



From Quantum Groups to Groups

Mehrdad Kalantar and Matthias Neufang

Abstract. In this paper we use the recent developments in the representation theory of locally compact quantum groups, to assign to each locally compact quantum group \mathbb{G} a locally compact group $\tilde{\mathbb{G}}$ that is the quantum version of point-masses and is an invariant for the latter. We show that “quantum point-masses” can be identified with several other locally compact groups that can be naturally assigned to the quantum group \mathbb{G} . This assignment preserves compactness as well as discreteness (hence also finiteness), and for large classes of quantum groups, amenability. We calculate this invariant for some of the most well-known examples of non-classical quantum groups. Also, we show that several structural properties of \mathbb{G} are encoded by $\tilde{\mathbb{G}}$; the latter, despite being a simpler object, can carry very important information about \mathbb{G} .

1 Introduction

One of Murray and von Neumann’s primary motivations to define and study operator algebras was the representation theory of groups, and in fact, those operator algebras associated with groups have played a prominent role in the theory of operator algebras ever since.

Also, using different group properties in constructing and studying various types of operator algebras has led to some of the deepest results in the subject.

The aim of this paper is to investigate operator algebras associated with locally compact quantum groups \mathbb{G} by studying the properties of an assigned locally compact group $\tilde{\mathbb{G}}$. The latter, in the classical case, is the initial group from which those operator algebras are constructed. We see that, despite possibly being a much smaller object in the non-commutative setting, the group $\tilde{\mathbb{G}}$ can carry very important information about the quantum group \mathbb{G} .

Locally compact quantum groups, as introduced and studied by Kustermans and Vaes in [17], provide a category which comprises both classical group algebras and group-like objects arising in mathematical physics such as Woronowicz’s famous quantum group $SU_\mu(2)$.

To find a Pontryagin-type duality theorem which holds for all locally compact groups rather than just abelian ones, one has to pass to the larger category of locally compact quantum groups. In order to embed locally compact groups in this larger category, one has to work with the algebras associated with a group. So, instead of working with a locally compact group G , we study $L^\infty(G)$, and we consider $VN(G)$ as its Pontryagin dual.

Also, there are many other occasions in which one prefers, or even has to pass from a group to an associated (operator) algebra. However, there are several equivalent

Received by the editors March 27, 2012.

Published electronically February 6, 2013.

AMS subject classification: 46L89.

Keywords: locally compact quantum group, locally compact group, von Neumann algebra.

ways to recover the initial group from these algebras: G is topologically isomorphic, for example, to

- the spectrum of $C_0(G)$;
- the spectrum of $A(G)$;
- the set of all group-like elements in the symmetric quantum group $\text{VN}(G)$.

In [27] Wendel proved that for a locally compact group G , every positive isometric linear (left or right) $L^1(G)$ -module map on $L^1(G)$, *i.e.*, every positive isometric (right or left) multiplier, has to be the convolution by a point-mass. Moreover, the set of point-masses regarded as maps on $L^1(G)$ with the strong operator (*i.e.*, the point-norm) topology is homeomorphic to the group G . In other words, he showed how a locally compact group can be recovered from its measure algebra.

In [14], Junge, Neufang and Ruan defined an analogue of measure algebra for locally compact quantum groups, and studied its structure and representation theory; in fact, by means of a representation theorem they investigated the quantum group analogue of the class of completely bounded multiplier algebras that play an important role in Fourier analysis over groups. The latter result enables the authors to express quantum group duality precisely in terms of a commutation relation. If $\mathbb{G} = L^\infty(G)$ for a locally compact group G , then the algebra of completely bounded multipliers $M_{cb}(L^1(\mathbb{G}))$ defined in [14] is the measure algebra of the group G . So we can regard the algebra $M_{cb}(L^1(\mathbb{G}))$ as a quantization of the measure algebra. This motivated us to look for objects similar to point-masses in the classical case, so-to-speak “quantum point-masses”.

Following this path, in this paper we start with assigning to each locally compact quantum group \mathbb{G} , a locally compact group $\tilde{\mathbb{G}}$ that is an invariant for the latter; we will later prove that this assignment preserves compactness as well as discreteness (hence also finiteness), and, for large classes of quantum groups, amenability.

We first prove some basic properties of this group before arriving at one of our main results in Section 3, Theorem 3.12, by establishing identifications between several different locally compact groups which can be assigned to a locally compact quantum group, including the intrinsic group of the dual quantum group, as well as the spectrum of the universal C^* -algebra and of the L^1 -algebra of the dual quantum group.

In Section 4, we calculate this associated group for some well-known examples of locally compact quantum groups. For Woronowicz’s class of compact matrix pseudogroups, we always obtain a compact Lie group—which in the case of $\text{SU}_\mu(2)$ is precisely the circle group.

In the last section of this paper, we present various applications of studying this group. In particular, we show that for a large class of locally compact quantum groups, the associated locally compact group cannot be “small”, and in fact, the smallness of the latter forces the former to be of a very specific type. We also see that this group carries some natural properties inherited from the locally compact quantum group, which shows that this assignment is natural.

The results in this paper are based on [15], written under the supervision of the second-named author.

2 Preliminaries

We recall from [17] and [26] that a (von Neumann algebraic) *locally compact quantum group* \mathbb{G} is a quadruple $(L^\infty(\mathbb{G}), \Gamma, \varphi, \psi)$, where $L^\infty(\mathbb{G})$ is a von Neumann algebra with a co-associative co-multiplication $\Gamma: L^\infty(\mathbb{G}) \rightarrow L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{G})$, φ and ψ are (normal faithful semi-finite) left and right Haar weights on $L^\infty(\mathbb{G})$, respectively. We write $\mathcal{M}_\varphi^+ = \{x \in L^\infty(\mathbb{G})^+ : \varphi(x) < \infty\}$ and $\mathcal{N}_\varphi = \{x \in L^\infty(\mathbb{G})^+ : \varphi(x^*x) < \infty\}$, and we denote by Λ_φ the inclusion of \mathcal{N}_φ into the GNS Hilbert space $L^2(\mathbb{G}, \varphi)$ of φ . According to [17, Proposition 2.11], we can identify $L^2(\mathbb{G}, \varphi)$ and $L^2(\mathbb{G}, \psi)$, and we simply use $L^2(\mathbb{G})$ for this Hilbert space in the rest of this paper.

