ON ORDER PARACOMPACT SPACES

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In (**2**), Fitzpatrick and Ford defined a class of spaces which they called *order totally paracompact.* This class of spaces is significant since the order totally paracompact *metric* spaces constitute the largest known class of spaces for which the small and large inductive dimensions coincide. In this paper we shall consider a class of spaces containing the class of order totally paracompact spaces; we shall call these spaces *order paracompact.* We shall establish the relationship of this class to the more familiar classes of spaces and obtain an invariance theorem for order paracompact spaces analogous to theorems obtained for various other classes.

In what follows, no separation axioms are assumed unless specifically mentioned. Thus, for example, a Lindelöf space is not taken to be Hausdorff without making the further assumption. Therefore, a Lindelöf space is not necessarily paracompact. The word *refinement* always means "open refinement" and is always a covering of the space involved.

Definition 1. A topological space X is called *order paracompact* if and only if every open covering $\mathscr{U}$ of X has a (linearly) ordered refinement $(\mathscr{H}, <)$ such that if $H \in \mathscr{H}$, then the collection of all members of $\mathscr{H}$ preceding H is locally finite at each point of $\bar{H}$.

An equivalent definition (say, Definition 1′) is obtained if the phrase "every open covering $\mathscr{U}$" in Definition 1 is replaced by "every basis $\mathscr{B}$". It follows immediately from Definition 1′ that every order totally paracompact space (**2**, p. 33) is order paracompact. It is also easy to see that every paracompact space is order paracompact: If X is paracompact and $\mathscr{U}$ is any open covering of X, then $\mathscr{U}$ has a locally finite refinement $\mathscr{H}$. If we well-order $\mathscr{H}$ by a relation $<$, then the ordered refinement $(\mathscr{H}, <)$ satisfies the above definition, showing that X is order paracompact. More generally, we have the following theorem.

THEOREM 1. *Let X be a space with the following property: Every open covering of X has a refinement which can be decomposed into an at most countable collection of locally finite families. Then X is order paracompact.*

Proof. First note that since we are *not* assuming X to be regular, this condition is not equivalent to paracompactness, as is the case with Michael's theorem (**1**, p. 163).

Now, let Z^+ denote the sequence of positive integers. Let $\mathscr{U}$ be an open covering of X and let $\mathscr{V} = \bigcup\{\mathscr{V}_n\colon n \in Z^+\}$ be a refinement of $\mathscr{U}$, where

Received November 28, 1967 and in revised form, May 29, 1968. This research was supported by NASA Research Grant NSG–568 at Kent State University.

https://doi.org/10.4153/CJM-1969-042-4 Published online by Cambridge University Press

each $\mathscr{V}_n$ is a locally finite family. We can assume that the families $\mathscr{V}_n$ and $\mathscr{V}_m$ have no members in common if $n \neq m$.

Next, well-order the members of each $\mathscr{V}_n$, and order $\mathscr{V}$ by the rule: if $U, V \in \mathscr{V}$, then $U < V$ if and only if there exists an $n \in Z^+$ such that $U, V \in \mathscr{V}_n$ and $U < V$ in the ordering of $\mathscr{V}_n$, otherwise $U \in \mathscr{V}_n$ and $V \in \mathscr{V}_m$, where $n < m$. Then the ordered collection $(\mathscr{V}, <)$ satisfies the condition of Definition 1: For, suppose that $U \in \mathscr{V}$. Then there exists an $n \in Z^+$ such that $U \in \mathscr{V}_n$. Let $x \in \bar{U}$. Then for each i $(i = 1, \ldots, n)$ there exists a neighbourhood O_i of x which intersects at most a finite number of the elements of $\mathscr{V}_i$. Hence, $O = \bigcap\{O_i\colon i = 1, \ldots, n\}$ is a neighbourhood of x which intersects at most a finite number of elements of $\bigcup\{\mathscr{V}_i\colon i = 1, \ldots, n\}$. Therefore, the refinement $\mathscr{V}$ has the required properties, and the proof is complete.

As a consequence we have the following result.

COROLLARY 1. *Every Lindelöf space is order paracompact.*

Proof. Every open covering of a Lindelöf space has an at most countable refinement.

A proof almost exactly like the proof of Theorem 1 yields the following theorem.

