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# REMAINDER TERM ESTIMATES IN A CONDITIONAL CENTRAL LIMIT THEOREM FOR INTEGER-VALUED RANDOM VARIABLES

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**Abstract**

A Berry-Esseen type result is given for the conditional distribution of a weighted sum of i.i.d. integer-valued r.v.'s given that their unweighted sum equals its expectation. The examples include the case of sampling without replacement from a finite population.

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## 1. Introduction and main results

Let $X_1', \ldots, X_n'$ be independent and identically distributed integer-valued random variables with maximal span 1. Suppose $EX_1' = \mu$, $E(X_1' - \mu)^2 = \sigma^2 > 0$, and set $X_k = (X_k' - \mu)/\sigma$, $k = 1, \ldots, n$, $\rho = E|X_1|^3$. Let $a_1, \ldots, a_n$ be real constants satisfying $\sum_k a_k = 0$, $\sum_k a_k^2 = n$. We seek bounds on the quantity

$$\Delta = \sup_x \left| P\left( n^{-1/2} \sum_k a_k X_k \leqslant x \Big| \sum_k X_k' = n\mu \right) - (2\pi)^{-1/2} \int_{-\infty}^x e^{-u^2/2}\, du \right|;$$

for $\Delta$ to be well defined, of course, $n\mu$ must be integral, so that the distribution of $X_1'$ itself may depend on $n$. In general our bound on $\Delta$ is a rather "unnatural" one, involving an estimate of the absolute value of the characteristic function $g(w) = Ee^{iwX_1}$, which is provided by the following basic lemma. However in the special cases considered later, the estimate is easy to calculate and manipulate.





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LEMMA 1. *For any* $b \in [0, \pi)$ *there exists a constant* $\eta_b \in (0, \frac{1}{2})$ *such that* $|g(w)| < e^{-\eta_b w^2}$, $0 < |w| < (\pi + b)\sigma$.

Set $d_n = (2\pi\sigma n^{1/2} P(\Sigma_k X'_k = n\mu))^{-1}$ and $T = n^{3/2}(\rho \Sigma_k |a_k|^3)^{-1}$.

THEOREM. *If* $\varepsilon$, $\delta$ *and* $\gamma$ *are positive constants with* $\varepsilon < 2^{-1/2}$, $\gamma < \frac{1}{2}(\sqrt{5} - 1)$, $b = 2\delta/\gamma < \pi$, *then*

$$\Delta \leqslant \frac{1}{(4\alpha - 2)T}\left\{Q_1 + Q_2 + Q_3 + Q_4 + n^{-1/2}Q_5\right\}, \tag{1}$$

*where, choosing H so that*

$$\alpha = \frac{1}{\pi}\int_{-H}^{H}\left(\frac{\sin y}{y}\right)^2 dy > \frac{1}{2}, \tag{2}$$

$$Q_1 = 1.596\alpha H/\delta,$$

$$Q_2 = \frac{2d_n}{\pi\eta_b^2}\left\{\frac{1}{4}\left(\frac{1}{2} + \frac{\varepsilon}{1 - 2\varepsilon^2}\right)\left(\frac{\pi}{\beta_1} + \frac{4}{\beta_2} + \frac{\pi}{\beta_3}\right) + \frac{1}{2\varepsilon}\left(\frac{1}{\beta_2} + \frac{c_b}{\beta_2^*} + \frac{\pi c_b}{2\beta_3^*}\right)\right\},$$

$$Q_3 = \frac{d_n c_b}{6\pi\eta_b^2}\left\{\frac{\pi}{2\beta_1^*} + \frac{3}{\beta_2^*} + \frac{3\pi}{2\beta_3^*} + \frac{\pi^{1/2}}{\varepsilon\beta_3^*(\alpha_2^*\eta_b)^{1/2}} + \frac{3\pi^{1/2}}{4\beta_1^*(\alpha_1\eta_b)^{1/2}}\right\},$$

$$Q_4 = \frac{0.133 d_n}{\varepsilon\eta_0^2(1 - 1/n)^2},$$

$$Q_5 = \frac{0.071 d_n c_b}{\alpha_1^{1/2}\eta_b^{5/2}}\left\{\frac{2}{\beta_2^*} + \frac{3}{\beta_3^*} + \frac{4}{\alpha_2^{*2}} + \frac{1.33}{\alpha_2^{*5/2}\eta_b^{1/2}\varepsilon}\right\} + \frac{1.061 d_n}{\eta_0^{5/2}(1 - 1/n)^{5/2}\varepsilon},$$