For each locally compact quantum group \mathbb{G} , there exists a *left fundamental unitary operator* W on $L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$ that satisfies the *pentagonal relation*

$$W_{12}W_{13}W_{23} = W_{23}W_{12}$$

and such that the co-multiplication Γ on $L^\infty(\mathbb{G})$ can be expressed as

$$\Gamma(x) = W^*(1 \otimes x)W \quad (x \in L^\infty(\mathbb{G})).$$

Let $L^1(\mathbb{G})$ be the predual of $L^\infty(\mathbb{G})$. Then the pre-adjoint of Γ induces on $L^1(\mathbb{G})$ an associative completely contractive multiplication

$$\star: L^1(\mathbb{G}) \widehat{\otimes} L^1(\mathbb{G}) \ni f_1 \otimes f_2 \mapsto f_1 \star f_2 = (f_1 \otimes f_2) \circ \Gamma \in L^1(\mathbb{G}).$$

The *left regular representation* $\lambda: L^1(\mathbb{G}) \rightarrow \mathcal{B}(L^2(\mathbb{G}))$ is defined by

$$\lambda: L^1(\mathbb{G}) \ni f \mapsto \lambda(f) = (f \otimes \iota)(W) \in \mathcal{B}(L^2(\mathbb{G})),$$

which is an injective and completely contractive algebra homomorphism from $L^1(\mathbb{G})$ into $\mathcal{B}(L^2(\mathbb{G}))$. Then

$$L^\infty(\widehat{\mathbb{G}}) = \{\lambda(f) : f \in L^1(\mathbb{G})\}''$$

is the von Neumann algebra associated with the dual quantum group $\widehat{\mathbb{G}}$ of \mathbb{G} . It follows that $W \in L^\infty(\mathbb{G}) \widehat{\otimes} L^\infty(\widehat{\mathbb{G}})$. We also define the completely contractive injection

$$\widehat{\lambda}: L^1(\widehat{\mathbb{G}}) \ni \widehat{f} \mapsto \widehat{\lambda}(\widehat{f}) = (\iota \otimes \widehat{f})(W) \in L^\infty(\mathbb{G}).$$

The *reduced quantum group C^* -algebra* $C_0(\mathbb{G}) = \overline{\widehat{\lambda}(L^1(\widehat{\mathbb{G}}))}^{\|\cdot\|}$ is a weak* dense C^* -subalgebra of $L^\infty(\mathbb{G})$ with the co-multiplication

$$\Gamma: C_0(\mathbb{G}) \rightarrow M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$$

given by the restriction of the co-multiplication on $L^\infty(\mathbb{G})$ to $C_0(\mathbb{G})$, where we denote by $M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$ the multiplier C^* -algebra of the minimal C^* -algebra tensor product $C_0(\mathbb{G}) \otimes C_0(\mathbb{G})$.

Let $M(\mathbb{G})$ denote the operator dual $C_0(\mathbb{G})^*$. There exists a completely contractive multiplication on $M(\mathbb{G})$ given by the convolution

$$\star: M(\mathbb{G}) \widehat{\otimes} M(\mathbb{G}) \ni \mu \otimes \nu \mapsto \mu \star \nu = \mu(\iota \otimes \nu)\Gamma = \nu(\mu \otimes \iota)\Gamma \in M(\mathbb{G})$$

such that $M(\mathbb{G})$ contains $L^1(\mathbb{G})$ as a norm closed two-sided ideal. If G is a locally compact group, then $C_0(\mathbb{G}_a)$ is the C^* -algebra $C_0(G)$ of continuous functions on G vanishing at infinity, and $M(\mathbb{G}_a)$ is the measure algebra $M(G)$ of G . Correspondingly, $C_0(\widehat{\mathbb{G}}_a)$ is the left reduced group C^* -algebra $C_\lambda^*(G)$ of G . Hence, we have $M(\widehat{\mathbb{G}}_a) = B_\lambda(G)$.

For a locally compact quantum group \mathbb{G} , we denote by S its *antipode*, which is the unique σ -strong* closed linear map on $L^\infty(\mathbb{G})$ satisfying $(\omega \otimes \iota)(W) \in \mathcal{D}(S)$ and $S(\omega \otimes \iota)(W) = (\omega \otimes \iota)(W^*)$ for all $\omega \in \mathcal{B}(L^2(\mathbb{G}))_*$ and such that the elements $(\omega \otimes \iota)(W)$ form a σ -strong* core for S . The antipode S has a polar decomposition $S = R\tau_{-\frac{i}{2}}$, where R is an anti-automorphism of $L^\infty(\mathbb{G})$ and (τ_t) is a strongly continuous one-parameter group of automorphisms of $L^\infty(\mathbb{G})$. We call R the *unitary antipode* and (τ_t) the *scaling group* of \mathbb{G} .

There exists a strictly positive operator δ , affiliated with M , called *modular element*, such that

$$\psi(x) = \varphi(\delta^{1/2}x\delta^{1/2})$$

for all $x \in \mathcal{M}_\psi$. Also, we have $\Gamma(\delta^{it}) = \delta^{it} \otimes \delta^{it}$ for all $t \in \mathbb{R}$. We say that a locally compact quantum group \mathbb{G} is *unimodular* if $\delta = 1$.

If we define a strictly positive operator P on $L^2(\mathbb{G})$ such that $P^{it}\Lambda_\varphi(x) = \Lambda_\varphi(\tau_t(x))$ for all $t \in \mathbb{R}$ and $x \in \mathcal{N}_\varphi$, then we have $\tau_t(x) = P^{it}xP^{-it}$, and

$$(2.1) \quad P = \widehat{P} \quad \text{and} \quad \Delta_{\widehat{\varphi}}^{it} = P^{it}J_\varphi\delta^{it}J_\varphi,$$

where $\Delta_{\widehat{\varphi}}$ is the modular operator associated to $\widehat{\varphi}$ and J_φ is the modular conjugate associated to φ .