THEOREM 2. *Let X be a space such that every basis can be decomposed into an at most countable collection of locally finite families. Then X is order totally paracompact. If X is also T_1 and regular, then X is a metrizable order totally paracompact space.*

Proof. Let $\mathscr{B}$ be any basis for X. By hypothesis, $\mathscr{B}$ can be decomposed into an at most countable collection of locally finite families. Then, as in the proof of Theorem 1, $\mathscr{B}$ can be linearly ordered so as to satisfy Definition 1′. The only additional requirement for order total paracompactness is satisfied is this case (**2**, p. 33). Hence, X is order totally paracompact. The last assertion follows from the Nagata-Smirnov Metrization Theorem (**1**, p. 194).

We next give an example which shows that an order paracompact space need not be paracompact.

Example 1. Gustin (**3**, p. 102) has given an example of a countable connected Hausdorff space X. This space is not regular, but being second countable, it is Lindelöf. Therefore, by Corollary 1, X is order paracompact. However, X is not paracompact, since a paracompact Hausdorff space is normal.

As Example 1 shows, an order paracompact Hausdorff space need not be normal. We shall presently prove that an order paracompact regular space is, in fact, collectionwise normal. For this purpose, we need some preliminary results.

Let Y be a subspace of a space X. Then, if $A \subset Y$, there is a well-known relationship between $\bar{A}$, the closure of A in X, and $\mathrm{cl}_Y(A)$, the closure of A in Y, namely $\mathrm{cl}_Y(A) = \bar{A} \cap Y$. But if A is not contained in Y, this relationship is not particularly useful. However, we can show that if Y is an *open* subspace of X, then a similar relationship holds between $\bar{A}$ and $\mathrm{cl}_Y(A \cap Y)$.

LEMMA 1. *Let X be a space, Y an open subspace of X, and A any subset of X. Then*

$$\mathrm{cl}_Y(A \cap Y) = \bar{A} \cap Y.$$

Proof. If $A \cap Y = \emptyset$, then $\bar{A} \cap Y = \emptyset$, and therefore

$$\mathrm{cl}_Y(A \cap Y) = \bar{A} \cap Y.$$

Hence, suppose that $A \cap Y \neq \emptyset$. By the above formula for subsets of Y we have that

$$\mathrm{cl}_Y(A \cap Y) = \overline{A \cap Y} \cap Y \subset (\bar{A} \cap \bar{Y}) \cap Y = \bar{A} \cap Y;$$

thus, containment holds in one direction. To prove containment in the other direction, let $x \in \bar{A} \cap Y$. If $x \in A$, then $x \in A \cap Y$, and therefore $x \in \overline{A \cap Y} \cap Y = \mathrm{cl}_Y(A \cap Y)$. Therefore, we can assume that $x \in A' - A$ (where A' is the set of limit points of A). Let U be any neighbourhood of x. Then $U \cap Y$ is a neighbourhood of x, and hence

$$(U \cap Y) \cap (A - \{x\}) = U \cap (Y \cap A - \{x\}) \neq \emptyset.$$

It therefore follows that

$$x \in (Y \cap A)' \cap Y \subset (\overline{Y \cap A}) \cap Y = \mathrm{cl}_Y(A \cap Y)$$

and the equality $\mathrm{cl}_Y(A \cap Y) = \bar{A} \cap Y$ is proved.

We are now in a position to prove the following theorem.

THEOREM 3. *Every regular order paracompact space is collectionwise normal.*

Proof. Let $\mathscr{F} = \{F_\lambda\}$ be a discrete collection of closed subsets of a regular order paracompact space X. Let $\mathscr{B} = \{B_\alpha\}$ be a basis for X. Then the subcollection

$$\mathscr{B}' = \{B_\alpha\colon \bar{B}_\alpha \cap F_\lambda \neq \emptyset \text{ for at most one } F_\lambda \in \mathscr{F}\}$$

is a basis for X: If U is any open set of X and $x \in U$, then there exists a neighbourhood G of x which intersects at most one of the F_λ's. By regularity, there exists a $B_\alpha \in \mathscr{B}$ such that $x \in B_\alpha \subset \bar{B}_\alpha \subset U \cap G$. Hence, $\bar{B}_\alpha$ can intersect at most one of the F_λ's; therefore $B_\alpha \in \mathscr{B}'$. Thus, $\mathscr{B}'$ is a basis, as asserted.

