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Locally compact groups appearing as ranges of cocycles of ergodic $\mathbb{Z}$-actions

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Abstract. The paper contains the proof of the fact that every solvable locally compact separable group is the range of a cocycle of an ergodic automorphism. The proof is based on the theory of representations of canonical anticommutation relations and the orbit theory of dynamical systems. The slight generalization of reasoning shows further that this result holds for amenable Lie groups as well and can be also extended to almost connected amenable locally compact separable groups.

1. The study of cocycles of dynamical systems is an important trend in the modern ergodic theory (see review papers [**7**], [**8**] and [**9**]). In particular, if an ergodic non-singular action of a locally compact group Γ on a Lebesgue space (X, μ) and a 1-cocycle $\alpha(x, \gamma)$ $(x \in X, \gamma \in \Gamma)$ of this action with values in a locally compact group G are given, then an ergodic action of the group G can be constructed by a standard procedure (see [**9**]). According to [**9**], this new dynamical system is called the *range* of the cocycle α of the ergodic Γ-action.

Mackey formulated the problem of the description of dynamical systems which are the ranges of cocycles of an ergodic automorphism (the Poincaré flows) [**9**]. Paper [**3**] contributed considerably to the solution of this problem. On the other hand, R. Zimmer proved that if a locally compact separable group (l.c.s. group) G is the range of a cocycle of an ergodic $\mathbb{Z}$-action (see a strict definition below), then any ergodic action of G is a Poincaré flow [**14**]. Thus the problem arises of describing the l.c.s. groups which are the ranges of cocycles of ergodic $\mathbb{Z}$-actions. Zimmer proved that any l.c.s. group which is the range of a cocycle of $\mathbb{Z}$ is amenable [**15**] and established that compact, discrete Abelian, connected nilpotent Lie groups and finite direct products of such groups appear as ranges of cocycles of an ergodic automorphism [**13**], [**14**].

In § 3 we use the representation theory of canonical anticommutation relations [**4**] (namely the proof of theorem 5, § 4) and the orbit theory of dynamical systems [**11**] to prove that any l.c.s. solvable group is the range of a cocycle of an ergodic $\mathbb{Z}$-action. Similar results will be presented for compact extensions of a solvable group (§ 6), solvable extensions of a compact group (§ 5) and for similar extensions of these newly formed groups (§ 7). It follows from the structure of l.c.s. groups [**10**] that the class of groups which are the ranges of cocycles of an ergodic automorphism includes, in particular, amenable Lie groups and amenable l.c.s. groups whose quotient by the topological component of the identity is compact (almost connected l.c.s. groups).

For each of these groups the paper presents a method of explicit construction of a 1-cocycle of an ergodic action of the group $\bigoplus_{i=1}^{\infty} \mathbb{Z}_2$ and, therefore, of $\mathbb{Z}$ (according to [**11**]), whose range is the given group.

The authors were informed by the referee of this paper that the results obtained by M. Herman with a different method [**6**] imply that any amenable l.c.s. group is the range of a cocycle. By developing further the methods proposed in this paper and using [**2**], we construct in [**5**] the cocycles of some special form so as to obtain the above result for an arbitrary amenable l.c.s. group. Our technique also permits the uniqueness of cocycles with a dense range in a given amenable l.c.s. group to be proved [**5**, theorem 5.1].

We wish to thank the referee for calling our attention to some errors in an earlier version of the paper.

2. A Borel map $\alpha: X \times \Gamma \to G$ such that

$$\alpha(x, \gamma_1\gamma_2) = \alpha(x, \gamma_1)\alpha(x \cdot \gamma_1, \gamma_2)$$

for all $\gamma_1, \gamma_2 \in \Gamma$ at almost every (a.e.) $x \in X$ is called a cocycle of the dynamical system (X, μ, Γ) with values in an l.c.s. group G.

Let μ_G be a right Haar measure of G. Then we can consider an action of the group Γ on $(X \times G, \mu \times \mu_G)$:

$$(x, g)\gamma = (x \cdot \gamma, g\alpha(x, \gamma)), \qquad \text{for } \gamma \in \Gamma, (x, g) \in X \times G. \tag{1}$$

If this action is ergodic, G is said to be the *range* of the cocycle α of the ergodic Γ-action on (X, μ).

It is known that any ergodic $\mathbb{Z}$-action is orbit equivalent to an action of the group $\Gamma = \bigoplus_{i=1}^{\infty} \mathbb{Z}^{(i)}, \mathbb{Z}^{(i)} \cong \mathbb{Z}_2$, on the space $(X, \mu) = (Y, \nu)^{\mathbb{N}}$, where $Y = \{0; 1\}$; $\nu(\{0\}) = \nu(\{1\}) = \frac{1}{2}$ [**11**]. According to [**4**, theorem 2, § 1], any cocycle of this action is given by the equation:

$$c(x, \delta_k) = f_1(x)^{-x_1} \cdot f_2(x)^{-x_2} \cdot \ \ldots \ \cdot f_{k-1}(x)^{-x_{k-1}} \cdot f_k(x)^{(-1)^{x_k}}$$
$$\cdot f_{k-1}(x\delta_k)^{x_{k-1}} \cdot \ \ldots \ \cdot f_1(x\delta_k)^{x_1}, \tag{2}$$

where $f_k: X \to G$ is a Borel map, invariant with respect to $\delta_1, \ldots, \delta_k$, and $\delta_k \in \Gamma$ is a sequence in which the k'th place is taken by 1 and the others by 0's. Therefore, we shall construct below the cocycles of the above action of Γ instead of cocycles of an ergodic $\mathbb{Z}$-action.