Moreover, the following hold for all $t \in \mathbb{R}$.

$$(2.2) \quad \Gamma \circ \tau_t = (\tau_t \otimes \tau_t) \circ \Gamma \quad \Gamma \circ \sigma_t^\varphi = (\tau_t \otimes \sigma_t^\varphi) \circ \Gamma;$$

$$(2.3) \quad \Gamma \circ \tau_t = (\sigma_t^\varphi \otimes \sigma_{-t}^\psi) \circ \Gamma \quad \Gamma \circ \sigma_t^\psi = (\sigma_t^\psi \otimes \tau_{-t}) \circ \Gamma.$$

For more details, we refer the reader to [17].

It is known that a locally compact quantum group $\mathbb{G} = (M, \Gamma, \varphi, \psi)$ is a *Kac algebra* if and only if the antipode S is bounded and the modular element δ is affiliated with the center of M .

Let $L_*^1(\widehat{\mathbb{G}}) = \{\widehat{\omega} \in L^1(\widehat{\mathbb{G}}) : \exists \widehat{f} \in L^1(\widehat{\mathbb{G}}) \text{ such that } \widehat{\lambda}(\widehat{\omega})^* = \widehat{\lambda}(\widehat{f})\}$. Then $L_*^1(\widehat{\mathbb{G}}) \subseteq L^1(\widehat{\mathbb{G}})$ is norm dense, and with the involution $\widehat{\omega}^* = \widehat{f}$, and the norm $\|\widehat{\omega}\|_u = \max\{\|\widehat{\omega}\|, \|\widehat{\omega}^*\|\}$, the space $L_*^1(\widehat{\mathbb{G}})$ becomes a Banach $*$ -algebra (for details see [16]). We obtain the *universal quantum group C^* -algebra* $C_u(\mathbb{G})$ as the universal enveloping C^* -algebra of the Banach algebra $L_*^1(\widehat{\mathbb{G}})$. There is a universal $*$ -representation $\widehat{\lambda}_u : L_*^1(\widehat{\mathbb{G}}) \rightarrow \mathcal{B}(H_u)$ such that

$$C_u(\mathbb{G}) = \overline{\widehat{\lambda}_u(L_*^1(\widehat{\mathbb{G}}))}^{\|\cdot\|}.$$

There is a universal co-multiplication

$$\Gamma_u : C_u(\mathbb{G}) \rightarrow M(C_u(\mathbb{G}) \otimes C_u(\mathbb{G})),$$

and the operator dual $M_u(\mathbb{G}) := C_u(\mathbb{G})^*$, which can be regarded as the space of all *quantum measures* on \mathbb{G} , is a unital completely contractive Banach algebra with multiplication given by

$$\omega \star_u \mu = \omega(\iota \otimes \mu)\Gamma_u = \mu(\omega \otimes \iota)\Gamma_u.$$

By universal property of $C_u(\mathbb{G})$, there is a unique surjective $*$ -homomorphism $\pi : C_u(\mathbb{G}) \rightarrow C_0(\mathbb{G})$ such that $\pi(\widehat{\lambda}_u(\widehat{\omega})) = \widehat{\lambda}(\widehat{\omega})$ for all $\widehat{\omega} \in L^1_*(\widehat{\mathbb{G}})$ (see [1], [3] and [16]).

A linear map m on $L^1(\mathbb{G})$ is called a *left centralizer* of $L^1(\mathbb{G})$ if it satisfies

$$m(f \star g) = m(f) \star g$$

for all $f, g \in L^1(\mathbb{G})$. We denote by $C^l_{cb}(L^1(\mathbb{G}))$ the space of all completely bounded left centralizers of $L^1(\mathbb{G})$. We have the natural inclusions

$$L^1(\mathbb{G}) \hookrightarrow M(\mathbb{G}) \hookrightarrow M_u(\mathbb{G}) \rightarrow C^l_{cb}(L^1(\mathbb{G})).$$

These algebras are typically not equal. We have

$$M(\mathbb{G}) = M_u(\mathbb{G}) = C^l_{cb}(L^1(\mathbb{G}))$$

if and only if \mathbb{G} is co-amenable, *i.e.*, $L^1(\mathbb{G})$ has a contractive (or bounded) approximate identity (cf. [1], [3], and [12]). If \mathbb{G}_a is the commutative quantum group associated with a locally compact group G , then \mathbb{G}_a is always co-amenable since $L^1(\mathbb{G}_a) = L^1(G)$ has a contractive approximate identity. On the other hand, if $\mathbb{G}_s = \widehat{\mathbb{G}}_a$ is the co-commutative dual quantum group of \mathbb{G}_a , it is co-amenable, *i.e.*, the Fourier algebra $A(G)$ has a contractive approximate identity, if and only if the group G is amenable.

The left regular representation λ can be extended to $C^l_{cb}(L^1(\mathbb{G}))$ such that $\langle \widehat{\omega}, \lambda(\omega) \rangle = \langle \widehat{\lambda}_u(\widehat{\omega}), \omega \rangle$ for all $\omega \in M_u(\mathbb{G})$ and $\widehat{\omega} \in L^1_*(\widehat{\mathbb{G}})$.

A normal completely bounded map Φ on $L^\infty(\mathbb{G})$ is called *covariant* if it satisfies

$$(\Phi \otimes \iota) \circ \Gamma = \Gamma \circ \Phi.$$

We denote by $\mathcal{CB}^\sigma_{cov}(L^\infty(\mathbb{G}))$ the algebra of all normal completely bounded covariant maps on $L^\infty(\mathbb{G})$. It is easy to see that a normal completely bounded map Φ on $L^\infty(\mathbb{G})$ is in $\mathcal{CB}^\sigma_{cov}(L^\infty(\mathbb{G}))$ if and only if it is a left $L^1(\mathbb{G})$ -module map on $L^\infty(\mathbb{G})$. Therefore, a map T is in $C^l_{cb}(L^1(\mathbb{G}))$ if and only if T^* is in $\mathcal{CB}^\sigma_{cov}(L^\infty(\mathbb{G}))$.

