Canad. Math. Bull. Vol. 62 (1), 2019 pp. 1–9
http://dx.doi.org/10.4153/CMB-2018-031-8

Embeddings of Müntz Spaces in $L^\infty(\mu)$

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Abstract. In this paper, we discuss the properties of the embedding operator $i_\mu^\Lambda\colon M_\Lambda^\infty \hookrightarrow L^\infty(\mu)$, where μ is a positive Borel measure on $[0,1]$ and M_Λ^∞ is a Müntz space. In particular, we compute the essential norm of this embedding. As a consequence, we recover some results of the first author. We also study the compactness (resp. weak compactness) and compute the essential norm (resp. generalized essential norm) of the embedding $i_{\mu_1,\mu_2}\colon L^\infty(\mu_1) \hookrightarrow L^\infty(\mu_2)$, where μ_1, μ_2 are two positive Borel measures on $[0,1]$ with μ_2 absolutely continuous with respect to μ_1.

1 Introduction

Throughout this paper, we will consider a non-zero positive Borel measure μ on $[0,1]$ and will denote by $L^\infty(A,\mu)$ the Banach space of essentially bounded measurable functions on $A \subset [0,1]$. We endow this space with the usual norm

$$\|f\|_{L^\infty(A,\mu)} = \inf\{c > 0\,;\, |f(x)| \le c, \text{ for } \mu\text{-almost every } x \in A.\}$$

In particular, we set $L^\infty(\mu) := L^\infty([0,1],\mu)$ and $L^\infty := L^\infty(dx)$ where dx is the Lebesgue measure.

Let $\Lambda = (\lambda_0 < \lambda_1 < \cdots)$ be an increasing sequence of positive real numbers satisfying the *Müntz condition*, $\sum_{\lambda\in\Lambda\setminus\{0\}} 1/\lambda < \infty$ and $M_\Lambda = \operatorname{span}\{x^\lambda,\ \lambda \in \Lambda\}$. The *Müntz space* M_Λ^p $(1 \le p \le \infty)$ is the closure of M_Λ in $L^p := L^p([0,1], dx)$. By the Müntz theorem [3], M_Λ^p is a proper subspace of L^p.

Chalendar, Fricain, and Timotin [4] studied the question whether the Müntz space M_Λ^1 can be included in $L^1(\mu)$ for a finite Borel positive measure μ. They called such a measure a Λ-*embedding*. In particular, several results were obtained on the compactness and the essential norm of the embedding $i_\mu^\Lambda\colon f \in M_\Lambda^1 \mapsto f \in L^1(\mu)$. Recall here that the essential norm of an operator $T\colon X \to Y$ is the distance from this operator to the space of compact operators and is defined by $\|T\|_e = \inf \|T - S\|$ where the infimum runs over the compact operators $S\colon X \to Y$. The embedding $M_\Lambda^2 \hookrightarrow L^2(\mu)$ was also studied in [7, 8], and in [5] this study is extended to the framework of L^p spaces for any finite values of $p \ge 1$.

The aim of this paper is to study the remaining extreme case, *i.e.*, the embedding $i_\mu^\Lambda\colon M_\Lambda^\infty \hookrightarrow L^\infty(\mu)$.

In Section 2, we focus on the Müntz framework and on the more interesting problem of characterizing the compactness of the Λ-embedding (see Proposition 2.3). We

Received by the editors November 21, 2017; revised July 1, 2018.
Published electronically December 1, 2018.
This work is part of the project CEDRE ESFO. The authors would like to thank the program PHC CEDRE and the Lebanese University for their support.
AMS subject classification: **32C22**, 47B33, 30B10.
Keywords: Müntz space, embedding, essential norm, compact operator.

actually compute the essential norm of this embedding and prove that it is equal to 1 when i_μ^Λ is not compact (see Theorem 2.5). As a consequence of our results, we recapture the results of [1] in the particular case of composition operators.