3. THEOREM 3.1. *Every solvable l.c.s. group G is the range of a cocycle of an ergodic $\mathbb{Z}$-action.*

Proof. To simplify our notation, suppose that G is a 2-step solvable l.c.s. group. As will be seen below, the general case can be considered in the same manner.

Let H be a countable subgroup dense in G (2-step solvable too). Denoting its maximum commutative normal subgroup by K, we note that the quotient group

H/K is also commutative. Let $\{k_j\}_{j=1}^{\infty}$ be a sequence of elements of K, in which every element of K occurs infinitely many times. We also construct a similar sequence $\{\bar{h}_j\}_{j=1}^{\infty}$ for the quotient H/K and then a sequence $\{h_j\}_{j=1}^{\infty}$ such that $\psi(h_j)=\bar{h}_j$, $j\in\mathbb{N}$, where $\psi\colon H\to H/K$ is the natural projection map. Now we construct the sequence of integers

$$\{n(j)\}_{j=1}^{\infty}\colon n(1)=1,\, n(j+1)=n(j)+2^j+1.$$

Finally, denote by $\{\zeta_r^j\}_{r=1}^{2^j}$ the set of sequences of 0's and 1's of length j. Let f_k be as follows:

$$\begin{aligned} f_{n(j)} &= h_j,\\ f_{n(j)+r} &= h_j^{\zeta_r^j(j)}\cdot h_{j-1}^{\zeta_r^j(j-1)}\cdot\ \ldots\ \cdot h_2^{\zeta_r^j(2)}\cdot h_1^{\zeta_r^j(1)}(k_j), \end{aligned}\tag{3}$$

where $g(k)=gkg^{-1}$, $j\in\mathbb{N}$, $r=1,\ldots,2^j$. Thus we complete the construction of the cocycle $c\colon X\times\Gamma\to G$ given by f_k according to (2).

LEMMA 3.2. *Let $A\subset X$, $\mu(A)>0$ and $k\in K$, then there exist $D\subset A$, $\mu(D)>0$ and $\gamma\in\Gamma$ such that $D\cdot\gamma\subset A$ and $c(x,\gamma)=k$ for a.e. $x\in D$.*

Proof. Let $a\in\prod_{i=1}^{N}\{0;1\}$ such that $\mu(A\cap I_N(a))>\frac{3}{4}\mu(I_N(a))$ where

$$I_N(a)=\{x\in X\colon x_i=a_i,\ i=1,\ldots,N\}.$$

Choose $j\in\mathbb{N}$ so that $n(j)>N$ and $k_j=k$. Then there exists a sequence $b\in\prod_{i=1}^{n(j)}\{0;1\}$ in which $b_1=a_1,\ldots,b_N=a_N$ and $\mu(A\cap I_{n(j)}(b))>\frac{3}{4}\mu(I_{n(j)}(b))$. Now choose $r=1,\ldots,2^j$ so that $\zeta_r^j(s)=b_{n(s)}$, $s=1,\ldots,j$. Then for $x\in I_{n(j)}(b)$

$$c(x,\delta_{n(j)+r})=k^{(-1)^{x_{n(j)+r}}}.\tag{4}$$

Write

$$A(0)=A\cap I_{n(j)}(b)\cap\{x\colon x_{n(j)+r}=0\};$$
$$A(1)=A\cap I_{n(j)}(b)\cap\{x\colon x_{n(j)+r}=1\}.$$

Then $\mu(A(0))>\frac{1}{4}\mu(I_{n(j)}(b))$, $\mu(A(1))>\frac{1}{4}\mu(I_{n(j)}(b))$. Take $D=A(0)\cap A(1)\cdot\delta_{n(j)+r}$. Evidently, $\mu(D)>0$ and $c(x,\delta_{n(j)+r})=k$ for a.e. $x\in D$, which was to be proved. □

LEMMA 3.3. *Let $A\subset X$, $\mu(A)>0$, $h\in H$. Then there exist $D\subset A$, $\mu(D)>0$, $k\in K$ and $\gamma\in\Gamma$ such that $D\cdot\gamma\subset A$ and $c(x,\gamma)=kh$ for a.e. $x\in D$.*

Proof. It follows from the commutativity of the quotient group H/K that

$$c(x,\delta_{n(j)})=\beta(x_1,\ldots,x_{n(j)})h_j^{(-1)^{x_{n(j)}}},\tag{5}$$

where $\beta(x_1,\ldots,x_{n(j)})\in K$. In other respects the proof is similar to that of lemma 3.2. □

LEMMA 3.4. *Let $A\subset X$, $\mu(A)>0$, $h\in H$. Then there exist $D\subset A$, $\mu(D)>0$ and $\gamma\in\Gamma$ such that $D\cdot\gamma\subset A$ and $c(x,\gamma)=h$ for a.e. $x\in D$.*

Proof. By lemma 3.3 we can construct $D_1\subset A$ and $\gamma_1\in\Gamma$ such that $D_1\cdot\gamma_1\subset A$ and $c(x,\gamma_1)=kh$ for a.e. $x\in D_1$. Then we apply lemma 3.2 to D_1 and k^{-1}. □

For Γ we consider the full group $[\Gamma]$ of automorphisms of (X,μ):

$$[\Gamma]=\{\gamma\in\mathrm{Aut}\,(X,\mu)\colon x\cdot\gamma=x\cdot\delta(x,\gamma),\ \delta(x,\gamma)\in\Gamma\}$$

[1]. Let $\alpha(x,\gamma)=c(x,\delta(x,\gamma))$. One can check that α is a cocycle for $[\Gamma]$ and $\alpha(x,\gamma)=c(x,\gamma)$ for $\gamma\in\Gamma$. Now we consider a subgroup L of the full group $[\Gamma]$ consisting of those automorphisms γ, for which $\alpha(x,\gamma)=e$ for a.e. $x\in X$.