Let $X, Y \subseteq \mathcal{B}(H)$. We denote by $\mathcal{CB}^{\sigma, X}_Y(\mathcal{B}(H))$ the algebra of all normal completely bounded Y -bimodule maps Φ on $\mathcal{B}(H)$ that leave X invariant. It was proved

in [14] that for a locally compact quantum group \mathbb{G} , the adjoint of every left centralizer $T \in C_{cb}^l(L^1(\mathbb{G}))$ has a unique extension to a map

$$\Phi_T \in \mathcal{CB}_{L^\infty(\widehat{\mathbb{G}})}^{\sigma, L^\infty(\mathbb{G})}(\mathcal{B}(L^2(\mathbb{G}))).$$

The next result, which can be seen as a quantum version of Wendel’s theorem ([27, Theorem 3]), is the starting point for our work. In fact, this theorem will suggest how to define the objects that should be called quantum point-masses.

In the sequel we shall denote by $\text{Ad}(u)$ the map $x \mapsto uxu^*$, for a unitary operator u .

Theorem 2.1 ([14, Theorem 4.7]) *Let \mathbb{G} be a locally compact quantum group, and let T be a complete contraction in $C_{cb}^l(L^1(\mathbb{G}))$. Then the following are equivalent:*

- (i) T is a completely isometric linear isomorphism on $L^1(\mathbb{G})$;
- (ii) T has a completely contractive inverse in $C_{cb}^l(L^1(\mathbb{G}))$;
- (iii) Φ_T is a $*$ -automorphism in $\mathcal{CB}_{L^\infty(\widehat{\mathbb{G}})}^{\sigma, L^\infty(\mathbb{G})}(\mathcal{B}(L^2(\mathbb{G})))$;
- (iv) *there exist a unitary operator $\widehat{u} \in L^\infty(\widehat{\mathbb{G}})$ and a complex number $\lambda \in \mathbb{T}$ such that $\Phi_T(x) = \lambda \text{Ad}(\widehat{u})(x)$. If, in addition, T is completely positive, then so is T^{-1} . In this case, we have $\Phi_T = \text{Ad}(\widehat{u})$.*

3 Assigning a Group to a Quantum Group

In view of Theorem 2.1 and [27, Theorem 3], we define the following assignment.

Definition 3.1 Let \mathbb{G} be a locally compact quantum group. Define $\widetilde{\mathbb{G}}$ to be the set of all completely positive maps $m \in C_{cb}^l(L^1(\mathbb{G}))$ which satisfy one of the equivalent conditions of Theorem 2.1. We endow $\widetilde{\mathbb{G}}$ with the strong operator topology.

Example 3.2 As a consequence of [27, Theorem 3], we see that if $\mathbb{G} = L^\infty(G)$ for a locally compact group G , then $\widetilde{\mathbb{G}}$ is topologically isomorphic to G .

Example 3.3 ([21, Theorem 2]) If $\mathbb{G} = \text{VN}(G)$ for an amenable locally compact group G , then $\widetilde{\mathbb{G}}$ is topologically isomorphic to \widehat{G} , the set of all continuous characters on G with the compact-open topology.

As we shall see later, amenability is not necessary.

We thus obtain an assignment $\mathbb{G} \rightarrow \widetilde{\mathbb{G}}$, from the category of locally compact quantum groups to the category of groups, which is inverse to the usual embedding of the latter category into the former. The main purpose of this paper is to investigate how much information about \mathbb{G} one can get from studying $\widetilde{\mathbb{G}}$, and also study the preservation of several natural properties under this assignment

Note that in the classical case, for $m \in \widetilde{\mathbb{G}}$, the adjoint map $m^*: L^\infty(G) \rightarrow L^\infty(G)$ is just the left translation. The next proposition shows that for a locally compact quantum group \mathbb{G} and $m \in \widetilde{\mathbb{G}}$, the adjoint map m^* can be regarded as quantum left translation.

Proposition 3.4 Let $m \in \widetilde{\mathbb{G}}$ and φ be the left Haar weight on \mathbb{G} . Then we have $\varphi \circ m^* = \varphi$.

Proof For $x \in \mathcal{M}_\varphi^+$ we have $m^*(x) \in \mathcal{M}_\varphi^+$ as well, and

$$\begin{aligned} \varphi \circ m^*(x)1 &= (\iota \otimes \varphi)\Gamma(m^*(x)) = (\iota \otimes \varphi)(m^* \otimes \iota)\Gamma(x) \\ &= m^*(\iota \otimes \varphi)\Gamma(x) = m^*(\varphi(x)1) = \varphi(x)1. \end{aligned} \quad \blacksquare$$

Proposition 3.5 $\widetilde{\mathbb{G}}$ is a topological group.

Proof Let $m_\alpha \rightarrow \iota$ and $n_\alpha \rightarrow \iota$, where (m_α) and (n_α) are nets in $\widetilde{\mathbb{G}}$. Then, for all $f \in L^1(\mathbb{G})$, we have

$$\begin{aligned} \|(m_\alpha n_\alpha - \iota)(f)\| &\leq \|(m_\alpha n_\alpha - m_\alpha)(f)\| + \|(m_\alpha - \iota)(f)\| \\ &\leq \|m_\alpha\| \|(n_\alpha - \iota)(f)\| + \|(m_\alpha - \iota)(f)\| \rightarrow 0. \end{aligned}$$

We also have

$$\begin{aligned} \|m_\alpha^{-1}(f) - f\| &= \|m_\alpha^{-1}(f) - m_\alpha^{-1}(m_\alpha(f))\| \\ &= \|m_\alpha^{-1}(f - m_\alpha(f))\| \leq \|f - m_\alpha(f)\| \rightarrow 0. \end{aligned} \quad \blacksquare$$

Let (M, Γ) be a Hopf-von Neumann algebra. The intrinsic group of (M, Γ) is defined as

$$\text{Gr}(M, \Gamma) := \{x \in M : \Gamma(x) = x \otimes x \text{ and } x \text{ is invertible}\}.$$

We endow this group with the induced weak* topology. It can be easily seen (cf. [8, Proposition 1.2.3]) that each $x \in \text{Gr}(M, \Gamma)$ is in fact a unitary. For a locally compact quantum group $\mathbb{G} = (M, \Gamma, \varphi, \psi)$ we denote $\text{Gr}(M, \Gamma)$ simply by $\text{Gr}(\mathbb{G})$. For $x \in \text{Gr}(\mathbb{G})$, by [17, Proposition 5.33], we have $x \in \mathcal{D}(S)$, and $S(x) = x^*$. This implies that $\tau_{-i}(x) = S^2(x) = x$, and it follows that $\tau_t(x) = x$ for all $x \in \text{Gr}(\mathbb{G})$ and $t \in \mathbb{R}$. Hence, we obtain $R(x) = x^*$ for all $x \in \text{Gr}(\mathbb{G})$.