$$\alpha_1 = 1 - \gamma - \gamma^2, \quad \alpha_2 = 1 - \gamma^2 - \gamma^3, \quad \alpha_2^* = \alpha_2 - 2/n,$$

$$\beta_1 = \alpha_1^{3/2}\alpha_2^{1/2}, \quad \beta_2 = \alpha_1\alpha_2, \quad \beta_3 = \alpha_1^{1/2}\alpha_2^{3/2},$$

$$\beta_1^* = \alpha_1^{3/2}\alpha_2^{*1/2}, \quad \beta_2^* = \alpha_1\alpha_2^*, \quad \beta_3^* = \alpha_1^{1/2}\alpha_2^{*3/2},$$

*and*

$$c_b = \exp\left(2\eta_b\delta^2/\gamma^2\right).$$

The theorem is given in this cumbersome form to facilitate computation of the bound in particular cases (see Section 2). The following corollaries may suffice in some applications.

COROLLARY 1. *There exists a constant C such that* $\Delta \leqslant Cd_n/T$.

The factor $d_n$ may also be removed in many situations, where the distributions under consideration are "uniformly aperiodic", as the following corollary shows.

COROLLARY 2. *If* $\eta_0 \geqslant \eta_0^* > 0$, *then* $\Delta \leqslant C/T$.

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Finally we remark that if $X_1'$ has maximal span $\lambda > 1$, then the above results can be applied to the variables $\lambda^{-1}X_k'$. In the next section we look at some examples, and in Section 3 we provide proofs.

## 2. Two examples

First we show how Corollary 2 can be used in a simple example. Suppose

$$P(X_k' = +1) = P(X_k' = -1) = p < \tfrac{1}{2}, \qquad P(X_k' = 0) = 1 - 2p$$

for $k = 1, 2, \ldots$. Then $g(w)$ and $\eta_0 > 0$ do not depend on $n$, and with $a_k = (-1)^k$, Corollary 2 gives

$$\Delta = \sup_x \left| P\left( \sum_{k=1}^{2n} (-1)^k X_k' \leqslant 2(np)^{1/2} x \,\Big|\, \sum_{k=1}^{2n} X_k' = 0 \right) - (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-u^2/2}\, du \right|$$
$$= O(n^{-1/2}).$$

For a given value of $p$, the theorem itself could be used to find a numerical constant $C_p$ such that $\Delta < C_p n^{-1/2}$, by methods which we will now demonstrate in a different context.

Suppose $P(X_1' = 1) = p$, $P(X_1' = 0) = q = 1 - p$. The problem is then equivalent to that of approximating the distribution of a sample of size $np$ drawn without replacement from the finite population $(a_1, \ldots, a_n)$. Höglund (1976) has shown that the order term is $O(T^{-1})$; we will use our theorem to show $\Delta < 145/T$. The characteristic function satisfies $|g(w)|^2 = 1 + 2pq(\cos(w/\sigma) - 1)$, so

$$|g(w)| \leqslant e^{pq(\cos(w/\sigma) - 1)} \leqslant \exp(-\eta_b^* w^2), \quad |w| \leqslant (\pi + b)\sigma,$$

where $\eta_b^* = (1 - \cos(\pi + b))/(\pi + b)^2$. Bhattacharya and Ranga Rao (1976, Lemma 12.3) show that it is always the case that $\Delta \leqslant 0.5416$. So we may take $T > 200$, whereby the bound (21) below (which is used in proving Corollary 2) gives $d_n \leqslant 0.416$. A numerical integration of (2) was performed (this can also be evaluated from one of the many tables of $\mathrm{Si}(x)$, since $\int_0^x ((\sin y)/y)^2\, dy = \mathrm{Si}(2x) - (\sin^2 x)/x$. See for example Abramowitz and Stegun (1965, page 236) for references to tables), and the right hand side of (1) was minimized (with $Q_5 = 0$) over the range $0.86 < H < 3$; the resulting minimum of 144.4 was found at $H = 2.16$, $\alpha = 0.876$, $\varepsilon = 0.45$, $\delta = 0.04$, $\gamma = 0.22$. Here $Q_5 < 84$ and it follows that $\Delta < 145/T$.

Finally, we note that other applications may be found in Holst (1979), where a corresponding central limit theorem is proved.