Since X is order paracompact, there exists an ordered refinement $(\mathscr{H} = \{H_\mu\}, <)$ of $\mathscr{B}'$ such that if $H_\mu \in \mathscr{H}$, then the collection of all $H_{\mu'}$ preceding H_μ is locally finite at each point of $\bar{H}_\mu$. For each λ define

$$\mathscr{H}_\lambda = \{H_\mu\colon H_\mu \cap F_\lambda \neq \emptyset\}.$$

Notice that if $H_\mu \in \mathscr{H}_\lambda$, then $\bar{H}_\mu \cap F_\gamma = \emptyset$ for all $\gamma \neq \lambda$: if $\bar{H}_\mu \cap F_\gamma \neq \emptyset$ for some γ, then $\bar{B}_\alpha \cap F_\gamma \neq 0$, where $H_\mu \subset B_\alpha \in \mathscr{B}'$, since $\mathscr{H}$ is a refinement of $\mathscr{B}'$. Hence, we must have that $\gamma = \lambda$ by the definition of $\mathscr{B}'$. It follows that a given set H_μ can belong to at most one of the collections $\mathscr{H}_\lambda$.

Next, for each $H_\mu \in \mathscr{H}_\lambda$, define

$$O_\mu = H_\mu - \overline{\bigcup\{H_{\mu'}: H_{\mu'} \notin \mathscr{H}_\lambda \text{ and } H_{\mu'} < H_\mu\}}.$$

Each O_μ is open in X and by Lemma 1 we obtain

$$\begin{aligned} O_\mu &= H_\mu - \overline{\bigcup\{H_{\mu'}: H_{\mu'} \notin \mathscr{H}_\lambda \text{ and } H_{\mu'} < H_\mu\}} \cap H_\mu \\ &= H_\mu - \mathrm{cl}_{H_\mu}[\bigcup\{H_{\mu'}: H_{\mu'} \notin \mathscr{H}_\lambda \text{ and } H_{\mu'} < H_\mu\} \cap H_\mu] \\ &= H_\mu - \mathrm{cl}_{H_\mu}[\bigcup\{H_{\mu'} \cap H_\mu: H_{\mu'} \notin \mathscr{H}_\lambda \text{ and } H_{\mu'} < H_\mu\}]. \end{aligned}$$

Now, the collection of all sets $H_{\mu'} < H_\mu$ is locally finite at each point of H_μ; hence, the collection of all sets $H_{\mu'} \cap H_\mu$ with $H_{\mu'} < H_\mu$ is locally finite (in H_μ) at each point of H_μ. Therefore,

$$O_\mu = H_\mu - \bigcup\{\mathrm{cl}_{H_\mu}(H_{\mu'} \cap H_\mu): H_{\mu'} \notin \mathscr{H}_\lambda \text{ and } H_{\mu'} < H_\mu\};$$

thus, by Lemma 1 again,

$$\begin{aligned} O_\mu &= H_\mu - \bigcup\{\bar{H}_{\mu'} \cap H_\mu: H_{\mu'} \notin \mathscr{H}_\lambda \text{ and } H_{\mu'} < H_\mu\} \\ &= H_\mu - (\bigcup\{\bar{H}_{\mu'}: H_{\mu'} \notin \mathscr{H}_\lambda \text{ and } H_{\mu'} < H_\mu\}) \cap H_\mu \\ &= H_\mu - \bigcup\{\bar{H}_{\mu'}: H_{\mu'} \notin \mathscr{H}_\lambda \text{ and } H_{\mu'} < H_\mu\}. \end{aligned}$$

Next, define

$$O_\lambda = \bigcup\{O_\mu: H_\mu \in \mathscr{H}_\lambda\}.$$

Then O_λ is open in X and $F_\lambda \subset O_\lambda$: If $x \in F_\lambda$, then there exists an $H_\mu \in \mathscr{H}_\lambda$ such that $x \in H_\mu$. If $H_{\mu'} \notin \mathscr{H}_\lambda$, then $\bar{H}_{\mu'} \cap F_\lambda = \emptyset$; thus $x \notin \bar{H}_{\mu'}$. Therefore

$$x \notin \bigcup\{\bar{H}_{\mu'}: H_{\mu'} \notin \mathscr{H}_\lambda \text{ and } H_{\mu'} < H_\mu\};$$

thus, it follows that $x \in O_\mu \subset O_\lambda$.