Using the left fundamental unitary W , we can define the map

$$\widetilde{\Gamma}: \mathcal{B}(L^2(\mathbb{G})) \ni x \mapsto W^*(1 \otimes x)W \in \mathcal{B}(L^2(\mathbb{G})) \overline{\otimes} \mathcal{B}(L^2(\mathbb{G})),$$

which extends the co-multiplication Γ to $\mathcal{B}(L^2(\mathbb{G}))$.

Proposition 3.6 We have $\text{Gr}(\mathbb{G}) = \text{Gr}(\mathcal{B}(L^2(\mathbb{G})), \widetilde{\Gamma})$.

Proof Obviously $\text{Gr}(\mathbb{G}) \subseteq \text{Gr}(\mathcal{B}(L^2(\mathbb{G})), \widetilde{\Gamma})$. If $x \in \text{Gr}(\mathcal{B}(L^2(\mathbb{G})), \widetilde{\Gamma})$, then $\widetilde{\Gamma}(x) = x \otimes x$; but since $\widetilde{\Gamma}(\mathcal{B}(L^2(\mathbb{G}))) \subseteq L^\infty(\mathbb{G}) \overline{\otimes} \mathcal{B}(L^2(\mathbb{G}))$, we have $x \in L^\infty(\mathbb{G})$. \blacksquare

The following theorem was proved by de Cannière [5, Definition 2.5] for the case of Kac algebras.

Theorem 3.7 *Let \mathbb{G} be a locally compact quantum group. Then we have a group homeomorphism*

$$\widetilde{\mathbb{G}} \cong \text{Gr}(\widehat{\mathbb{G}}).$$

Proof Let $m \in \widetilde{\mathbb{G}}$. Then m^* is a $*$ -automorphism on $L^\infty(\mathbb{G})$, and since $\varphi \circ m^* = \varphi$ (by Proposition 3.4), we can extend the map

$$\Lambda_\varphi(x) \mapsto \Lambda_\varphi(m^*(x)) \quad (x \in \mathcal{N}_\varphi)$$

to a unitary \widehat{u}_m on $L^2(\mathbb{G})$. Moreover, we have

$$\begin{aligned} m^*(x)\Lambda_\varphi(a) &= \Lambda_\varphi(m^*(x)a) = \Lambda_\varphi(m^*(xm^{*-1}(a))) \\ &= \widehat{u}_m\Lambda_\varphi(xm^{*-1}(a)) = \widehat{u}_mx\Lambda_\varphi(m^{*-1}(a)) \\ &= \widehat{u}_mx\widehat{u}_m^*\Lambda_\varphi(a), \end{aligned}$$

which implies that $m^* = \text{Ad}(\widehat{u}_m)$, where $\text{Ad}(\widehat{u}_m)(x) = \widehat{u}_mx\widehat{u}_m^*$, $x \in \mathcal{B}(L^2(\mathbb{G}))$. We show that $\widehat{u}_m \in \text{Gr}(\widehat{\mathbb{G}})$. For all $x, y \in \mathcal{N}_\varphi$ we have

$$\begin{aligned} (\widehat{u}_m \otimes 1)W^*(\Lambda_\varphi(x) \otimes \Lambda_\varphi(y)) &= (\widehat{u}_m \otimes 1)\Lambda_{\varphi \otimes \varphi}(\Gamma(y)(x \otimes 1)) = \Lambda_{\varphi \otimes \varphi}((m^* \otimes \iota)(\Gamma(y)(x \otimes 1))) \\ &= \Lambda_{\varphi \otimes \varphi}(\Gamma(m^*(y))(m^*(x) \otimes 1)) = W^*(\Lambda_\varphi(m^*(x)) \otimes \Lambda_\varphi(m^*(y))) \\ &= W^*(\widehat{u}_m \otimes \widehat{u}_m)(\Lambda_\varphi(x) \otimes \Lambda_\varphi(y)). \end{aligned}$$

Hence, we obtain $W(\widehat{u}_m \otimes 1)W^* = \widehat{u}_m \otimes \widehat{u}_m$ which implies

$$\begin{aligned} \widehat{W}^*(1 \otimes \widehat{u}_m)\widehat{W} &= \chi(W)(1 \otimes \widehat{u}_m)\chi(W^*) = \chi(W(\widehat{u}_m \otimes 1)W^*) \\ &= \chi(\widehat{u}_m \otimes \widehat{u}_m) = \widehat{u}_m \otimes \widehat{u}_m. \end{aligned}$$

Thus $\widehat{u}_m \in \text{Gr}(\mathcal{B}(L^2(\mathbb{G})), \widehat{\Gamma})$, and so $\widehat{u}_m \in \text{Gr}(\widehat{\mathbb{G}})$ by Proposition 3.6.

Now define $\Psi: \widetilde{\mathbb{G}} \rightarrow \text{Gr}(\widehat{\mathbb{G}})$, $\Psi(m) = \widehat{u}_m$. It is easily seen that Ψ is a well-defined group homomorphism. If $\Psi(m) = 1$, we have $m^* = \text{Ad} 1 = \iota$, which implies $m = \iota$. Hence Ψ is injective.

To see that Ψ is also surjective, let $\widehat{u} \in \text{Gr}(\widehat{\mathbb{G}})$. Then the above calculations show that for all $\omega \in \mathcal{B}(L^2(\mathbb{G}))_*$, we have

$$\begin{aligned} \text{Ad}(\widehat{u})((\iota \otimes \omega)W^*) &= (\iota \otimes \omega)((\widehat{u} \otimes 1)W^*(\widehat{u}^* \otimes 1)) = (\iota \otimes \omega)(W^*(\widehat{u} \otimes \widehat{u})(\widehat{u}^* \otimes 1)) \\ &= (\iota \otimes \omega)(W^*(1 \otimes \widehat{u})) = (\iota \otimes (\widehat{u} \cdot \omega))W^*. \end{aligned}$$

Since $\{(\iota \otimes \omega)W^* : \omega \in \mathcal{B}(L^2(\mathbb{G}))_*\}$ is weak* dense in $L^\infty(\mathbb{G})$, we see that

$$\text{Ad}(\widehat{u})(L^\infty(\mathbb{G})) \subseteq L^\infty(\mathbb{G}),$$

whence $\text{Ad}(\widehat{u}) \in \mathcal{CB}_{L^\infty(\widehat{\mathbb{G}})}^{\sigma, L^\infty(\mathbb{G})}(\mathcal{B}(L^2(\mathbb{G})))$. Hence, by Theorem 2.1, there exists $m \in \widetilde{\mathbb{G}}$ such that $m^* = \text{Ad}(\widehat{u})$, which implies that $\Psi(m) = \widehat{u}$.