LEMMA 3.5. *L acts ergodically on (X,μ).*

Proof. Let $E\subset X$, $0<\mu(E)<1$. Since Γ acts ergodically on X, there exist $B\subset E$, $A\subset X\setminus E$ $(\mu(A),\mu(B)>0)$, $h^{-1}\in H$ and $\gamma_1\in\Gamma$ such that $B\cdot\gamma_1=A$ and $c(x,\gamma_1)=h^{-1}$ for a.e. $x\in B$. Now we apply lemma 3.4 to $A\subset X$ and $h\in H$ and find $D\subset A$, $\mu(D)>0$ and $\gamma\in\Gamma$ such that $D\cdot\gamma\subset A$ and $c(x,\gamma)=h$ for a.e. $x\in D$. Then for $\gamma_2=\gamma_1\gamma$ and a.e. $x\in D\cdot\gamma_1^{-1}$ we have:

$$\alpha(x,\gamma_2)=c(x,\gamma_2)=c(x,\gamma_1)c(x\cdot\gamma_1,\gamma)=e. \tag{6}$$

Let $D_1=D\cdot\gamma_1^{-1}$, then $\mu(D_1)>0$, $D_1\subset E$, $D_1\cdot\gamma_2\subset X\setminus E$. Define an automorphism $\gamma_2'\in[\Gamma]$:

$$\gamma_2'=\begin{cases}\gamma_2 & \text{for } x\in D_1,\\ \gamma_2^{-1} & \text{for } x\in D_1\cdot\gamma_2,\\ \mathrm{id} & \text{for } x\in X\setminus(D_1\cup D_1\cdot\gamma_2).\end{cases} \tag{7}$$

Evidently, $D_1\cdot\gamma_2'=D_1\cdot\gamma_2$ and $\alpha(x,\gamma_2')=e$ now for a.e. $x\in X$. □

Completion of the proof of theorem 3.1. It is to be proved that the action of Γ on $(X\times G,\mu\times\mu_G)$ defined by (1) is ergodic, with $\alpha(x,\gamma)$ being the cocycle constructed above. Let $f\in L^\infty(X\times G,\mu\times\mu_G)$ and f be Γ-invariant. Then f is L-invariant, and it follows from the ergodicity of L that $f(x,g)$ is independent of $x\in X$. Since $\{\alpha(x,\gamma)\}_{x\in X}^{\gamma\in\Gamma}=H$, f must be H-invariant, and it follows from the density of H in G that $f=\text{const}$. This completes the proof. □

In the case of an n-step solvable group a sequence of elements from $G^{(k)}$ is constructed in a similar way, where $G^{(k)}$ is a maximum k-step solvable normal subgroup. When constructing a cocycle, the elements of lower subgroups are conjugated with higher subgroup elements. The general proof is quite the same.

4. According to Zimmer's results [13], any compact group is the range of a cocycle of an ergodic $\mathbb{Z}$-action. In this section we shall give a new version of the proof of this fact. The methods developed below will be used in subsequent sections.

THEOREM 4.1. *Let K be a second countable compact group. Then it is a range of a cocycle of an ergodic $\mathbb{Z}$-action.*

To prove this we require the following standard proposition:

LEMMA 4.2. *Let K be a second countable compact group and U a neighbourhood of the identity in K. Then there exists a neighbourhood of the identity V such that for all $h\in K$, $h^{-1}Vh\subset U$.*

Proof. Suppose that the statement of the lemma is not valid and $\{V_n\}_{n=1}^\infty$ is a fundamental system of neighbourhoods of the identity in K. Then for each $n\in\mathbb{N}$ we can find $h_n\in K$, $x_n\in V_n$ such that

$$h_n^{-1}x_nh_n\notin U. \tag{8}$$

Evidently, $\lim_{n\to\infty} x_n = e$. By picking a subsequence convergent in K to some limit h, and proceeding thus to the limit in (8), we find that $e \notin \operatorname{int} U$; that is, U is not a neighbourhood of the identity. This is a contradiction. □

The following is straightforward.

COROLLARY 4.3. *Let K be a second countable compact group, U and V neighbourhoods of the identity in K satisfying the requirements of lemma* 4.2. *Let $\{f_i\}_{i=1}^n$ form a V-net in K. Then for any $h \in K$, $\{h^{-1}f_ih\}_{i=1}^n$ form a U-net in K.*

Proof of theorem 4.1. Let $\{W_i\}_{i=1}^\infty$ be a fundamental system of neighbourhoods of the identity in K. For each $i \in \mathbb{N}$ we construct a finite W_i-net in K. We shall regard $f_k(x)$, $k \in \mathbb{N}$, as constant and define these so that for each $i \in \mathbb{N}$ there will be a sequence of natural numbers $\{n_i(k)\}_{k=1}^\infty$ such that $\{f_{n_i(k)+j}\}_{j=1}^{q(i)}$ is coincident with the W_i-net constructed above. The cocycle $c: X \times \Gamma \to K$ is therefore determined by (2).