In Section 3, for the sake of completeness, we focus on the non Müntz situation and study the embedding $i_{\mu_1,\mu_2}\colon L^\infty(\mu_1) \hookrightarrow L^\infty(\mu_2)$, where μ_1, μ_2 are two positive Borel measures on $[0,1]$ (when μ_2 is absolutely continuous relative to μ_1). We characterize its compactness: the embedding i_{μ_1,μ_2} is compact if and only if the measure μ_2 is a linear combination of Dirac measures (see Theorem 3.3). We also estimate its essential norm and prove that it is equal to 1 when the embedding is not compact (see Theorem 3.4).

Definition 1.1 Let μ_1, μ_2 be two positive Borel measures on $[0,1]$ and let X be a closed subspace of $L^\infty(\mu_1)$. We say that X is embedded into $L^\infty(\mu_2)$ if $X \subset L^\infty(\mu_2)$ and there exists a constant $C > 0$ such that, for every f in X, we have

$$\|f\|_{L^\infty(\mu_2)} \leq C\|f\|_{L^\infty(\mu_1)}.$$

We denote by $i_{X,\mu_1,\mu_2}\colon f \in X \mapsto f \in L^\infty(\mu_2)$ the embedding operator.

We shall mainly focus on the following particular cases: when $\mu_1 = dx$, $X = M_\Lambda^\infty$, and $\mu_2 = \mu$, the embedding operator is then simply denoted by $i_\mu^\Lambda\colon M_\Lambda^\infty \hookrightarrow L^\infty(\mu)$. When $X = L^\infty(\mu_1)$, we denote the embedding operator by i_{μ_1,μ_2}.

2 Embedding of Müntz Spaces

In this section, we study the embedding i_μ^Λ, *i.e.* in the case where $X = M_\Lambda^\infty$. For this, we begin with the following proposition.

Proposition 2.1 *For any positive measure μ on $[0,1]$, M_Λ^∞ is embedded in $L^\infty(\mu)$ with norm $\|i_\mu^\Lambda\| \leq 1$. Moreover,we have the following.*

(i) *If $\mu((1-\varepsilon,1]) \neq 0$ for every $\varepsilon > 0$, then $\|i_\mu^\Lambda\| = 1$.*
(ii) *If we assume that $0 \in \Lambda$, then $\|i_\mu^\Lambda\| = 1$.*

Proof First recall that any function $f \in M_\Lambda^\infty$ has a (unique) representative $\widetilde{f}$ that is continuous on $[0,1]$ (in passing, usually M_Λ^∞ denotes the space of continuous functions obtained taking the closed linear space spanned by M_Λ). Actually we can view the embedding as

$$f \in M_\Lambda^\infty \longmapsto \widetilde{f} \in \overline{M_\Lambda}^{C([0,1])} \longmapsto i_\mu^\Lambda(f) = \widetilde{f} \in L^\infty(\mu).$$

In the sequel we write simply f instead of $\widetilde{f}$ and we have

$$\|i_\mu^\Lambda(f)\|_{L^\infty(\mu)} \leq \sup_{x\in[0,1]}|f(x)| = \|f\|_{L^\infty}.$$

Hence, we deduce that $\|i_\mu^\Lambda\| \leq 1$.

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Moreover, (i) for every $\varepsilon > 0$, $\mu((1-\varepsilon, 1]) \neq 0$. Then

$$\|i_\mu^\Lambda\| = \sup_{f \in M_\Lambda^\infty \setminus \{0\}} \frac{\|f\|_{L^\infty(\mu)}}{\|f\|_{L^\infty}} \geq \limsup_{n\to\infty} \|x^{\lambda_n}\|_{L^\infty([1-\lambda_n^{-2},1],\mu)}$$
$$\geq \lim_{n\to\infty} (1-\lambda_n^{-2})^{\lambda_n} = 1.$$

(ii) Just consider the function $f = \mathbb{1}$ and the result follows immediately.
This finishes the proof. ∎

Remark 2.2 If we want a non trivial estimate for the norm of the embedding, we cannot avoid an assumption on Λ or on the support of the measure μ. Indeed, if $\mu = \delta_0$ (the Dirac measure at zero) and $\lambda_0 > 0$, we have $\|f\|_{L^\infty(\mu)} = 0$ for every $f \in M_\Lambda^\infty$.