We now show that Ψ is a homeomorphism with respect to the corresponding topologies. For all $\xi \in L^2(\mathbb{G})$, $y \in L^\infty(\mathbb{G})$ and $m \in \widetilde{\mathbb{G}}$, and $m^* = \text{Ad}(\widehat{u}_m)$, we have

$$\begin{aligned} \langle m(\omega_\xi), y \rangle &= \langle \omega_\xi, m^*(y) \rangle = \langle \omega_\xi, \widehat{u}_m y \widehat{u}_m^* \rangle = \langle \widehat{u}_m y \widehat{u}_m^* \xi, \xi \rangle \\ &= \langle y \widehat{u}_m^* \xi, \widehat{u}_m^* \xi \rangle = \langle \omega_{\widehat{u}_m^* \xi}, y \rangle, \end{aligned}$$

which implies $m(\omega_\xi) = \omega_{\widehat{u}_m^* \xi}$. Next, denote by $\mathcal{P} \subseteq L^2(\mathbb{G})$ the positive cone associated to the standard representation of $L^\infty(\mathbb{G})$ on $L^2(\mathbb{G})$ (see [10] for details). Since \widehat{u}_m is the canonical unitary implementation of the $*$ -automorphism m^* ([10, proof of Proposition 3.7]), it follows from [10, Theorem 3.2] that \widehat{u}_m leaves \mathcal{P} invariant. So, we have

$$(3.1) \quad \widehat{u}(\mathcal{P}) \subseteq \mathcal{P}.$$

for all $\widehat{u} \in \text{Gr}(\widehat{\mathbb{G}})$.

Now, let (m_α) be a net in $\widetilde{\mathbb{G}}$ such that $m_\alpha \rightarrow \iota$, with $m_\alpha^* = \text{Ad}(\widehat{u}_\alpha)$. Then, using (3.1), [10, Lemma 2.10], and the fact that \mathcal{P} generates $L^2(\mathbb{G})$ [10, Remark 2.7], we get

$$\begin{aligned} m_\alpha \rightarrow \iota &\Leftrightarrow \|m_\alpha(\omega_\xi) - \omega_\xi\| \rightarrow 0 \quad \forall \xi \in \mathcal{P} \\ &\Leftrightarrow \|\omega_{\widehat{u}_\alpha^* \xi} - \omega_\xi\| \rightarrow 0 \quad \forall \xi \in \mathcal{P} \\ &\Leftrightarrow \|\widehat{u}_\alpha^* \xi - \xi\| \rightarrow 0 \quad \forall \xi \in \mathcal{P} \\ &\Leftrightarrow \widehat{u}_\alpha^* \xrightarrow{\text{ sot }} 1 \Leftrightarrow \widehat{u}_\alpha \xrightarrow{\text{ sot }} 1 \Leftrightarrow \widehat{u}_\alpha \xrightarrow{w^*} 1. \end{aligned}$$

Hence, Ψ is a homeomorphism. ■

Our next result is a generalization of the Heisenberg commutation relation, which has been known to hold for Kac algebras [8, Corollary 4.6.6].

Theorem 3.8 *Let \mathbb{G} be a locally compact quantum group. Let $u \in \text{Gr}(\mathbb{G})$ and $\widehat{u} \in \text{Gr}(\widehat{\mathbb{G}})$. Then there exists $\lambda \in \mathbb{T}$ such that*

$$u\widehat{u} = \lambda\widehat{u}u.$$

Proof One can obtain similar results as [14, Theorem 5.1 and Corollary 5.3] for left multipliers, using same arguments, which then together with Theorems 2.1 and 3.7 imply that $\text{Ad}(u)$ and $\text{Ad}(\widehat{u})$ commute as operators on $\mathcal{B}(L^2(\mathbb{G}))$. Therefore we have

$$\text{Ad}(u)\text{Ad}(\widehat{u}) = \text{Ad}(\widehat{u})\text{Ad}(u) \Rightarrow \text{Ad}(u\widehat{u}u^*\widehat{u}^*) = \iota \Rightarrow u\widehat{u}u^*\widehat{u}^* \in \mathbb{C}1,$$

which yields the conclusion. ■

Next theorem is as well a generalization of a result known in the Kac algebra case [8, Theorem 3.6.10]. The latter proof uses boundedness of the antipode (which does not hold in the case of general locally compact quantum groups) in an essential way. In [15, Theorem 3.2.11] we proved this result for the general case of locally compact quantum groups. Here we present a new proof which is also shorter.

For a Banach algebra \mathcal{A} we denote by $\text{sp}(\mathcal{A})$ its spectrum, *i.e.*, the set of all non-zero bounded multiplicative linear functionals on \mathcal{A} .

Theorem 3.9 *Let \mathbb{G} be a locally compact quantum group. Then we have*

$$\text{Gr}(\mathbb{G}) = \text{sp}(L^1(\mathbb{G})).$$

Proof Let $x \in \text{Gr}(\mathbb{G})$. Then $\Gamma(x) = x \otimes x$ and $x \neq 0$, and for $\omega, \omega' \in L^1(\mathbb{G})$ we have

$$\langle \omega \star \omega', x \rangle = \langle \omega \otimes \omega', \Gamma(x) \rangle = \langle \omega \otimes \omega', x \otimes x \rangle = \langle \omega, x \rangle \langle \omega', x \rangle,$$

which implies $x \in \text{sp}(L^1(\mathbb{G}))$. Hence, $\text{Gr}(\mathbb{G}) \subseteq \text{sp}(L^1(\mathbb{G}))$.