To prove the theorem, it suffices to show that for any cocycle β equivalent to c, the closed subgroup generated by the set $\{\beta(x, \gamma)\}_{x\in X}^{\gamma\in\Gamma}$ should coincide with K [**12**, corollary 3.8]. To do so, it is enough to prove that for any neighbourhood of the identity U in K a U-net can be constructed of elements of the subgroup generated by the set $\{\beta(x, \gamma)\}_{x\in X}^{\gamma\in\Gamma}$. Now we examine the proof of the latter statement.

Let $\beta: X \times \Gamma \to K$ be a cocycle equivalent to c; that is, there exists a Borel function $g: X \to K$ such that

$$\beta(x, \gamma) = g(x)c(x, \gamma)g(x\cdot\gamma)^{-1}.$$

Assume also that an arbitrary neighbourhood of the identity U in K is given. We choose a neighbourhood of the identity V, so that $V = V^{-1}$ and $V \cdot V \subset U$. Then we choose $i \in \mathbb{N}$ so that for any $h \in K$, $h^{-1}W_ih \subset V$. Let W be a neighbourhood of the identity for which

$$\underbrace{W\cdot W\cdot \cdots \cdot W}_{2q(i)\text{ times}} \subset V.$$

Now we define a neighbourhood of the identity $\bar{W}$ so that $\bar{W} = \bar{W}^{-1}$ and for any $h \in K$ there is an inclusion $h^{-1}\bar{W}h \subset W$. Finally, we take a neighbourhood of the identity W' such that $W' = W'^{-1}$ and $W' \cdot W' \subset \bar{W}$.

Let $A \subset X$, $\mu(A) > 0$, be a set on which $g(x)$ is continuous and x_0 be a point of density of the set A. Then we find $k \in \mathbb{N}$ such that for all $y \in I_{n_i(k)}(x_0) \cap A$ we have $g(y) \in g(x_0)W'$,

$$\mu(I_{n_i(k)}(x_0) \cap A) > (q(i)/(1+q(i)))\mu(I_{n_i(k)}(x_0)).$$

Consider a set

$$P = \bigcap_{j=n_i(k)+1}^{n_i(k)+q(i)} (I_{n_i(k)}(x_0) \cap A)\cdot\delta_j \cap I_{n_i(k)}(x_0) \cap A.$$

We can see that $P \neq \emptyset$ by the above choice of k and $P\cdot\delta_j \subset I_{n_i(k)}(x_0) \cap A$ for $j = n_i(k)+1, \ldots, n_i(k)+q(i)$. Let $x \in P$, then for the same j,

$$\begin{aligned} g(x\cdot\delta_j) &\in g(x_0)W' \subset g(x)W'^{-1}W' \\ &= g(x)W'\cdot W' \subset g(x)\bar{W}. \end{aligned}$$

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Consider

$$\omega_j = c(x, \delta_{n_i(k)+1})^{-x_{n_i(k)+1}} \cdot \ldots \cdot c(x, \delta_{n_i(k)+j-1})^{-x_{n_i(k)+j-1}}$$
$$\times c(x, \delta_{n_i(k)+j})^{(-1)^{x_{n_i(k)+j}}} \cdot c(x, \delta_{n_i(k)+j-1})^{x_{n_i(k)+j-1}} \cdot \ldots \cdot c(x, \delta_{n_i(k)+1})^{x_{n_i(k)+1}}$$

for $j = 1, \ldots, q(i)$. Transformation of this expression gives

$$\omega_j = \varphi_{n_i(k)}(x)^{-1} f_{n_i(k)+j} \cdot \varphi_{n_i(k)}(x), \qquad j = 1, \ldots, q(i),$$

where $\varphi_s(x) = f_s^{x_s} \cdot f_{s-1}^{x_{s-1}} \cdot \ldots \cdot f_2^{x_2} \cdot f_1^{x_1}$.

Now consider

$$\tilde{\omega}_j = \beta(x, \delta_{n_i(k)+1})^{-x_{n_i(k)+1}} \cdot \ldots \cdot \beta(x, \delta_{n_i(k)+j-1})^{-x_{n_i(k)+j-1}}$$
$$\times \beta(x, \delta_{n_i(k)+j})^{(-1)^{x_{n_i(k)+j}}} \cdot \beta(x, \delta_{n_i(k)+j-1})^{x_{n_i(k)+j-1}} \cdot \ldots \cdot \beta(x, \delta_{n_i(k)+1})^{x_{n_i(k)+1}}.$$