Finally, notice that $O_\lambda \cap O_\gamma = \emptyset$ if $\lambda \neq \gamma$: If $x \in O_\lambda \cap O_\gamma$, then there exist sets $H_\mu \in \mathscr{H}_\lambda$ and $H_{\mu'} \in \mathscr{H}_\gamma$ such that $x \in O_\mu = H_\mu - \bigcup\{\bar{H}_{\mu'}: H_{\mu'} \notin \mathscr{H}_\lambda$ and $H_{\mu'} < H_\mu\}$ and also $x \in O_{\mu'} = H_{\mu'} - \bigcup\{\bar{H}_\mu: H_\mu \notin \mathscr{H}_\gamma$ and $H_\mu < H_{\mu'}\}$. But this is clearly impossible. This contradiction completes the proof of the theorem.

We now proceed to prove an invariance theorem for order paracompact spaces. For this purpose we define a class of maps which we call *projections.*

Definition 2. A continuous surjection $f: X \to Y$ is called a *projection* if and only if f is both open and closed and satisfies the condition: U, V open in X and $U \cap V = \emptyset$ imply $f(U) \cap f(V) = \emptyset$.

Clearly, a projection must be a homeomorphism if X is a Hausdorff space. However, there is a fairly large class of projections defined on T_1-spaces which are not homeomorphisms. We give a simple example.

Example 2. Let $X = [0, 1] \cup \{3/2\}$ and define a topology $\mathscr{G}$ on X by $G \in \mathscr{G}$ if and only if $G = \emptyset$, $G = \{0\}$, or $X - G$ is at most countable. Let $Y = [0, 1]$ with the topology $\mathscr{G}'$ defined by $G' \in \mathscr{G}'$ if and only if $G' = \emptyset$,

$G' = \{0\}$, or $Y - G'$ is at most countable. Then X and Y are both T_1-spaces. Define a map $f: X \to Y$ by

$$f(x) = x \quad \text{if } x \in [0, 1], \qquad f(3/2) = 1.$$

Then f is a projection but not a homeomorphism. Moreover, the condition "U, V open in X and $U \cap V = \emptyset$ imply $f(U) \cap f(V) = \emptyset$" is non-vacuously fulfilled in this example.

We are now in a position to prove the following invariance theorem.

THEOREM 4. *Order paracompactness is invariant under projections.*

Proof. Let X be order paracompact and let $f: X \to Y$ be a projection. To prove that Y is order paracompact, let $\mathscr{U} = \{U_\alpha\}$ be any open covering of Y. Then $\mathscr{V} = \{f^{-1}(U_\alpha)\}$ is an open covering of X. Since X is order paracompact, $\mathscr{V}$ has an ordered refinement $\mathscr{H} = (\{H_\lambda\}, <)$ such that if $H_\lambda \in \mathscr{H}$, the set of all $H_{\lambda'}$ preceding H_λ is locally finite at each point of $\bar{H}_\lambda$. Consider the collection $\mathscr{I} = \{f(H_\lambda)\}$. Since f is an open map and $\mathscr{H}$ is a refinement of $\mathscr{V}$, $\mathscr{I}$ is a refinement of $\mathscr{U}$. From each collection of H_λ's having the same image $f(H_\lambda)$, choose a single H_μ. This selection yields a subcollection $\mathscr{H}_0 = \{H_\mu\}$ of $\mathscr{H}$ such that if H_μ, $H_{\mu'} \in \mathscr{H}_0$ and $\mu \neq \mu'$, then $f(H_\mu) \neq f(H_{\mu'})$. Then $\mathscr{I}_0 = \{f(H_\mu): H_\mu \in \mathscr{H}_0\}$ is a refinement of $\mathscr{U}$. Define an ordering in $\mathscr{I}_0$ by: $f(H_{\mu'}) < f(H_\mu)$ if and only if $H_{\mu'} < H_\mu$. Then the ordered refinement $(\mathscr{I}_0, <)$ of $\mathscr{U}$ satisfies the requirement of Definition 1: Suppose that $f(H_\mu) \in \mathscr{I}_0$ and let $y \in \overline{f(H_\mu)}$. Since f is a closed map, it follows that $\overline{f(H_\mu)} = f(\bar{H}_\mu)$. Hence, there exists an $x \in \bar{H}_\mu$ such that $y = f(x)$. Now, the set of all $H_{\mu'}$ preceding H_μ is locally finite at the point x. Thus, there exists a neighbourhood U of x which intersects at most finitely many of the $H_{\mu'}$ preceding H_μ. Therefore, $f(U)$ is a neighbourhood of y which intersects at most finitely many of the $f(H_{\mu'})$ preceding $f(H_\mu)$: If $U \cap H_{\mu'} = \emptyset$, then $f(U) \cap f(H_{\mu'}) = \emptyset$ since f is a projection. Hence, we have shown that the set of all $f(H_{\mu'})$ preceding $f(H_\mu)$ is locally finite at the point y. We conclude that Y is order paracompact.