## 3. Proofs

First we prove Lemma 1. Since $X_1$ has maximal span $\sigma^{-1}$, there exists for any $\varepsilon > 0$ a number $k_\varepsilon > 0$ such that $|g(w)| < e^{-k_\varepsilon}$, $\varepsilon < |w| < 2\pi\sigma - \varepsilon$ (Gnedenko (1963), page 297). Also, if $\varepsilon < \varepsilon_0 = \min(1.5/\rho, 2^{1/2})$, then for $|w| < \varepsilon$, $|g(w)| < 1 - \frac{1}{2}w^2 + \frac{1}{6}\rho|w|^3 < e^{-w^2/4}$. Assume without loss of generality that $e^{-k_\varepsilon} \geqslant e^{-\varepsilon^2/4}$. Then for $b = \pi - \sigma^{-1}\varepsilon > b_0 = \pi - \sigma^{-1}\varepsilon_0$, the lemma holds with $\eta_b = k_\varepsilon/(\pi + b)^2\sigma^2$. If $b_0 > 0$ then for $b < b_0$ the lemma holds with $\eta_b = \eta_{b_0}$.

We now turn to the proof of the theorem. Let

$$k(t) = \begin{cases} 1 - \dfrac{|t|}{\delta T}, & |t| \leqslant \delta T, \\ 0, & \text{otherwise.} \end{cases}$$

This is the characteristic function of the probability measure with density

$$f(x) = \frac{\delta T}{2\pi}\left(\frac{\sin\frac{1}{2}\delta T x}{\frac{1}{2}\delta T x}\right)^2, \qquad -\infty < x < \infty.$$

Taking $H$ and $\alpha$ as in (2), Lemma (12.2) of Bhattacharya and Ranga Rao gives

$$(3) \qquad \Delta \leqslant \frac{1}{2\alpha - 1}\left[\frac{1}{2\pi}\int_{|t| \leqslant \delta T} |t|^{-1}|\psi(t) - e^{-t^2/2}|\,dt + \frac{2\alpha H}{(2\pi)^{1/2}\delta T}\right]$$

where $\psi(t) = E\{\exp(itn^{-1/2}\Sigma_k a_k X_k | \Sigma_k X'_k = n\mu)\}$. It follows from Bartlett (1938) that

$$(4) \qquad \psi(t) = \left\{2\pi P\left(\sum_k X'_k = n\mu\right)\right\}^{-1}$$
$$\times \int_{-\pi}^{\pi} E\left\{\exp\left(ix\left(\sum_k X'_k - n\mu\right) + itn^{-1/2}\sum_k a_k X_k\right)\right\} dx$$
$$= d_n \int_{-\pi\sigma n^{1/2}}^{\pi\sigma n^{1/2}} \prod_k g_k(t, v)\,dv,$$

where $g_k(t, v) = g(n^{-1/2}(ta_k + v))$. For $t > 0$,

$$(5) \quad |\psi(t) - e^{-t^2/2}| = e^{-t^2/2}\left|\int_0^t \frac{d}{ds} e^{s^2/2}\psi(s)\,ds\right| \leqslant |t| e^{-t^2/2} \sup_{|s| \leqslant |t|}\left|\frac{d}{ds} e^{s^2/2}\psi(s)\right|,$$

and similarly this inequality also obtains for $t < 0$. We will use (4) and (5) to bound the integral in (3).

The next lemma is due to Höglund (1976).

LEMMA 2. *Let* $K_\gamma = \{k\colon 1 \leqslant k \leqslant n,\ \gamma|a_k| \leqslant \Sigma_j |a_j|^3/n\}$, $0 < \gamma < 1$. *Then* $\Sigma_{k \in K_\gamma}(a_k x + y)^2/n \geqslant \alpha_1 x^2 + \alpha_2 y^2$, *where* $\alpha_1$ *and* $\alpha_2$ *are defined in the theorem and are positive if* $2\gamma < \sqrt{5} - 1$.