Transform this expression for $x_{n_i(k)+j} = 0$:

$$\begin{aligned}
\tilde{\omega}_j &= g(x \cdot \delta_{n_i(k)+1})^{x_{n_i(k)+1}} \cdot c(x, \delta_{n_i(k)+1})^{-x_{n_i(k)+1}} \cdot g(x)^{-x_{n_i(k)+1}} \\
&\quad \times \cdots \times g(x \cdot \delta_{n_i(k)+j-1})^{x_{n_i(k)+j-1}} \cdot c(x, \delta_{n_i(k)+j-1})^{-x_{n_i(k)+j-1}} \cdot g(x)^{-x_{n_i(k)+j-1}} \\
&\quad \times g(x) c(x, \delta_{n_i(k)+j}) \cdot g(x \cdot \delta_{n_i(k)+j})^{-1} \cdot g(x)^{x_{n_i(k)+j-1}} \\
&\quad \times c(x, \delta_{n_i(k)+j-1})^{x_{n_i(k)+j-1}} \cdot g(x \cdot \delta_{n_i(k)+j-1})^{-x_{n_i(k)+j-1}} \cdot \ldots \cdot g(x)^{x_{n_i(k)+1}} \\
&\quad \times c(x, \delta_{n_i(k)+1})^{x_{n_i(k)+1}} \cdot g(x \cdot \delta_{n_i(k)+1})^{-x_{n_i(k)+1}} \\
&\in g(x)^{x_{n_i(k)+1}} \cdot \bar{W} \cdot c(x, \delta_{n_i(k)+1})^{-x_{n_i(k)+1}} \cdot \bar{W} \cdot \ldots \cdot \bar{W} \\
&\quad \times c(x, \delta_{n_i(k)+j-1})^{-x_{n_i(k)+j-1}} \cdot \bar{W} \cdot c(x, \delta_{n_i(k)+j}) \cdot \bar{W} \cdot c(x, \delta_{n_i(k)+j-1})^{x_{n_i(k)+j-1}} \\
&\quad \times \bar{W} \cdot \ldots \cdot \bar{W} \cdot c(x, \delta_{n_i(k)+1})^{x_{n_i(k)+1}} \cdot \bar{W} \cdot g(x)^{-x_{n_i(k)+1}} \\
&\subset g(x)^{x_{n_i(k)+1}} \cdot c(x, \delta_{n_i(k)+1})^{-x_{n_i(k)+1}} \cdot \ldots \cdot c(x, \delta_{n_i(k)+j-1})^{-x_{n_i(k)+j-1}} \\
&\quad \times c(x, \delta_{n_i(k)+j}) \cdot c(x, \delta_{n_i(k)+j-1})^{x_{n_i(k)+j-1}} \cdot \ldots \cdot c(x, \delta_{n_i(k)+1})^{x_{n_i(k)+1}} \\
&\quad \times g(x)^{-x_{n_i(k)+1}} \cdot \underbrace{W \cdot W \cdot \ldots \cdot W \cdot W}_{2q(i)-1 \text{ times}} \\
&\subset g(x)^{x_{n_i(k)+1}} \cdot \varphi_{n_i(k)}(x)^{-1} \cdot f_{n_i(k)+j} \cdot \varphi_{n_i(k)}(x) \cdot g(x)^{-x_{n_i(k)+1}} \cdot V.
\end{aligned}$$

The same inclusion can be proved similarly for $x_{n_i(k)+j} = 1$. By corollary 4.3 we have that

$$\bar{\omega}_j = g(x)^{x_{n_i(k)+1}} \cdot \varphi_{n_i(k)}(x)^{-1} \cdot f_{n_i(k)+j} \cdot \varphi_{n_i(k)}(x) \cdot g(x)^{-x_{n_i(k)+1}}$$

for $j = 1, \ldots, q(i)$ form a V-net. Since in this case $\tilde{\omega}_j \in \bar{\omega}_j V$, evidently $\{\tilde{\omega}_j\}_{j=1}^{q(i)}$ form a U-net in K. □

5. In this section we combine the methods employed when proving theorems 3.1 and 4.1 to establish the following:

THEOREM 5.1. *Let G be an l.c.s. group, K a normal compact subgroup in G, whose quotient group G/K is solvable. Then G is the range of a cocycle of an ergodic automorphism.*

Proof. For simplicity's sake, we shall assume that G/K is an Abelian group. The general case may be considered in a similar manner.

Let $\{h'_j\}_{j=1}^{\infty}=H$ be a countable dense subgroup in G/K. Its elements may be arranged into a sequence $\{\bar{h}_j\}_{j=1}^{\infty}$ in which every h'_j occurs infinitely many times. Finally, we construct a sequence $\{h_j\}_{j=1}^{\infty}$ so that $\psi(h_j)=\bar{h}_j$, where $\psi: G\to G/K$ is the natural map.

Then, let $\{W_i\}_{i=1}^{\infty}$ be a fundamental system of neighbourhoods of the identity in G. For each $i\in\mathbb{N}$ we construct a finite W_i-net $\{k_s^i\}_{s=1}^{q(i)}\subset K$. Now we consider a surjective map $p:\mathbb{N}\to\mathbb{N}$ which takes every value infinitely many times, and construct the sequence of natural numbers $\{n(j)\}_{j=1}^{\infty}$: $n(1)=1$, $n(j+1)=n(j)+2^j\cdot q(p(j))+1$. Finally, we denote by $\{\zeta_r^j\}_{r=0}^{2^j-1}$ the set of all sequences of 0's and 1's of length j. Define f_k as follows:

$$\begin{aligned} f_{n(j)}&=h_j,\\ f_{n(j)+rq(p(j))+s}&=h_j^{\zeta_r^j(j)}\cdot h_{j-1}^{\zeta_r^j(j-1)}\cdot\ \dots\ \cdot h_2^{\zeta_r^j(2)}\cdot h_1^{\zeta_r^j(1)}(k_s^{p(j)}), \end{aligned}\tag{9}$$

$j\in\mathbb{N}$, $r=0,\dots,2^j-1$, $s=1,\dots,q(p(j))$. Thus a cocycle $c: X\times\Gamma\to G$ is defined.