Let us consider Example 2 again. Since both X and Y are Lindelöf, these spaces are also order paracompact. Therefore, this example shows that the situation in Theorem 4 can occur without f being a homeomorphism.

Next, let us examine subspaces of order paracompact spaces. As the following example shows, an arbitrary subspace of an order paracompact space need not be order paracompact.

Example 3. Let X be a locally compact Hausdorff space which is not normal (**1**, p. 239). Then X is regular but not order paracompact, since otherwise Theorem 3 would imply that X is normal. Let X^* be a compactification of X. Then X^*, being Lindelöf, is order paracompact, but X, which is an open subspace of X^*, is not.

However, we have the following theorem.

Theorem 5. (a) *A closed subspace of an order paracompact space is order paracompact.*

https://doi.org/10.4153/CJM-1969-042-4 Published online by Cambridge University Press

(b) *If every open subspace of a space X is order paracompact, then X is hereditarily order paracompact.*

Proof. (a) Let X be order paracompact and Y a closed subspace of X. Let $\mathscr{U} = \{U_\alpha\}$ be any covering of Y by open sets of Y. Then for each α, there exists an open set V_α of X such that $U_\alpha = V_\alpha \cap Y$. Hence, $\mathscr{V} = \{V_\alpha\colon U_\alpha \in \mathscr{U}\} \cup \{X - Y\}$ is an open covering of X. By the order paracompactness of X, $\mathscr{V}$ has an ordered refinement $(\mathscr{H} = \{H_\lambda\}, <)$ such that if $H_\lambda \in \mathscr{H}$, the set of all $H_{\lambda'}$ preceding H_λ is locally finite at each point of $\bar{H}_\lambda$. Define an ordered refinement $(\mathscr{G} = \{G_\lambda\}, <)$ of $\mathscr{U}$ by taking $G_\lambda = H_\lambda \cap Y$ for each λ, and specifying $G_{\lambda'} < G_\lambda$ if and only if $H_{\lambda'} < H_\lambda$. Then $(\mathscr{G}, <)$ satisfies the condition of Definition 1, namely: If $G_\lambda \in \mathscr{G}$ and $x \in \text{cl}_Y(G_\lambda) = \bar{G}_\lambda \cap Y$, then $x \in \bar{H}_\lambda$. Hence, there exists a neighbourhood O of x in X which intersects at most finitely many of the $H_{\lambda'}$ preceding H_λ. It follows that $O \cap Y$ is a neighbourhood of x in Y which intersects at most finitely many of the $G_{\lambda'}$ preceding G_λ. Hence (a) is proved.

(b) Let Y be any subspace of the space X. To show that Y is order paracompact, let $\mathscr{U} = \{U_\alpha\}$ be any covering of Y by sets open in Y. Then for each α, there exists an open set V_α of X such that $U_\alpha = V_\alpha \cap Y$. Define $V = \bigcup V_\alpha$. Then V, as an open subspace of X, is order paracompact. The fact that the subspace Y of V is order paracompact now follows almost exactly as in (a). Therefore, X is hereditarily order paracompact.

Finally, we consider products of order paracompact spaces. As our final example shows, such a product need not be order paracompact, even in the simplest case of two factors.

Example 4. Let X be the set of real numbers with the upper limit topology (**4**). Then X, being Lindelöf, is order paracompact. It is known that $X \times X$ is regular but not normal. Hence, $X \times X$ is not order paracompact, since otherwise Theorem 3 would imply that $X \times X$ is normal.

Added in proof. It can easily be shown (as has been pointed out to me by H. W. Martin) that the product of an order paracompact space and a compact space is order paracompact. On the other hand, H. Tamano has shown (*On paracompactness*, Pacific J. Math. **10** (1960), 1043–1047) that if $X \times \beta X$ is normal, then X is paracompact. Using these results and Theorem 3 above, we obtain the following theorem: *A regular T_1-space is order paracompact if and only if it is paracompact.*

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