PROOF. Hölder's inequality gives $\sum_{k \in K_\gamma} 1 \geqslant (1-\gamma^3)n$, $\sum_{k \in K_\gamma} a_k^2 \geqslant (1-\gamma)n$, and $\sum_{k \notin K_\gamma} |a_k| \leqslant \gamma^2 n$. So

$$\begin{aligned}\sum_{k \in K_\gamma} (a_k x + y)^2/n &\geqslant (1-\gamma)x^2 + (1-\gamma^3)y^2 - 2\sum_{k \notin K_\gamma} a_k xy/n \\ &\geqslant (1-\gamma)x^2 + (1-\gamma^3)y^2 - 2|xy|\gamma^2 \\ &\geqslant (1-\gamma-\gamma^2)x^2 + (1-\gamma^2-\gamma^3)y^2.\end{aligned}$$

LEMMA 3. *Let $(d_j, -\infty < j < \infty)$ be constants, $d_j < d_{j+1}$ for all $j$; let $Z$ be a r.v. with $P(Z = d_j) = p_j$ for all $j$, $\sum_j p_j = 1$, $EZ = \mu$, $\min_j(d_j - d_{j-1}) = \varepsilon > 0$. Then*

$$\frac{E|Z-\mu|^2}{E|Z-\mu|^3} \leqslant \frac{2}{\varepsilon}.$$

PROOF. Let $n$ be the integer such that $d_n \leqslant \mu < d_{n+1}$; let $\alpha = \mu - d_n$, $\beta = d_{n+1} - \mu$, $\delta = \alpha + \beta$, $\alpha' = \alpha/\delta$, $\beta' = \beta/\delta$. Then $E(Z-\mu) = 0$ implies

$$\alpha p_n - \beta p_{n+1} = \sum_{j<n} (d_j - \mu)p_j + \sum_{j>n+1} (d_j - \mu)p_j.$$

Since $\alpha' + \beta' = 1$,

$$\alpha'^3 p_n + \beta'^3 p_{n+1} = (\alpha'^2 + \beta'^2)(\alpha'^2 p_n + \beta'^2 p_{n+1}) + \alpha'\beta'(\alpha' - \beta')(\alpha' p_n - \beta' p_{n+1})$$

so

$$\begin{aligned}E|Z-\mu|^3 &= \alpha^3 p_n + \beta^3 p_{n+1} + \sum_{j \neq n, n+1} |d_j - \mu|^3 p_j \\ &\geqslant \delta^3\left\{(\alpha'^2 + \beta'^2)(\alpha'^2 p_n + \beta'^2 p_{n+1}) + \alpha'\beta'(\alpha' - \beta')(\alpha' p_n - \beta' p_{n+1})\right\} \\ &\quad + \sum_{j \neq n, n+1} |d_j - \mu|^3 p_j \\ &\geqslant (\alpha'^2 + \beta'^2)\delta\left\{\alpha^2 p_n + \beta^2 p_{n+1}\right\} - \delta^2\alpha'\beta'(\alpha' - \beta')\sum_{j<n}\left((d_j - \mu)^2/\alpha\right)p_j \\ &\quad + \sum_{j \neq n, n+1} |d_j - \mu|^3 p_j \quad (\text{if } \alpha > \beta) \\ &\geqslant \tfrac{1}{2}\delta\left\{\alpha^2 p_n + \beta^2 p_{n+1}\right\} + \left\{\varepsilon + \alpha - \delta\beta'(\alpha' - \beta')\right\}\sum_{j<n}(d_j - \mu)^2 p_j \\ &\quad + (\varepsilon + \beta)\sum_{j>n+1}(d_j - \mu)^2 p_j \\ &\geqslant \tfrac{1}{2}\delta(\alpha^2 p_n + \beta^2 p_{n+1}) + (\varepsilon + \tfrac{1}{2}\delta)\sum_{j<n}(d_j - \mu)^2 p_j + \varepsilon\sum_{j>n+1}(d_j - \mu)^2 p_j \\ &\geqslant \tfrac{1}{2}\varepsilon\sigma^2.\end{aligned}$$

A similar argument applies if $\alpha \leqslant \beta$.

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REMARK. This inequality is sharp when $\varepsilon = 1$ for a r.v. $Z$ with $P(Z=0) = P(Z=1) = \frac{1}{2}$.

LEMMA 4. *For* $|s| \leqslant \delta T$, $|v| \leqslant \pi\sigma n^{1/2}$, $|\prod_k g_k(s,v)| \leqslant \exp(-\eta_b(\alpha_1 s^2 + \alpha_2 v^2))$ *so long as* $b = 2\delta/\gamma < \pi$.

