial\Omega_t,\ |\alpha| \leqq m - 1$$

is monotone decreasing in t, and

$$\lim_{t\to 0+} \mu_0(t) = +\infty.$$

We can also assume that the smallest eigenvalue varies continuously when the domain G is deformed "continuously" in a sense similar to that specified in [**1**].

The following lemma can be easily proved by repeated application of Leibniz' rule:

LEMMA 2. *If $u = u(r)$ is an m-times differentiable function for all r in $(0, \infty)$, then the following inequality holds:*

$$\sum_{|\alpha|=m} (D^\alpha u)^2 \leqq \sum_{k=1}^{m} m_k r^{2k-2m} (u^{(k)}(r))^2$$

for $r > 1$, where $u^{(k)}(r) = d^k u/dr^k$, and each m_k is a positive constant, $k = 1, 2, \ldots, m$.

Let $\lambda(x)$ denote, as before, the largest eigenvalue of the coefficient matrix

$(a_{\alpha_i\alpha_j})$. Let

$$\tilde{\lambda}(r, \theta_1, \ldots, \theta_{n-1}) = \lambda(x)$$

$$\tilde{c}(r, \theta_1, \ldots, \theta_{n-1}) = c(x)$$

$$\Lambda(r) = \int_{W_n} \tilde{\lambda}(r, \theta_1, \ldots, \theta_{n-1}) dw_n$$

$$c(r) = \int_{W_n} \tilde{c}(r, \theta_1, \ldots, \theta_{n-1}) dw_n$$

where $r, \theta_1, \ldots, \theta_{n-1}$ are the hyperspherical polar coordinates, and W_n is the surface area of the unit ball in E^n.

For each pair of real numbers $\{a, b\}$ such that $0 < a < b < \infty$, let M_a^b be the quadratic functional defined by

$$M_a^b[u] = \int_a^b \sum_{k=1}^m [m_k r^{2k-2m} \Lambda(r)(u^{(k)}(r))^2 - c(r)u^2] r^{n-1} dr$$

with domain consisting of all $u \in C^m(a, b)$, where m_k and $c(r)$ are as defined above. The proof of the following lemma may be found in [**5**].

LEMMA 3. *If $v = v(r)$ is a function defined on the interval $[a, b]$, having the properties*

(i) $v(r) \in C^{m-1}[a, b]$

(ii) $v^{(m)} \in L_2(a, b)$

(iii) $v^{(i)}(a) = v^{(i)}(b) = 0, i = 0, 1, 2, \ldots, m-1$,

then for any $\delta > 0$ there exists a function $u \in C^{2m}(a, b)$ which satisfies the conditions

$$u^{(i)}(a) = u^{(i)}(b) = 0, i = 0, 1, 2, \ldots, 2m-1, \text{ and}$$

$$|M_a^b[u] - M_a^b[v]| < \delta.$$

4. Oscillation criteria.

THEOREM 4. *Equation* (1) *is oscillatory in an unbounded domain $G \subset E^n$ if G contains a sequence of spherical annuli defined by*

$$N_k(x_k; a_k; b_k) = \{x \in E^n : 0 < a_k < |x_k - x| < b_k\};$$

$k = 1, 2, \ldots$, *having the following properties:*

(a) *There exists a function $v_k = v_k(|x - x_k|)$ on each N_k which satisfy, on the interval $[a_k, b_k]$, the properties* (i), (ii), (iii) *of Lemma 3; and $M_{a_k}^{b_k}[v_k] < 0$ for all sufficiently large k; and*

(b) *For arbitrary $r > 0$ there exists a number $n(r)$ such that $N_k \subset G_r = \{x \in G : |x| > r\}$ for all $k > n(r)$.*

Proof. Let $\mu(t)$ denote the smallest eigenvalue of the problem

$$Lu = \mu(t)u \text{ in } N_k(x_k; a_k; t),$$

$$D^\alpha u = 0 \text{ on } \partial N_k(x_k; a_k; t) \text{ for all } |\alpha| \leqq m-1,$$

where $a_k < t \leqq b_k$. By hypothesis (a) and Lemma 3 exists a function $w_k = w_k(|x - x_k|) \in C^{2m}(a_k, b_k)$ satisfying

$$w_k^{(i)}(a_k) = w_k^{(i)}(b_k) = 0, \quad i = 1, 2, \ldots, m - 1, \quad M_{a_k}{}^{b_k}[w_k] < 0.$$