LEMMA 5.2. *Let $A\subset X$, $\mu(A)>0$, $\bar{h}\in H$. Then there exists $D\subset A$, $\mu(D)>0$ and $\gamma\in\Gamma$ such that $D\cdot\gamma\subset A$ and $\psi(c(x,\gamma))=\bar{h}$ for a.e. $x\in D$.*

Proof. It follows from commutativity of the quotient group that

$$c(x,\delta_{n(j)})=\beta(x_1,\dots,x_{n(j)})h_j^{(-1)^{x_{n(j)}}},$$

where $\beta(x_1,\dots,x_{n(j)})\in K$ and therefore,

$$\psi(c(x,\delta_{n(j)}))=\bar{h}_j^{(-1)^{x_{n(j)}}}.\tag{10}$$

In other respects the proof is exactly the same as that of lemma 3.2. □

Define a cocycle $\alpha: X\times[\Gamma]\to G$: $\alpha(x,\gamma)=c(x,\delta(x,\gamma))$ where $x\cdot\gamma=x\cdot\delta(x,\gamma)$, $\gamma\in[\Gamma]$, $\delta(x,\gamma)\in\Gamma$.

LEMMA 5.3. *The subgroup $L=\{\gamma\in[\Gamma]$: $\alpha(x,\gamma)\in K$ for a.e. $x\in X\}$ acts ergodically on (X,μ).*

The proof is similar to that of lemma 3.5 for a cocycle $\psi(\alpha(x,\gamma))$ but using lemma 5.2 instead of lemma 3.4.

LEMMA 5.4. *K is the range of the cocycle $\alpha'=\alpha|_{X\times L}$ of an ergodic action of the group L on (X,μ).*

Proof. Note first that for $p\neq n(j), j\in\mathbb{N}$, $c(x,\delta_p)\in K$ for all $x\in X$, and since $\alpha(x,\delta_p)=c(x,\delta_p)$, all δ_p with $p\neq n(j)$, $j\in\mathbb{N}$, lie in L. Moreover, when $x_{n(1)}=\zeta_r^j(1),\dots,x_{n(j)}=\zeta_r^j(j)$, we have

$$\begin{aligned}\alpha(x,\delta_{n(j)+rq(p(j))+s})&=\varepsilon(x_1,\dots,x_{n(j)+rq(p(j))+s})^{-1}(k_s^{p(j)})^{(-1)^{x_{n(j)+rq(p(j))+s}}}\\ &\quad\cdot\varepsilon(x_1,\dots,x_{n(j)+rq(p(j))+s}),\end{aligned}\tag{11}$$

where $\varepsilon(x_1,\dots,x_{n(j)+rq(p(j))+s})\in K$, so that in this case lemma 4.2 can be applied. In other respects the proof is similar to that of theorem 4.1. □

Completion of the proof of theorem 5.1. It is to be proved that the action of Γ on $(X\times G,\mu\times\mu_G)$ defined by (1) is ergodic. Let $f\in L^{\infty}(X\times G,\mu\times\mu_G)$ and f be Γ-invariant. Then f is L-invariant. By lemma 5.4, the action of L on $(X\times gK,\mu\times\mu_K)$

is ergodic for each fixed $g\in G$. Therefore, f is constant on every such set and hence it can be considered as a function $\bar f$ of G/K. Since $\{\psi(\alpha(x,\gamma))\}_{x\in X}^{\gamma\in\Gamma}=H$, $\bar f$ must be H-invariant. Now it follows from density of H in G/K that $\bar f=\text{const}$ and hence $f=\text{const}$. This completes the proof. □

6. THEOREM 6.1. *Let G be an l.c.s. group, R a solvable normal subgroup in G with the compact quotient group G/R. Then G is the range of a cocycle of an ergodic automorphism.*

Proof. Assume again that R is an Abelian group. Let $\{W_i\}_{i=1}^{\infty}$ be a fundamental system of neighbourhoods of the identity in G and $\psi: G\to G/R$ the natural map. For each $i\in\mathbb{N}$ we construct a finite $\psi(W_i)$-net in G/R. Then construct a sequence $\{\varphi_j\}_{j=1}^{\infty}$, $\varphi_j\in G/R$ with the following property: for each $i\in\mathbb{N}$ there exists a sequence of natural numbers $\{p_i(k)\}_{k=1}^{\infty}$ such that $\{\varphi_{p_i(k)+m}\}_{m=1}^{q(i)}$ is coincident with the $\psi(W_i)$-net constructed above. Now define the sequence of natural numbers $\{n(j)\}_{j=1}^{\infty}$: $n(1)=1$, $n(j+1)=n(j)+2^j+1\}$. Finally, take $f_{n(j)}\in G$ so that $\psi(f_{n(j)})=\varphi_j$.

Let $H=\{r_j'\}_{j=1}^{\infty}$ be a countable dense subgroup in R. We construct a sequence $\{r_j\}_{j=1}^{\infty}$ so that each r_j' occurs in it infinitely many times. Denote by $\{\zeta_l^j\}_{l=1}^{2^j}$ the set of all sequences of 0's and 1's of length j. Define

$$f_{n(j)+l}=f_{n(j)}^{\zeta_l^j(j)}\cdot f_{n(j-1)}^{\zeta_l^j(j-1)}\cdot\ldots\cdot f_2^{\zeta_l^j(2)}\cdot f_1^{\zeta_l^j(1)}(r_j),\tag{12}$$

$j\in\mathbb{N}$, $l=1,\ldots,2^j$. Thus a cocycle $c: X\times\Gamma\to G$ is given.