In the following proposition, we will characterize the compactness of the embedding $i_\mu^\Lambda : M_\Lambda^\infty \hookrightarrow L^\infty(\mu)$.

Proposition 2.3 *Let $\Lambda = (\lambda_0 < \lambda_1 < \cdots)$ be a sequence of real positive numbers satisfying $\sum_{\lambda \in \Lambda \setminus \{0\}} 1/\lambda < \infty$ and let μ be a positive Borel measure on $[0,1]$. Then the embedding i_μ^Λ is compact if and only if there exists $\varepsilon > 0$ such that $\mu((1-\varepsilon, 1)) = 0$.*

Proof Assume that for some $\varepsilon > 0$, we have $\mu((1-\varepsilon,1)) = 0$. Write

$$K_\varepsilon = [0, 1-\varepsilon] \cup \{1\}.$$

We already know from [1, Corollary 2.5] that the restriction operator

$$f \in M_\Lambda^\infty \longmapsto f_{|K_\varepsilon} \in C(K_\varepsilon)$$

is compact. And i_μ^Λ can be expressed as the composition of this operator and the identity from $C(K_\varepsilon)$ to $L^\infty(K_\varepsilon, \mu)$, which can be naturally identified with $L^\infty(\mu)$.

For the converse, we consider the sequence $(x^{\lambda_n})_n$ that belongs to the unit ball of M_Λ^∞. Using the compactness of i_μ^Λ, there exists a subsequence $(x^{\lambda_{n_k}})_k$ that converges to $h \in L^\infty(\mu)$. Then $h = 0$ μ-almost everywhere on $[0,1)$ and

$$\|x^{\lambda_{n_k}}\|_{L^\infty([0,1),\mu)} \underset{k\to\infty}{\longrightarrow} 0.$$

Hence, we compute

$$\left\|x^{\lambda_{n_k}}\right\|_{L^\infty([0,1),\mu)} \geq \left\|x^{\lambda_{n_k}}\right\|_{L^\infty([1-\lambda_{n_k}^{-2},1),\mu)} \geq (1-\lambda_{n_k}^{-2})^{\lambda_{n_k}} \|1\|_{L^\infty([1-\lambda_{n_k}^{-2},1),\mu)}.$$

As $k \to \infty$, we obtain that $\|1\|_{L^\infty([1-\lambda_{n_k}^{-2},1),\mu)}$ tends to zero. But

$$\|1\|_{L^\infty([1-\lambda_{n_k}^{-2},1),\mu)} = \begin{cases} 0 & \text{if } \mu([1-\lambda_{n_k}^{-2},1)) = 0, \\ 1 & \text{otherwise,} \end{cases}$$

and this ends the proof. ∎

Remark 2.4 In the previous statement, the open interval $(1-\varepsilon, 1)$ cannot be closed at point 1. Indeed, when $\mu = \delta_1$, the embedding i_μ^Λ is compact. However, $\mu((1-\varepsilon,1]) = 1 \neq 0$ for any $\varepsilon > 0$.

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Now we will extend the previous result and compute the essential norm of the embedding i_μ^Λ.

Theorem 2.5 *Let $\Lambda = (\lambda_0 < \lambda_1 < \cdots)$ be a sequence of real positive numbers satisfying $\sum_{\lambda\in\Lambda\setminus\{0\}} 1/\lambda < \infty$ and let μ be a positive Borel measure on $[0,1]$. We have*

$$\|i_\mu^\Lambda\|_e = \begin{cases} 1 & \text{if for every } \varepsilon > 0, \mu((1-\varepsilon,1)) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof By Proposition 2.3, it is sufficient to prove that the essential norm is equal to 1 if $\mu((1-\varepsilon,1)) \neq 0$ for any $\varepsilon > 0$. Using Proposition 2.1, we have that $\|i_\mu^\Lambda\|_e \le \|i_\mu^\Lambda\| \le 1$. To get the lower estimate, we will use [2, Lemma 3.4] which states that, given an operator $T\colon X \to Y$ and $\alpha > 0$, if the range of the unit ball $T(B_X)$ contains a 2α-separated sequence, then $\|T\|_e \ge \alpha$. In our framework, for any $\alpha \in (0,1)$ we will find a normalized sequence $(f_n)_n$ in M_Λ^∞ satisfying $\|i_\mu^\Lambda(f_m) - i_\mu^\Lambda(f_n)\|_{L^\infty(\mu)} \ge 2\alpha$ for any $m \neq n$, hence $\|i_\mu^\Lambda\|_e \ge \alpha$. Actually, given any $\alpha \in (0,1)$, we are going to fulfill these conditions with a subsequence of the sequence $2x^{\lambda_n} - 1 \in M_\Lambda^\infty$ for a suitable sequence (λ_n) in Λ.