PROOF. If $k \in K_\gamma$, then

$$n^{-1/2}|sa_k| \leqslant \delta/\rho\gamma \leqslant \delta/\gamma, \tag{6}$$

so $n^{-1/2}|sa_k + v| \leqslant (2\delta/\gamma + \pi)\sigma$ from Lemma 3. So Lemma 1 gives

$$\left|\prod_k g_k(s,v)\right| \leqslant \left|\prod_{k\in K_\gamma} g_k(s,v)\right| \leqslant \exp\left(-\eta_b \sum_{k\in K_\gamma}(sa_k+v)^2/n\right)$$

and the result follows from Lemma 2.

We can now consider the integral in (3). Let

$$I_1 = \int_{-\varepsilon U}^{\varepsilon U} |t|^{-1}|\psi(t) - e^{-t^2/2}|\,dt \leqslant d_n \int_{-\varepsilon U}^{\varepsilon U} e^{-t^2/2}(J_{11}(t) + J_{12}(t))\,dt \tag{7}$$

using (4) and (5), where $0 < \varepsilon < 2^{-1/2}$, $U = n^{1/2}/\max_k|a_k|$, and

$$J_{11}(t) = \int_{-\varepsilon n^{1/2}}^{\varepsilon n^{1/2}} \sup_{|s|\leqslant|t|}\left|\frac{d}{ds}e^{s^2/2}\prod_k g_k(s,v)\right| dv,$$

$$J_{12}(t) = \int_{\varepsilon n^{1/2} < |v| < \pi\sigma n^{1/2}} \sup_{|s|\leqslant|t|}\left|\frac{d}{ds}e^{s^2/2}\prod_k g_k(s,v)\right| dv.$$

Here and in the sequel we assume that $b = 2\delta/\gamma < \pi$ as in Lemma 4.

LEMMA 5. *For* $|s| \leqslant \min(\varepsilon U, \delta T)$, $|v| \leqslant \varepsilon n^{1/2}$,

$$\left|\frac{d}{ds}e^{s^2/2}\prod_k g_k(s,v)\right| \leqslant \left(\tfrac{1}{2}\rho + \frac{\varepsilon}{1-2\varepsilon^2}\right)n^{-3/2}\sum_k |a_k|(sa_k+v)^2 \exp\left(\tfrac{1}{2}s^2 - \eta_b\left(\alpha_1 s^2 + \alpha_2 v^2\right)\right).$$

PROOF. First, note that

$$|g_k(s,v) - 1| = |E\{\exp(iX_k(sa_k+v)n^{-1/2})\} - 1| \leqslant \tfrac{1}{2}(sa_k+v)^2/n \tag{8}$$

$$\leqslant \left((sa_k)^2 + v^2\right)/n \leqslant 2\varepsilon^2. \tag{9}$$

Let $h(s,v) = \sum_k \log g_k(s,v) + \frac{1}{2}s^2 + \frac{1}{2}v^2$. Then

(10)

$$\left|\frac{d}{ds}h(s,v)\right| \leqslant \left|\sum_k \frac{d}{ds}g_k(s,v) + s\right| + \left|\sum_k g_k^{-1}(s,v)(1 - g_k(s,v))\frac{d}{ds}g_k(s,v)\right|$$

$$= A_1 + A_2$$

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say, and

$$(11)\quad A_1 = \left|\sum_k E\left\{iX_k a_k n^{-1/2}\left(e^{iX_k(sa_k+v)}n^{-1/2} - 1 - iX_k(sa_k+v)n^{-1/2}\right)\right\}\right|$$
$$\leqslant \tfrac{1}{2}\sum_k (sa_k+v)^2|a_k|\rho n^{-3/2}.$$

From (8) and (9),

(12)
$$A_2 \leqslant (1-2\varepsilon^2)^{-1}\sum_k |1-g_k(s,v)|\cdot\left|\frac{d}{ds}g_k(s,v)\right|$$
$$\leqslant \varepsilon(1-2\varepsilon^2)^{-1}n^{-1/2}\sum_k |sa_k+v|\cdot\left|E\left\{ia_kX_kn^{-1/2}\left(e^{iX_k(sa_k+v)n^{-1/2}}-1\right)\right\}\right|$$
$$\leqslant \varepsilon(1-2\varepsilon^2)^{-1}n^{-3/2}\sum_k (sa_k+v)^2|a_k|.$$

Since

$$\left|\frac{d}{ds}e^{s^2/2}\prod_k g_k(s,v)\right| = \left|\frac{d}{ds}e^{h(s,v)-v^2/2}\right| \leqslant \left|\frac{d}{ds}h(s,v)\right|\cdot\left|e^{h(s,v)-v^2/2}\right|$$
$$= e^{s^2/2}\left|\frac{d}{ds}h(s,v)\right|\cdot\left|\prod_k g_k(s,v)\right|,$$

the lemma follows from (10)–(12) and Lemma 4.