Then

$$\int_{N_k(x_k; a_k; b_k)} w_k L w_k dx \leqq M_{a_k}{}^{b_k}[w_k] < 0$$

follows from integration by parts. From the last inequality and by a well known variational principle [**4**] we see that $\mu(b_k) \leqq 0$. By Lemma 1 there exists t, $a_k < t \leqq b_k$, such that $\mu(t) = 0$. Hence the domain $N_k(x_k; a_k; t)$ is a nodal domain of a nontrivial solution of (1) for sufficiently large k. By hypothesis (b), for arbitrary $r > 0$ there exists a number $n(r)$ such that $N_k(x_r; a_k; t) \subset G_r$. This completes the proof of Theorem 3.

THEOREM 5. *Equation* (1) *is oscillatory in an unbounded domain* $G \subset E^n$ *if* G *contains a sequence of spherical annuli* $\{N_k(x_k; a_{k/2}; 3a_k)\}$, $k = 1, 2, \ldots$, *with the following properties:*

(i) $\lim_{k\to\infty} (|x_k| - 3a_k) = \infty$;

(ii) $c(x)$ *is non-negative in each* N_k, *and*

$$\lim_{k\to\infty} a_k{}^{2m} \left[\int_{N_k(x_k; a_k/2; 3a_k)} \lambda(x) dx \right]^{-1} \int_{N_k(x_k; a_k; 2a_k)} c(x) dx = +\infty.$$

Proof. A sequence of functions will be constructed which satisfy hypothesis (a) of Theorem 4.

Let

$$v(t) = k \int_0^t s^{m-1}(1 - s)^{m-1} ds,$$

where k is chosen so that $v(1) = 1$. Let v_k be defined by

$$\begin{aligned} v_k(r) &= 0 && r < a_k/2 \\ &= v\left(\frac{2r - a_k}{a_k}\right) && a_k/2 \leqq r < a_k \\ &= 1 && a_k \leqq r < 2a_k \\ &= v\left(\frac{3a_k - r}{a_k}\right) && 2a_k \leqq r < 3a_k \\ &= 0 && r \geqq 3a_k \end{aligned}$$

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where $r = |x - x_k|$. Then,

$$M_{a_k/2}{}^{3a_k}[v_k] = \int_{a_k/2}^{a_k} \sum_{i=1}^{m} m_i r^{2i-2m} \Lambda(r) 2^{2i} a_k^{-2i} r^{n-1} (v_k^{(i)}(r))^2 dr$$

$$+ \int_{2a_k}^{3a_k} \sum_{i=1}^{m} m_i r^{2i-2m} \Lambda(r) 2^{2i} a_k^{-2i} r^{n-1} (v_k^{(i)}(r))^2 dr$$

$$- \int_{N_k(x_k; a_k/2; 3a_k)} v_k^2(x) c(x) dx$$

$$\leqq k_1 a_k^{-2m} \left[\int_{a_k/2}^{a_k} \Lambda(r) r^{n-1} dr + \int_{2a_k}^{3ak} \Lambda(r) r^{n-1} dr \right]$$

$$- \int_{N_k(x_k; a_k; 2a_k)} c(x) dx$$

for some positive constant K_1. Hence

$$a^{2m} \left[\int_{N_k(x_k; a_k/2; 3a_k)} \lambda(x) dx \right]^{-1} M_{a_k/2}{}^{3a_k}[v_k]$$

$$\leqq k_1 - a_k^{2m} \left[\int_{N_k(x_k; a_k/2; 3a_k)} \lambda(x) dx \right]^{-1} \int_{N_k(x_k; a_k; 2a_k)} c(x) dx.$$

Hypothesis (ii) then shows that $M_{a_k/2}{}^{3a_k}[v_k] < 0$ for sufficiently large k, and therefore hypothesis (a) of Theorem 4 is satisfied.

By (i), there exists a number $n(r)$ for each $r > 0$ such that $|x_k| - 3a_k > r$ whenever $k > n(r)$. Then $x \in N_k(x_k;\ a_k/2;\ 3a_k)$ implies that $|x| \geqq |x_k| - |x - x_k| > |x| - 3a_k > r$ so that $x \in G_r$, and $N_k(x_k;\ a_k/2;\ 3a_k) \subset G_r$ for all $k > n(r)$. Hence (1) is oscillatory by Theorem 4.