Let us fix $\alpha \in (0,1)$ and choose $\delta \in (0,1)$ and an increasing sequence (λ_n) in Λ such that

$$e^\delta \le \frac{1}{\sqrt{\alpha}}, \quad e^{\frac{1}{\delta}} \ge \frac{1}{1-\sqrt{\alpha}}, \quad \text{and} \quad \lambda_{n+2} \ge \lambda_n + \frac{1}{\delta^2}\lambda_{n+1}.$$

We first notice that the hypothesis gives that, for infinitely many values of n, the interval $J_n = (\exp(-\delta/\lambda_n), \exp(-\delta/\lambda_{n+1}))$ has a positive μ-measure. In the sequel we consider only a subsequence of these values, say $(k_n)_{n\ge1}$ with $k_{n+1} \ge k_n + 2$. Define $f_n(x) = 2x^{\lambda_{k_n}} - 1$. Clearly, we have that $\|f_n\|_\infty = 1$.

Now for any $m > n$ and $x \in J_{k_n}$, we have

$$\begin{aligned}(f_n - f_m)(x) &\ge 2e^{-\delta}\big(1 - \exp(-\delta(\lambda_{k_m} - \lambda_{k_n})/\lambda_{k_n+1})\big) \\ &\ge 2e^{-\delta}\Big(1 - \exp(-\delta(\lambda_{k_n+2} - \lambda_{k_n})/\lambda_{k_n+1})\Big) \ge 2\alpha,\end{aligned}$$

so that $\|f_n - f_m\|_{L^\infty(\mu)} \ge 2\alpha$.

We get that $\|i_\mu^\Lambda\|_e \ge \alpha$ for arbitrary $\alpha \in (0,1)$, which finishes the proof of the theorem. ∎

As a corollary, we deduce the following results that were proved in [1].

Corollary 2.6 *Let $\varphi\colon [0,1] \to [0,1]$ be a continuous function. Then the composition operator $C_\varphi\colon M_\Lambda^\infty \to C([0,1]); f \to f\circ\varphi$ is compact on M_Λ^∞ if and only if $1 \notin \operatorname{Im}\varphi$ or $\varphi = 1$. Moreover,*

$$\|C_\varphi\|_e = \begin{cases} 0 & \text{if } \varphi = 1 \text{ or } \|\varphi\|_\infty < 1, \\ 1 & \text{if } \|\varphi\|_\infty = 1 \text{ and } \varphi \neq 1. \end{cases}$$

Proof Let $(dx)_\varphi$ be the pullback measure of the Lebesgue measure dx by the function φ. It is then easy to check that for any function $f \in L^\infty((dx)_\varphi)$, the equality

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$\|f\|_{L^\infty((dx)_\varphi)} = \|f\circ\varphi\|_{L^\infty}$. This means that the operator

$$\mathrm{I}_\varphi\colon f\in L^\infty((dx)_\varphi)\longmapsto f\circ\varphi\in L^\infty$$

is an isometry. Denoting J_∞ the natural injection from $C([0,1])$ to L^∞, we have

$$\mathrm{I}_\varphi\circ i^\Lambda_{(dx)_\varphi} = J_\infty\circ C_\varphi,$$

which implies the equivalence between compactness for the operators $i^\Lambda_{(dx)_\varphi}$ and C_φ.

Now we have

$$(dx)_\varphi((1-\varepsilon,1)) = 0 \Longleftrightarrow \varphi^{-1}((1-\varepsilon,1)) = \varnothing \Longleftrightarrow 1\notin \operatorname{Im}\varphi \text{ or } \varphi = 1.$$

This finishes the first part of the proof of the corollary.