LEMMA 6. *For* $|s| \leqslant \min(\varepsilon U, \delta T)$, $\varepsilon n^{1/2} < |v| \leqslant \pi\sigma n^{1/2}$,

$$\left|\frac{d}{ds}e^{s^2/2}\prod_k g_k(s,v)\right| \leqslant |s|\exp\left(\tfrac{1}{2}s^2 - \eta_b\left(\alpha_1 s^2 + \alpha_2 v^2\right)\right)$$
$$+ (|s|+|v|)c_b\exp\left(\tfrac{1}{2}s^2 - \eta_b\left(\alpha_1 s^2 + (\alpha_2 - 2/n)v^2\right)\right).$$

PROOF. We have

$$\left|\frac{d}{ds}e^{s^2/2}\prod_k g_k(s,v)\right| \leqslant |s|e^{s^2/2}\left|\prod_k g_k(s,v)\right|$$
$$+ e^{s^2/2}\left|\sum_j \frac{d}{ds}g_j(s,v)\prod_{k\neq j} g_k(s,v)\right| = B_1 + B_2,$$

say. Lemma 4 gives $B_1 \leqslant |s| \exp(\frac{1}{2}s^2 - \eta_b(\alpha_1 s^2 + \alpha_2 v^2))$, and as in the derivation of (12) we get

$$B_2 \leqslant n^{-1} e^{s^2/2} \sum_j |a_j| \cdot |sa_j + v| \cdot \left| \prod_{k \neq j,\, k \in K_\gamma} g_k(s, v) \right|$$

$$\leqslant n^{-1} e^{s^2/2} \sum_j |a_j| \cdot |sa_j + v| \exp\left\{ -\eta_b\left(\alpha_1 s^2 + \alpha_2 v^2\right) + \eta_b \max_{k \in K_\gamma} \left(sa_k + v\right)^2/n \right\};$$

the lemma follows from (6), since $\rho \geqslant 1$.

The assumption $\gamma < \frac{1}{2}(\sqrt{5} - 1)$ ensures $\alpha_1, \alpha_2 > 0$. So using Lemmas 5 and 6 on (7) gives

(13)

$$I_1 \leqslant n^{-3/2} d_n \left( \tfrac{1}{2}\rho + \frac{\varepsilon}{1 - 2\varepsilon^2} \right) \int_{-\varepsilon U}^{\varepsilon U} \int_{-\varepsilon n^{1/2}}^{\varepsilon n^{1/2}} \sum_k |a_k| (ta_k + v)^2 \times \exp\left(-\eta_b\left(\alpha_1 t^2 + \alpha_2 v^2\right)\right) dv\, dt$$

$$+ d_n \int_{-\varepsilon U}^{\varepsilon U} \int_{\varepsilon n^{1/2} < |v| < \pi \sigma n^{1/2}} t \exp\left(-\eta_b\left(\alpha_1 t^2 + \alpha_2 v^2\right)\right) dv\, dt$$

$$+ d_n \int_{-\varepsilon U}^{\varepsilon U} \int_{\varepsilon n^{1/2} < |v| < \pi \sigma n^{1/2}} (|t| + |v|) c_b \exp\left(-\eta_b\left(\alpha_1 t^2 + (\alpha_2 - 2/n) v^2\right)\right) dv\, dt$$

$$\leqslant \tfrac{1}{2} d_n n^{-3/2} \left( \tfrac{1}{2}\rho + \frac{\varepsilon}{1 - 2\varepsilon^2} \right) \eta_b^{-2} \sum_k \left( \frac{\pi |a_k|^3}{\beta_1} + \frac{4a_k^2}{\beta_2} + \frac{\pi |a_k|}{\beta_3} \right)$$

$$+ \frac{d_n n^{-1/2}}{\varepsilon \eta_b^2} \left( \frac{1}{\beta_2} + \frac{c_b}{\alpha_1 \alpha_2^*} + \frac{\pi c_b}{2\alpha_1^{1/2} \alpha_2^{*3/2}} \right).$$