To show the inverse inclusion, first let $x \in \text{sp}(L^1(\mathbb{G})) \cap L^\infty(\mathbb{G})^+$. Then $x^{is} \in \text{Gr}(\mathbb{G})$ for all $s \in \mathbb{R}$, since

$$\begin{aligned} \Gamma(x^{is}) &= (x \otimes x)^{is} = e^{is \log(x \otimes x)} = e^{is(\log(x \otimes 1) + \log(1 \otimes x))} \\ &= e^{is \log(x \otimes 1)} \cdot e^{is \log(1 \otimes x)} = (e^{is \log x} \otimes 1)(1 \otimes e^{is \log x}) \\ &= e^{is \log x} \otimes e^{is \log x} = x^{is} \otimes x^{is}. \end{aligned}$$

So $\tau_t(x^{is}) = x^{is}$ for all $s, t \in \mathbb{R}$. Therefore, by the equations (2.3), there exists $c \in \mathbb{C}$ such that

$$\sigma_t^\varphi(x)^{is} = \sigma_t^\varphi(x^{is}) = c^s x^{is} = (c^{-i} x)^{is}$$

for all $s, t \in \mathbb{R}$, which implies $\sigma_t^\varphi(x) = c^{-i} x$ for all $t \in \mathbb{R}$. Since $x \geq 0$ and σ_t^φ in a $*$ -automorphism, we have $\sigma_t^\varphi(x) = x$ for all $t \in \mathbb{R}$. This yields, by [25, Theorem 2.6], that x is a multiplier of \mathcal{M}_φ , and we have $\varphi(ax) = \varphi(xa)$ for all $a \in \mathcal{M}_\varphi$.

Since φ is n.s.f., there exists $a \in \mathcal{M}_\varphi$ such that $\varphi(ax) = 1$, and we then have

$$1 = \varphi(ax)1 = (\iota \otimes \varphi)\Gamma(ax) = (\iota \otimes \varphi)(\Gamma(a)(x \otimes x)) = ((\iota \otimes \varphi)(\Gamma(a)(1 \otimes x)))x.$$

Hence, x has a left inverse. Similarly we can show that x has also a right inverse, and therefore x is invertible.

Now, let $x \in \text{sp}(L^1(\mathbb{G}))$. Then both xx^* and x^*x are in $\text{sp}(L^1(\mathbb{G}))$, whence invertible by the above, which implies that x is invertible. ■

If $\tilde{\Gamma}: \mathcal{B}(L^2(\mathbb{G})) \rightarrow \mathcal{B}(L^2(\mathbb{G})) \otimes \mathcal{B}(L^2(\mathbb{G}))$ is the extension of Γ via the fundamental unitary W , then $\tilde{\Gamma}_*$ defines a product on $\mathcal{B}(L^2(\mathbb{G}))_*$, which turns the latter to a completely contractive Banach algebra. We denote this Banach algebra by $\mathcal{J}_*(\mathbb{G})$.

Corollary 3.10 *We have $\text{Gr}(\mathcal{B}(L^2(\mathbb{G})), \tilde{\Gamma}) = \text{sp}(\mathcal{J}_*(\mathbb{G}))$.*

Proof The inclusion $\text{Gr}(\mathcal{B}(L^2(\mathbb{G})), \Gamma) \subseteq \text{sp}(\mathcal{T}_*(\mathbb{G}))$ is obvious. To see the converse, let $x \in \text{sp}(\mathcal{T}_*(\mathbb{G}))$, i.e., $\Gamma(x) = x \otimes x$ and $x \neq 0$. Then we have $x \in L^\infty(\mathbb{G})$, and so, by Theorem 3.9, $x \in \text{Gr}(\mathbb{G})$, which is equal to $\text{Gr}(\mathcal{B}(L^2(\mathbb{G})), \Gamma)$ by Proposition 3.6. ■

Theorem 3.11 *Let \mathbb{G} be a locally compact quantum group. Then we have a group homeomorphism*

$$\text{sp}(L^1(\widehat{\mathbb{G}})) \cong \text{sp}(C_u(\mathbb{G})).$$

Proof Let $\phi \in \text{sp}(C_u(\mathbb{G}))$. Then for all $\widehat{\omega}_1, \widehat{\omega}_2 \in L_*^1(\widehat{\mathbb{G}})$ we have

$$\begin{aligned} \langle \widehat{\omega}_1 \star \widehat{\omega}_2, \lambda(\phi) \rangle &= \langle \widehat{\lambda}_u(\widehat{\omega}_1 \star \widehat{\omega}_2), \phi \rangle = \langle \widehat{\lambda}_u(\widehat{\omega}_1) \widehat{\lambda}_u(\widehat{\omega}_2), \phi \rangle \\ &= \langle \widehat{\lambda}_u(\widehat{\omega}_1), \phi \rangle \langle \widehat{\lambda}_u(\widehat{\omega}_2), \phi \rangle = \langle \widehat{\omega}_1, \lambda(\phi) \rangle \langle \widehat{\omega}_2, \lambda(\phi) \rangle. \end{aligned}$$

Since $L_*^1(\widehat{\mathbb{G}})$ is norm dense in $L^1(\widehat{\mathbb{G}})$, and $\langle L^1(\widehat{\mathbb{G}}) \star L^1(\widehat{\mathbb{G}}) \rangle$ is norm dense in $L^1(\widehat{\mathbb{G}})$, we have that $\langle L_*^1(\widehat{\mathbb{G}}) \star L_*^1(\widehat{\mathbb{G}}) \rangle$ is norm dense in $L^1(\widehat{\mathbb{G}})$. Hence, $\lambda(\phi) \in \text{sp}(L^1(\widehat{\mathbb{G}}))$.