For the second part, if $\varphi = 1$ or $\|\varphi\|_\infty < 1$, then $i^\Lambda_{(dx)_\varphi}$ is compact, hence $\|C_\varphi\|_e = 0$. If $\|\varphi\|_\infty = 1$ and $\varphi \neq 1$, then obviously $1 \geq \|C_\varphi\|_e$. On the other hand, although the relation $\mathrm{I}_\varphi\circ i^\Lambda_{(dx)_\varphi} = J_\infty\circ C_\varphi$ does not immediately give the estimation on the essential norm (*a priori* it gives the essential norm of $J_\infty\circ C_\varphi$ only), the proof of Theorem 2.5 immediately adapts. Indeed, the same sequence (f_n) verifies for arbitrary $\alpha\in(0,1)$ and $n\neq m$ $\|f_n\circ\varphi - f_m\circ\varphi\|_\infty = \|f_n - f_m\|_{L^\infty((dx)_\varphi)} \geq 2\alpha$ and [2, Lemma 3.4] gives the conclusion in the same way. ∎

Let us mention that the generalized essential norm (relative to weakly compact operators) is equal to the essential norm. Indeed, the dual of M^∞_Λ has the Schur property, *i.e.* every weakly convergent sequence is norm convergent (see the proof of [1, Lemma 4.3.]).

The following corollary generalizes the result in [1, Corollary 4.10], which characterizes (weak forms of) compactness for the composition operator

$$C_\varphi\colon M^\infty_\Lambda \to C([0,1]);\qquad f\mapsto f\circ\varphi.$$

Corollary 2.7 *Under the same hypothesis as in Proposition 2.3, the following assertions are equivalent.*

- (i) i^Λ_μ *is a weakly compact operator.*
- (ii) i^Λ_μ *is a compact operator.*
- (iii) i^Λ_μ *is a Dunford–Pettis operator.*
- (iv) *There exists* $\varepsilon > 0$ *such that* $\mu((1-\varepsilon,1)) = 0$.
- (v) i^Λ_μ *is a nuclear operator.*
- (vi) i^Λ_μ *is an integral operator.*
- (vii) i^Λ_μ *is an absolutely summing operator.*

Proof We know from [1, Lemma 4.3] that compactness, weak compactness, and being Dunford–Pettis are equivalent for every operator T from M^∞_Λ to any Banach space. Therefore, the equivalences (i) ⇔ (ii) ⇔ (iii) ⇔ (iv) follow from that result and Theorem 2.5.

We show now that condition (iv) implies condition (v). Assume that for some $\delta > 0$ we have $\mu((1-\delta,1)) = 0$. For every $f\in M^\infty_\Lambda$, we can use its Erdös decomposition to write $i^\Lambda_\mu(f)(x) = f(x) = \sum_{k=0}^\infty a_k(f)x^{\lambda_k}$, where $x\in[0,1)$. Let $r = 1-\delta\in(0,1)$ and

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note that the series $\sum_{k=0}^{\infty} a_k(f)x^{\lambda_k}$ converges uniformly on $[0,r]$. Since $\mu((r,1)) = 0$, we can write

$$i_\mu^\Lambda(f) = f\mathbb{1}_{[0,r]} + f\mathbb{1}_{\{1\}} = \sum_{k=0}^{\infty} a_k(f)x^{\lambda_k}\mathbb{1}_{[0,r]} + f(1)\mathbb{1}_{\{1\}}.$$

Let $x_k^*: M_\Lambda^\infty \to \mathbb{C}$ be defined by $x_k^*(f) = a_k(f)$.