If $\varepsilon U < \delta T$ we have also to consider the integral

(14)
$$I_2 = \int_{\varepsilon U < |t| < \delta T} |t|^{-1} |\psi(t) - e^{-t^2/2}|\, dt$$

$$= \int_{\varepsilon U < |t| < \delta T} |t|^{-1} |\psi(t) - \psi(0) e^{-t^2/2}|\, dt$$

$$= d_n \int_{\varepsilon U < |t| < \delta T} |t|^{-1} \left| \int_{-\pi \sigma n}^{\pi \sigma n^{1/2}} \left( \prod_k g_k(t, v) - e^{-t^2/2} \prod_k g_k(0, v) \right) dv \right| dt$$

$$\leqslant d_n \int_{\varepsilon U < |t| < \delta T} |t|^{-1} \left( J_{21}(t) + J_{22}(t) \right) dt,$$

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where

$$J_{21}(t) = \int_{-\pi\sigma n^{1/2}}^{\pi\sigma n^{1/2}} \left| \prod_k g_k(t, v) - e^{-(t^2+v^2)/2} \right| dv$$

and

$$J_{22}(t) = \int_{-\pi\sigma n^{1/2}}^{\pi\sigma n^{1/2}} e^{-t^2/2} \left| g^n(vn^{-1/2}) - e^{-v^2/2} \right| dv.$$

The integrand in $J_{21}(t)$ is

$$(15)\quad \left| \prod_k g_k(t, v) - \prod_k e^{-(ta_k+v)^2/(2n)} \right|$$
$$\leqslant \sum_{j=1}^{n} \left| \prod_{k=1}^{j-1} g_k(t, v) \left\{ g_j(t, v) - e^{-(ta_j+v)^2/(2n)} \right\} \prod_{k=j+1}^{n} e^{-(ta_k+v)^2/(2n)} \right|.$$

We have

$$(16)\quad \left| g_j(t, v) - e^{-(ta_j+v)^2/(2n)} \right| \leqslant \left| g_j(t, v) - 1 + \tfrac{1}{2}(ta_j + v)^2/n \right|$$
$$+ \left| e^{-(ta_j+v)^2/(2n)} - 1 + \tfrac{1}{2}(ta_j + v)^2/n \right|$$
$$\leqslant \tfrac{1}{6}\rho |ta_j + v|^3 n^{-3/2} + \tfrac{1}{8}(ta_j + v)^4 n^{-2}.$$

For $k \in K_j$, $n^{-1/2}|ta_k + v| \leqslant (2\delta/\gamma + \pi)\sigma$ as in the proof of Lemma 4, so with $b$ as before,

$$(17)\quad \left| \prod_{k=1}^{j-1} g_k(t, v) \prod_{k=j+1}^{n} e^{-(ta_k+v)^2/(2n)} \right|$$
$$\leqslant \exp\left\{ -\eta_b \sum_{k<j, k\in K_\gamma} (ta_k + v)^2/n - \tfrac{1}{2} \sum_{k>j} (ta_k + v)^2/n \right\}$$
$$\leqslant \exp\left\{ -\eta_b \sum_{k\in K_\gamma} (ta_k + v)^2/n + \eta_b \max_{j\in K_\gamma} (ta_j + v)^2/n \right\}$$
$$\leqslant c_b \exp\left(-\eta_b\left(\alpha_1 t^2 + \alpha_2^* v^2\right)\right)$$

from (6) and Lemma 2. From (15)–(17),

(18)

$$J_{21}(t) \leqslant \int_{-\pi\sigma n^{1/2}}^{\pi\sigma n^{1/2}} \sum_j \left( \tfrac{1}{6}\rho |ta_j + v|^3 n^{-3/2} + \tfrac{1}{8}(ta_j + v)^4 n^{-2} \right) c_b e^{-\eta_b(\alpha_1 t^2 + \alpha_2^* v^2)}\, dv.$$

Now consider $J_{22}(t)$. Since $|vn^{-1/2}| < \pi\sigma$, (1) gives $|g(vn^{-1/2})| < e^{-\eta_0 v^2/n}$, and setting $t = 0$ in (16) gives

$$\left|g(vn^{-1/2}) - e^{-v^2/(2n)}\right| \leqslant \tfrac{1}{6}\rho|v|^3 n^{-3/2} + \tfrac{1}{8}v^4 n^{-2}.$$