Consider the space $(Y\times Z,\nu\times\eta)$, where $(Y,\nu)\cong(Z,\eta)\cong(X,\mu)$. Define the action of the group Γ on this space: $(y,z)\cdot\gamma=(\bar y,\bar z)$, where $\bar y_j=y_j+\gamma_{n(j)}$, $\bar z_j=z_j+\gamma_{\bar n(j)}$; $\{\bar n(j)\}_{j=1}^{\infty}$ is an increasing sequence of natural numbers which is complementary to $\{n(j)\}_{j=1}^{\infty}$. Define a map $\theta: X\to Y\times Z$ by $\theta(x)=(y,z)$, where $y_j=x_{n(j)}$, $z_j=x_{\bar n(j)}$. Evidently, θ is a Borel bijective map which preserves the measure and satisfies the requirement $\theta(x\cdot\gamma)=\theta(x)\cdot\gamma$, i.e. the actions of Γ on (X,μ) and $(Y\times Z,\nu\times\eta)$ are equivalent. The cocycle c constructed above may therefore be considered as defined on $(Y\times Z,\nu\times\eta)$.

Note also that partition of the space $Y\times Z$ into the sets $y\times Z$ is measurable, and the measure $\xi=\nu\times\eta$ may be disintegrated as follows:

$$\xi=\int_Y \eta_y\,d\nu(y),\tag{13}$$

where $\eta_y=p_{y*}(\eta)$; $p_y: Z\to y\times Z$ is the natural map.

Denote by Γ_Y and Γ_Z the subgroups of Γ generated by $\{\delta_{n(j)}\}_{j=1}^{\infty}$ and $\{\delta_{\bar n(j)}\}_{j=1}^{\infty}$, respectively.

LEMMA 6.2. *R is the range of the cocycle $c_y(z,\gamma)=c((y,z),\gamma)|_{y\times Z\times\Gamma_Z}$ of an ergodic Γ_Z-action on $(y\times Z,\eta_y)$ for any $y\in Y$.*

The proof is similar to that of theorem 3.1, with $K=\{e\}$.

LEMMA 6.3. *G/R is the range of the cocycle $\alpha(y,\gamma)=\psi(c((y,z),\gamma)|_{Y\times z\times\Gamma_Y})$, $z\in Z$, of an ergodic Γ_Y-action on (Y,ν).*

The proof essentially coincides with that of lemma 5.4.

Now we complete the proof of theorem 6.1. It is to be proved that the action of Γ on $(Y\times Z\times G, \nu\times\eta\times\mu_G)$ defined by

$$(y, z, g)\cdot\gamma=((y, z)\cdot\gamma, gc((y, z), \gamma)), \qquad \gamma\in\Gamma,$$

is ergodic. Let $f\in L^\infty(Y\times Z\times G, \nu\times\eta\times\mu_G)$ and f be Γ-invariant. Then for a.e. $(y, g)\in Y\times G$, the function $f_{y,g}=f|_{y\times Z\times gR}$ is measurable on $(Z\times R, \eta\times\mu_R)$ and invariant with respect to the action of Γ_Z: $(z, r)\cdot\gamma=(z\cdot\gamma, rc_y(z, \gamma))$. Therefore, by lemma 6.2, $f_{y,g}=\text{const}$. Hence, the function f is a measurable function $\bar f$ on $(Y\times G/R, \nu\times\mu_{G/R})$, invariant with respect to the action of Γ_Y: $(y, p)\cdot\gamma=(y\cdot\gamma, p\alpha(y, \gamma))$. By lemma 6.3, $\bar f=\text{const}$ and hence $f=\text{const}$, which was to be proved. □

It is well known that any almost connected amenable Lie group has the structure indicated in theorem 6.1. We obtain

COROLLARY 6.4. *Any almost connected amenable Lie group is the range of a cocycle of an ergodic $\mathbb{Z}$-action.*

7. It is known that any almost connected l.c.s. group G has the following structure: in an arbitrary neighbourhood of the identity there is a compact normal subgroup K such that G/K is isomorphic to a Lie group [**10**, p. 175].

THEOREM 7.1. *Any amenable almost connected l.c.s. group is the range of a cocycle of an ergodic $\mathbb{Z}$-action.*

Proof. The theorem can be proved in general by the methods developed above. We shall restrict ourselves to the construction of the cocycle and the main stages of the proof.

Let K be a compact normal subgroup in G such that G/K is a Lie group. By the assumptions concerning G, it must be amenable and almost connected, and hence it contains a solvable normal subgroup R (which will be regarded here as Abelian) so that $(G/K)/R$ is compact. Let $\varphi: G\to G/K$ and $\psi: G/K\to(G/K)/R$ be the natural maps, and $\{W_i\}_{i=1}^\infty$ a fundamental system of neighbourhoods of the identity in G.

(1) We construct $\{a_j\}_{j=1}^\infty\subset(G/K)/R$ for which the following is true: for any $i\in\mathbb{N}$, there exists a sequence of natural numbers $\{p_i(k)\}_{k=1}^\infty$ such that $\{a_{p_i(k)+j}\}_{j=1}^{q(i)}$ are $\psi\circ\varphi(W_i)$-nets in $(G/K)/R$, the same for all $k\in\mathbb{N}$.