We have the following coefficient estimate for Müntz functions [3, p.177, E.3.c]: for every $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that for every $f = \sum_{k=0}^{\infty} a_k(f)x^{\lambda_k} \in M_\Lambda^\infty$,

$$|a_k(f)|(1-\varepsilon)^{\lambda_k} \le c_\varepsilon \|f\|_\infty.$$

Choose $\varepsilon = (1-\sqrt{r})$ and write simply c instead of $c_{1-\sqrt{r}}$; we have $\|x_k^*\| \le c r^{-\frac{\lambda_k}{2}}$, and then

$$\sum_{k=0}^{\infty} \|x_k^*\| \|x^{\lambda_k}\|_{L^\infty([0,r],\mu)} \le c \sum_{k=0}^{\infty} r^{\frac{\lambda_k}{2}} < +\infty.$$

Then we get that i_μ^Λ is a normally convergent series of operators of rank 1, which means that i_μ^Λ is a nuclear operator.

The implications (v) ⇒ (vi) ⇒ (vi) ⇒ 1 are always true for any operator $T: X \to Y$ [6, III.F 22]. ∎

3 Embedding of $L^\infty(\mu)$.

In this section, we focus on the embedding i_{μ_1,μ_2} (recall the notations after Definition 1.1) in the case $X = L^\infty(\mu_1)$. The particular case $\mu_1 = dx$ can be viewed as the weak-star dense case, compared to Section 2, where we worked with non dense Müntz spaces. When the Müntz condition is not fulfilled, then $M_\Lambda^\infty = C([0,1])$, which is weak-star dense in L^∞.

We shall assume in the sequel that μ_2 is absolutely continuous with respect to μ_1. Let us point out that this assumption makes the definition of i_{μ_1,μ_2} unambiguous. If $f = g$ μ_1-almost everywhere, then $f = g$ μ_2-almost everywhere.

Lemma 3.1 *The embedding i_{μ_1,μ_2} is bounded with $\|i_{\mu_1,\mu_2}\| = 1$.*

Proof Let $f \in L^\infty(\mu_1)$ and $\alpha \ge \|f\|_{L^\infty(\mu_1)}$. Then there exists a Borel set B such that $\mu_1(B^c) = 0$ and $|f(x)| \le \alpha$ for every $x \in B$. Therefore $\mu_2(B^c) = 0$ and $\alpha \ge \|f\|_{L^\infty(\mu_2)}$. Hence, $\|f\|_{L^\infty(\mu_2)} \le \|f\|_{L^\infty(\mu_1)}$. To end the proof of the lemma, simply point out that

$$\|i_{\mu_1,\mu_2}\| \ge \|i_{\mu_1,\mu_2}(\mathbb{1})\|_{L^\infty(\mu_2)} = 1$$ ∎

Let us point out that the assumption of absolute continuity of μ_2 with respect to μ_1 is not restrictive. Indeed, as soon as i_{μ_1,μ_2} is defined, for every Borel set B such that $\mu_1(B) = 0$, the function f is equal to $+\infty$ on B and 0 on B^c belongs to $L^\infty(\mu_1)$, therefore to $L^\infty(\mu_2)$. This forces $\mu_2(B) = 0$.

We shall now focus on the compactness problem, but we first need the following lemma, where $U_{x,\varepsilon}$ denotes $((x-\varepsilon,x)\cup(x,x+\varepsilon))\cap[0,1]$ for any $x \in [0,1]$ and $\varepsilon > 0$.

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Lemma 3.2 *Let μ be a (non trivial) positive Borel measure on $[0,1]$. The following assertions are equivalent.*

(i) *There exist $\alpha_1,\dots,\alpha_m \in \mathbb{R}^{+*}$, $x_1,\dots,x_m \in [0,1]$ such that $\mu = \alpha_1\delta_{x_1}+\dots+\alpha_m\delta_{x_m}$.*
(ii) *For any $x \in [0,1]$, there exists $\varepsilon_x > 0$ such that $\mu(U_{x,\varepsilon_x}) = 0$.*

Proof The implication (i) $\Rightarrow$ (ii) is trivial.

We now prove (ii) $\Rightarrow$ (i). First, we point out that the measure is discrete: consider the decomposition $\mu = \mu_c + \mu_d$ (the sum of a continuous measure μ_c and a discrete measure μ_d, both positive). Since μ_c is continuous, we have $\mu_c(\{x\}) = 0$ for every $x \in [0,1]$. On the other hand, we have $0 = \mu(U_{x,\varepsilon_x}) \geq \mu_c(U_{x,\varepsilon_x}) = 0$. Hence, $\mu_c((x-\varepsilon_x, x+\varepsilon_x)\cap[0,1]) = 0$ and $\mu_c = 0$ by compactness. Therefore $\mu = \mu_d$ is discrete.