So since $|\alpha^n - \beta^n| \leqslant n|\alpha - \beta|(\max(|\alpha|, |\beta|))^{n-1}$,

(19)
$$J_{22}(t) \leqslant ne^{-t^2/2}\int_{-\pi\sigma n^{1/2}}^{\pi\sigma n^{1/2}} \left(\tfrac{1}{6}\rho|v|^3 n^{-3/2} + \tfrac{1}{8}v^4 n^{-2}\right) e^{-\eta_0(1-1/n)v^2}\,dv$$
$$\leqslant ne^{-t^2/2}\left(\frac{\rho n^{-3/2}}{6\eta_0^2(1-1/n)^2} + \frac{3\pi^{1/2}n^{-2}}{4\eta_0^{5/2}(1-1/n)^{5/2}}\right).$$

Combining (14) with (18) and (19) gives

(20)
$$I_2 \leqslant \tfrac{1}{6}d_n\rho n^{-3/2}c_b\eta_b^{-2}\sum_j\left(\frac{|a_j|^3\pi}{2\beta_1^*} + \frac{3a_j^2}{\beta_2^*} + \frac{3\pi|a_j|}{2\beta_3^*} + \frac{\pi^{1/2}}{\varepsilon U\alpha_1^{1/2}\alpha_2^{*2}\eta_b^{1/2}}\right)$$
$$+ \tfrac{1}{8}d_n n^{-2}c_b\eta_b^{-5/2}\frac{\pi^{1/2}}{\alpha_1^{1/2}}\sum_j\left(\frac{a_j^4}{\beta_1^*} + \frac{2|a_j|^3}{\beta_2^*} + \frac{3a_j^2}{\beta_3^*} + \frac{4|a_j|}{\alpha_2^{*2}} + \frac{3\pi^{1/2}}{4\varepsilon U\alpha_2^{*5/2}\eta_b^{1/2}}\right)$$
$$+ \frac{n^{-1/2}\rho d_n(2\pi)^{1/2}}{6\eta_0^2(1-1/n)^2\varepsilon U} + \frac{n^{-1}d_n 3\pi 2^{1/2}}{4\eta_0^{5/2}(1-1/n)^{5/2}\varepsilon U}.$$

Using (3), (13) and (20) and the inequalities $\rho \geqslant 1$, $\Sigma|a_k| \leqslant \Sigma a_k^2 \leqslant \Sigma|a_k|^3$, $T \leqslant n^{1/2}$, $U \geqslant 1$, $\max_k|a_k| \leqslant n^{1/2}$, we obtain the bound given in the theorem.

Proof of corollaries. Corollary 1 is obtained simply by choosing specific values for $\alpha$, $\varepsilon$, $\delta$ and $\gamma$; for example $\alpha = \frac{3}{4}$, $\varepsilon = \gamma = \delta = \frac{1}{2}$. Corollary 2 is proved as follows. Setting $t = 0$ in (4) gives

$$d_n^{-1} = \int_{-\pi\sigma n^{1/2}}^{\pi\sigma n^{1/2}} g^n(vn^{-1/2})\,dv,$$

so

$$|d_n^{-1} - (2\pi)^{1/2}| \leqslant \int_{-\pi\sigma n^{1/2}}^{\pi\sigma n^{1/2}} |g^n(vn^{-1/2}) - e^{-v^2/2}|\,dv + \int_{|v|>\pi\sigma n^{1/2}} e^{-v^2/2}\,dv$$
$$= K_1 + K_2,$$

say. $K_1$ is just $J_{22}(0)$, which from (19) is bounded for $n \geqslant 2$ by $2/(3\eta_0^2 T) + 7.52/(\eta_0^{5/2}n)$, whilst $K_2 \leqslant 2/(\pi\sigma n^{1/2}) < 1.3/T$ from Lemma 3. So if $\eta_0 \geqslant \eta_0^* > 0$, then for $T > T_0$,

(21)
$$d_n \leqslant \left[(2\pi)^{1/2} - K_1 - K_2\right]^{-1} \leqslant \left[(2\pi)^{1/2} - \frac{2}{3\eta_0^{*2}T_0} - \frac{7.52}{\eta_0^{*5/2}T_0^2} - \frac{1.3}{T_0}\right]^{-1},$$

and Corollary 2 follows on choosing $\alpha$, $\varepsilon$, $\delta$ and $\gamma$ as before.

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