(2) We construct the sequence $\{\bar r_j\}_{j=1}^\infty$ of elements from a countable subgroup H dense in R, in whch every element of H occurs infinitely many times.

(3) For every $i\in\mathbb{N}$ we construct $\{k_s^i\}_{s=1}^{d(i)}$ which form a W_i-net in K. Choose also a surjective map $u: \mathbb{N}\to\mathbb{N}$ which takes every value infinitely many times. Form two sequences of natural numbers $\{n(j)\}_{j=1}^\infty$ and $\{k(j)\}_{j=1}^\infty$ such that $n(1)=1$, $n(j+1)=n(j)+2^j\cdot d(u(j))+1$; $k(1)=1$, $k(j+1)=k(j)+2^j+1$. Define f_k so that

$$\psi\circ\varphi(f_{n(k(j))})=a_j, \qquad j\in\mathbb{N}; \tag{14}$$

$$f_{n(k(j)+l)}=f_{n(k(j))}^{\zeta_l^j(j)}\cdot f_{n(k(j-1))}^{\zeta_l^j(j-1)}\cdot\ \ldots\ \cdot f_{n(k(2))}^{\zeta_l^j(2)}\cdot f_{n(k(1))}^{\zeta_l^j(1)}(r_j), \tag{15}$$

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where $\varphi(r_j)=\bar{r}_j$, $l=1,\ldots,2^j$;

$$f_{n(k(j)+l)+(m-1)d(u(k(j)+l))+s}=f_{n(k(j)+l)}^{\zeta_s^{k(j)+l}(k(j)+l)}\cdot f_{n(k(j)+l-1)}^{\zeta_s^{k(j)+l}(k(j)+l-1)}\cdot\ldots\cdot f_{n(k(1))}^{\zeta_s^{k(j)+l}(k(1))}(k_s^{u(k(j)+l)}),\quad (16)$$

$m=1,\ldots,2^{k(j)+l}$, $s=1,\ldots,d(u(k(j)+l))$.

As in the preceding section, consider the space $(Y\times Z\times T,\ \nu\times\eta\times\rho)$ where $(Y,\nu)\cong(Z,\eta)\cong(T,\rho)\cong(X,\mu)$. Denote by $\{\bar{n}(j)\}_{j=1}^{\infty}$ and $\{\bar{k}(j)\}_{j=1}^{\infty}$ the increasing sequences of natural numbers which are complementary to $\{n(j)\}_{j=1}^{\infty}$ and $\{k(j)\}_{j=1}^{\infty}$, respectively, and then construct the action of Γ on $(Y\times Z\times T,\ \nu\times\eta\times\rho)$ by $(y,z,t)\cdot\gamma=(\bar{y},\bar{z},\bar{t})$, where $\bar{y}_j=y_j+\gamma_{n(k(j))}$, $\bar{z}_j=z_j+\gamma_{n(\bar{k}(j))}$, $\bar{t}_j=t_j+\gamma_{\bar{n}(j)}$. This action is equivalent to the action of Γ on (X,μ) because of the properties of the map $\theta: X\to Y\times Z\times T$: $\theta(x)=(y,z,t)$, where $y_j=x_{n(k(j))}$, $z_j=x_{n(\bar{k}(j))}$, $t_j=x_{\bar{n}(j)}$. Consider also the space $(y\times z\times T,\ \rho_{y,z})$, where $\rho_{y,z}=p_{y,z^*}(\rho)$, $p_{y,z}: T\to y\times z\times T$ is the natural map, and similarly, the space $(y\times Z,\ \eta_y)$, where $\eta_y=\pi_{y^*}(\eta)$, $\pi_y: Z\to y\times Z$ is the natural map. Finally, denote by Γ_Y, Γ_Z, Γ_T the subgroups of Γ generated by $\{\delta_{n(k(j))}\}_{j=1}^{\infty}$, $\{\delta_{n(\bar{k}(j))}\}_{j=1}^{\infty}$ and $\{\delta_{\bar{n}(j)}\}_{j=1}^{\infty}$, respectively.

LEMMA 7.2. *K is the range of the cocycle $c_{y,z}(t,\gamma)=c((y,z,t),\gamma)|_{y\times z\times T\times\Gamma_T}$ of an ergodic action of Γ_T on $(y\times z\times T,\ \rho_{y,z})$ for all $y\in Y$, $z\in Z$.*

LEMMA 7.3. *R is the range of the cocycle $\alpha_y(z,\gamma)=\varphi(c((y,z,t),\gamma)|_{y\times Z\times t\times\Gamma_Z})$, $t\in T$ of the ergodic action of Γ_Z on $(y\times Z,\ \eta_y)$ for all $y\in Y$.*

LEMMA 7.4. *$(G/K)/R$ is the range of the cocycle*

$$\beta(y,\gamma)=\psi\circ\varphi(c((y,z,t),\gamma)|_{Y\times z\times t\times\Gamma_Y}),\qquad z\in Z,\ t\in T$$

of the ergodic action of Γ_Y on (Y,ν).

The proof of theorem 7.1 is completed in the same manner as that of theorem 6.1.

COROLLARY 7.5. *Any connected amenable l.c.s. group is the range of a cocycle of an ergodic automorphism.*

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