Now we claim that there exists $x \in [0,1]$ such that $\mu(\{x\}) \neq 0$. Indeed, the negation of this statement would give that $\mu((x-\varepsilon_x, x+\varepsilon_x)\cap[0,1]) = 0$ for any $x \in [0,1]$. Hence by compactness of the interval $[0,1]$, this would imply that $\mu([0,1]) = 0$, which would contradict the fact that μ is a non-zero measure. Secondly, we shall prove that there is no infinite sequence of distinct real numbers $(x_n)_n$ such that $\mu(\{x_n\}) \neq 0$. By contradiction, the sequence $(x_n)_n$ would have an accumulation point $x \in [0,1]$. Then for any $\varepsilon > 0$, we would get that $\mu((x-\varepsilon, x)\cup(x, x+\varepsilon)) \neq 0$ because the last interval would contain some x_n for n large enough, which would give a contradiction. Hence, we deduce that there exist only a finite number of reals $x_1,\dots,x_n \in [0,1]$ such that $\mu(\{x_k\}) \neq 0$ for every $1 \leq k \leq n$ and then $\mu = \alpha_1\delta_{x_1}+\dots+\alpha_n\delta_{x_n}$, where $\alpha_1 = \mu(\{x_1\}),\dots,\alpha_n = \mu(\{x_n\})$. ∎

The following theorem characterizes the compactness and weak compactness of the embedding operator i_{μ_1,μ_2}. It occurs that the situation is very different compared to the Müntz framework (see the previous section). For this, we say that a sequence $(\widetilde{x}_n)_{n\in\mathbb{N}}$ is a block-subsequence of $(x_n)_{n\in\mathbb{N}}$ if there is a sequence of non empty finite subsets of integers $(I_m)_{m\in\mathbb{N}}$ with $\max I_m < \min I_{m+1}$ and $c_i \in [0,1]$ such that

$$\widetilde{x}_m = \sum_{j\in I_m} c_j x_j \quad \text{and} \quad \sum_{j\in I_m} c_j = 1.$$

Theorem 3.3 *Let μ_1, μ_2 be two positive Borel measures on $[0,1]$ with μ_2 absolutely continuous with respect to μ_1. The following assertions are equivalent.*

(i) *i_{μ_1,μ_2} is a compact operator.*
(ii) *i_{μ_1,μ_2} is a weakly compact operator.*
(iii) *There exist $\alpha_1,\dots,\alpha_m \in \mathbb{R}^{+*}$, and $x_1,\dots,x_m \in [0,1]$ such that*

$$\mu_2 = \alpha_1\delta_{x_1}+\dots+\alpha_m\delta_{x_m}.$$

Proof (i) $\Rightarrow$ (ii) is obvious.

For (ii) $\Rightarrow$ (iii), we fix $x \in [0,1)$, and consider $B_n = (x, x+\frac{1}{n})$. As the operator i_{μ_1,μ_2} is weakly compact, there exists a subsequence $(\mathbb{1}_{B_{n_k}})_k$ weakly converging to some f in $L^\infty(\mu_2)$. Hence, the sequence $(\mathbb{1}_{B_{n_k}})_k$ converges μ_2-almost everywhere to f, but since $\cap B_{n_k} = \varnothing$, we have $f = 0$. By the Banach–Mazur theorem, there exists a block-subsequence $(\widetilde{\mathbb{1}_{B_n}})_{n\in\mathbb{N}}$ of $(\mathbb{1}_{B_n})_{n\in\mathbb{N}}$ converging to 0 in $L^\infty(\mu_2)$. The fact that