COROLLARY 6. *Equation* (1) *is oscillatory in an unbounded domain* $G \subset E^n$ *if* G *contains a sequence of spherical annuli* $\{N_k(x_k; a_k/2; 3a_k)\}$, $k = 1, 2, \ldots$, *with the following properties:*

(i) $\lim_{k\to\infty} (|x_k| - 3a_k) = \infty$;
(ii) $(a_{\alpha\beta}(x))$ *is bounded* (*as a form*) *in* G;
(iii) $c(x)$ *is non-negative in each* $N(x_k; a_k/2; 3a_k)$, *and*

$$\lim_{k\to\infty} a_k^{2m-n} \int_{N_k(x_k; a_k; 2a_k)} c(x) dx = +\infty.$$

The above corollary generalizes a recent result of Swanson [**6**] to differential equations of arbitrary even order.

Example. Suppose G contains a sequence of open discs $\{N_k(x_k;\ a)\}$ such that $\lim_{k\to\infty}|x_k| = \infty$. Evidently this condition is satisfied if G contains an infinite cylinder, and also for a class of "spiral" domains containing no infinite ray.

The equation

$$(-)^m \Delta^m u + c(x)u = 0$$

is oscillatory in G if any one of the following conditions is satisfied:

(a) $c(x)$ is non-negative in each $N_k(x_k; a)$, and

$$\lim_{k\to\infty} \int_{N_k(x_k; a/3; 2a/3)} c(x)dx = +\infty;$$

(*b*) $c(x) \geqq c_k > 0$ in each $N_k(x_k; a)$ where $\lim_{k\to\infty} c_k = +\infty$;

(c) $\lim_{|x|\to\infty} c(x) = +\infty$ uniformly in G.

We shall consider now the special case when G is the whole space E^n. The following theorem generalizes a recent result of Kreith and Travis [**3**].

THEOREM 7. *The partial differential equation*

(2) $$Lu = \sum_{|\alpha|=|\beta|=2} D^\alpha(a_{\alpha\beta}(x)D^\beta u) - c(x)u = 0$$

is oscillatory in E^n *if the following ordinary differential equation is oscillatory at* $r = \infty$:

(3) $$lu = [r^{n-1}\Lambda(r)z'']'' - [2r^{n-3}\Lambda(r)z']' - r^{n-1}c(r)z = 0.$$

Proof. Suppose equation (3) is oscillatory at $r = \infty$. Let $I_1 = \{r: r_1 < r < t_1\}$ be a nodal domain for the operator l. By increasing t_1 if necessary and using lemma 1, we can assume that the smallest eigenvalue λ_1 of the problem

$$lu = \lambda u \text{ in } I_1,$$
$$u(r_1) = u'(r_1) = u(t_1) = u'(t_1) = 0$$

is negative. Let $z_1(r)$ be the corresponding eigenfunction. Suppose I_k, $z_k(r)$ have been chosen. Let $I_{k+1} = \{r: r_{k+1} < r < t_{k+1}\}$ be a nodal domain for the operator l such that $r_{k+1} > t_k$. By increasing t_{k+1} if necessary we can assume, as before, that the smallest eigenvalue λ_k of the problem

$$lu = \lambda u \text{ in } I_{k+1}$$
$$u(r_{k+1}) = u'(r_{k+1}) = u(t_{k+1}) = u'(t_{k+1}) = 0$$

is negative. Let $z_{k+1}(r)$ be the corresponding eigenfunction. By induction there exists a sequence of eigenfunctions on the intervals I_k, $k = 1, 2, \ldots$, such that

(4) $$lz_k = \lambda_k z_k \quad \text{in } I_k$$
$$z_k(r_k) = z_k'(r_k) = z_k(t_k) = z_k'(t_k) = 0,$$
$$\lambda_k < 0.$$

Take N_k in Theorem 4 to be the annular domain defined by

$$N_k = \{x \in E^n : r_k < |x| < t_k\}.$$

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Take $v_k(x) = z_k(|x|)$. Then $v(x) = \partial v/\partial x_i = 0$ on ∂N_k for all $k, i = 1, 2, \ldots, n$, and it is easily checked that

$$\int_{N_k} v_k L v_k dx \leqq M_{rk}{}^{t_k}[v] = \int_{r_k}^{t_k} \Lambda(r)(z_k''(r))^2 r^{n-1} + 2(z'(r))^2 r^{n-3} - r^{n-1}c(r)(z(r))^2 dr < 0.$$

The conclusion follows from Theorem 4.

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