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$\|\widetilde{\mathbb{1}_{B_n}}\|_{L^\infty(\mu_2)} \to 0$ implies that $\mu_2(B_n) = 0$ for n large enough, *i.e.*, for some $\varepsilon > 0$, $\mu_2((x, x+\varepsilon)) = 0$. In the same way, we show that $\mu_2((x-\varepsilon, x)) = 0$ for all $x \in (0,1]$, and we get that for any $x \in [0,1]$ there is $\varepsilon > 0$ such that $\mu_2(U_{x,\varepsilon}) = 0$. Hence, using Lemma 3.2, we deduce that there exists a finite number of reals $x_1, \dots, x_n \in [0,1]$ such that $\mu_2 = \alpha_1\delta_{x_1} + \cdots + \alpha_n\delta_{x_n}$, where $\alpha_1 = \mu_2(x_1), \dots, \alpha_n = \mu_2(x_n)$.

For (iii) ⇒ (i), we assume that $\mu_2 = \alpha_1\delta_{x_1} + \cdots + \alpha_n\delta_{x_m}$ with $\alpha_j \neq 0$ for every j. The operator i_{μ_1,μ_2} is compact because it has finite-dimensional range. Using Lemma 3.1, we have that $\|f\|_{L^\infty(\mu_2)} = \max_{1\le j\le m} |f(x_j)|$ so the space $L^\infty(\mu_2)$ is actually isometric to $\ell_m^\infty = (\mathbb{C}^m, \|\cdot\|_\infty)$. ∎

The next theorem computes the essential norm of the embedding operator i_{μ_1,μ_2}.

Theorem 3.4 *Under the same hypothesis as in Theorem 3.3, we have*

$$\|i_{\mu_1,\mu_2}\|_e = \begin{cases} 0 & \textit{if } \mu_2 \textit{ is a finite linear combination of Dirac measures,} \\ 1 & \textit{otherwise.} \end{cases}$$

Proof By Theorem 3.3, it is sufficient to prove that the essential norm is equal to 1 if μ_2 is not a finite linear combination of Dirac measures, which means (by Lemma 3.2) that there exists $x_0 \in [0,1]$ such that $\mu_2((x_0-\varepsilon, x_0) \cup (x_0, x_0+\varepsilon)) \neq 0$ for any $\varepsilon > 0$. Without loss of generality, we can assume that $x_0 \in (0,1]$ and $\mu_2((x_0-\varepsilon, x_0)) \neq 0$; otherwise it suffices to symmetrize and interchange $x \leftrightarrow 1-x$. Using Lemma 3.1, we already know that $\|i_{\mu_1,\mu_2}\|_e \le \|i_{\mu_1,\mu_2}\| = 1$.

Now we focus on the lower estimate and consider the sequence

$$f_n(x) = \begin{cases} 2\big(\frac{x}{x_0}\big)^{\lambda_n} - 1 & \text{if } x \in [0, x_0], \\ 1 & \text{if } x \in [x_0, 1]. \end{cases}$$

Clearly, we have that $\|f_n\|_{L^\infty(\mu_1)} \le 1$ and for any n, m,

$$\|f_n - f_m\|_{L^\infty(\mu_2)} = 2\|x^{\lambda_n} - x^{\lambda_m}\|_{L^\infty(\mu_2')},$$

where $\mu_2'(B) = \mu_2(x_0.B)$ for any B is a Borel subset of $[0,1]$. It suffices now to apply the argument of the first part of the proof of Theorem 2.5 to get $\|i_{\mu_1,\mu_2}\|_e \ge 1$. ∎

Remark 3.5 For the case $X = C([0,1]) = C$ we obtain the same results. The embedding $i_{C,\mu}\colon C([0,1]) \to L^\infty(\mu)$ is compact if and only if there exists $\alpha_1, \dots, \alpha_m \in \mathbb{R}^{+*}$, $x_1, \dots, x_m \in [0,1]$ such that $\mu = \alpha_1\delta_{x_1} + \cdots + \alpha_m\delta_{x_m}$. Moreover, we have

$$\|i_{C,\mu}\|_e = \begin{cases} 0 & \text{if } \mu \text{ is a finite linear combination of Dirac measures,} \\ 1 & \text{otherwise.} \end{cases}$$

Acknowledgment The authors wish to thank the referees for their very careful reading and their suggestions to improve the quality of the redaction.

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