Now, let $\widehat{x} \in \text{sp}(L^1(\widehat{\mathbb{G}}))$. Then $\widehat{x} \in \text{Gr}(\widehat{\mathbb{G}})$, by Theorem 3.9, and so we have $\widehat{S}(\widehat{x}) = \widehat{x}^*$. Therefore we get $\langle \widehat{\omega}^*, \widehat{x} \rangle = \langle \widehat{\omega}, \widehat{S}(\widehat{x})^* \rangle = \overline{\langle \widehat{\omega}, \widehat{x} \rangle}$ for all $\widehat{\omega} \in L_*^1(\widehat{\mathbb{G}})$. Hence, the map $\langle \cdot, \widehat{x} \rangle: L_*^1(\widehat{\mathbb{G}}) \rightarrow \mathbb{C}$ is a non-zero $*$ -homomorphism, and so by the universality of $C_u(\mathbb{G})$, we obtain a $*$ -homomorphism $\theta_{\widehat{x}}: C_u(\mathbb{G}) \rightarrow \mathbb{C}$ such that $\langle \widehat{\lambda}_u(\widehat{\omega}), \theta_{\widehat{x}} \rangle = \langle \widehat{\omega}, \widehat{x} \rangle$ for all $\widehat{\omega} \in L_*^1(\widehat{\mathbb{G}})$.

We show that the induced maps $\text{sp}(C_u(\mathbb{G})) \ni \phi \mapsto \lambda(\phi) \in \text{sp}(L^1(\widehat{\mathbb{G}}))$ and $\text{sp}(L^1(\widehat{\mathbb{G}})) \ni \widehat{x} \mapsto \theta_{\widehat{x}} \in \text{sp}(C_u(\mathbb{G}))$ are inverses to each other. Let $\widehat{x} \in \text{sp}(L^1(\widehat{\mathbb{G}}))$, then we have

$$\langle \widehat{\omega}, \lambda(\theta_{\widehat{x}}) \rangle = \langle \widehat{\lambda}_u(\widehat{\omega}), \theta_{\widehat{x}} \rangle = \langle \widehat{\omega}, \widehat{x} \rangle$$

for all $\widehat{\omega} \in L_*^1(\widehat{\mathbb{G}})$, which yields, by density of $L_*^1(\widehat{\mathbb{G}})$ in $L^1(\widehat{\mathbb{G}})$, that $\lambda(\theta_{\widehat{x}}) = \widehat{x}$. Conversely, assume that $\phi \in \text{sp}(C_u(\mathbb{G}))$, then we have

$$\langle \widehat{\lambda}_u(\widehat{\omega}), \theta_{\lambda(\phi)} \rangle = \langle \widehat{\omega}, \lambda(\phi) \rangle = \langle \widehat{\lambda}_u(\widehat{\omega}), \phi \rangle$$

for all $\widehat{\omega} \in L_*^1(\widehat{\mathbb{G}})$. The density of $\widehat{\lambda}_u(L_*^1(\widehat{\mathbb{G}}))$ in $C_u(\mathbb{G})$ implies that $\theta_{\lambda(\phi)} = \phi$.

Since $\lambda: C_u(\mathbb{G})^* \rightarrow L^\infty(\widehat{\mathbb{G}})$ is an algebra homomorphism, the map

$$\text{sp}(C_u(\mathbb{G})) \ni \phi \mapsto \lambda(\phi) \in \text{sp}(L^1(\widehat{\mathbb{G}}))$$

defines a bijective group homomorphism. Moreover, for a net (ϕ_α) in $\text{sp}(C_u(\mathbb{G}))$ and $\phi \in \text{sp}(C_u(\mathbb{G}))$, the density of $L_*^1(\widehat{\mathbb{G}})$ and $\widehat{\lambda}_u(L_*^1(\widehat{\mathbb{G}}))$ in $L^1(\widehat{\mathbb{G}})$ and $C_u(\mathbb{G})$, respectively, yield

$$\begin{aligned} \phi_\alpha \xrightarrow{w^*} \phi &\iff \langle \widehat{\lambda}_u(\widehat{\omega}), \phi_\alpha \rangle \longrightarrow \langle \widehat{\lambda}_u(\widehat{\omega}), \phi \rangle \quad \forall \widehat{\omega} \in L_*^1(\widehat{\mathbb{G}}) \\ &\iff \langle \widehat{\omega}, \lambda(\phi_\alpha) \rangle \longrightarrow \langle \widehat{\omega}, \lambda(\phi) \rangle \quad \forall \widehat{\omega} \in L_*^1(\widehat{\mathbb{G}}) \iff \lambda(\phi_\alpha) \xrightarrow{w^*} \lambda(\phi). \end{aligned}$$

Hence, the map $\text{sp}(C_u(\mathbb{G})) \ni \phi \mapsto \lambda(\phi) \in \text{sp}(L^1(\widehat{\mathbb{G}}))$ is a group homeomorphism. ■

Next theorem combines all the above identifications.

Theorem 3.12 *The following can be identified as locally compact groups:*

- (i) $\tilde{\mathbb{G}}$ with the strong operator topology;
- (ii) $\text{Gr}(\widehat{\mathbb{G}})$ with the weak* topology;
- (iii) $\text{sp}(L^1(\widehat{\mathbb{G}}))$ with the weak* topology;
- (iv) $\text{Gr}(\mathcal{B}(L^2(\mathbb{G})), \widehat{\Gamma})$ with the weak* topology;
- (v) $\text{sp}(\mathcal{T}_*(\widehat{\mathbb{G}}))$ with the weak* topology;
- (vi) $\text{sp}(C_u(\mathbb{G}))$ with the weak* topology.

Proof Since the spectrum of a Banach algebra is locally compact with weak* topology, all the above groups are locally compact groups. ■

Remark 3.13 Applying Theorem 3.12 to the case where $\mathbb{G} = \text{VN}(G)$ for a locally compact group G , we obtain a generalization of a Renault’s result (cf. [21, Theorem 2]), in which G is assumed amenable.

Theorem 3.14 *The assignment $\mathbb{G} \rightarrow \tilde{\mathbb{G}}$ preserves compactness, discreteness, and hence finiteness.*

Proof Let \mathbb{G} be compact. Then $\widehat{\mathbb{G}}$ is discrete, and in view of Theorem 3.12, we may equivalently show that $\text{Gr}(\widehat{\mathbb{G}})$ is compact. Let $\widehat{e} \in L^1(\widehat{\mathbb{G}})$ be the unit. Then, for any $\widehat{x} \in \text{Gr}(\widehat{\mathbb{G}})$, we have

$$\langle \widehat{f}, \widehat{x} \rangle = \langle \widehat{f} \star \widehat{e}, \widehat{x} \rangle = \langle \widehat{f} \otimes \widehat{e}, \widehat{\Gamma}(\widehat{x}) \rangle = \langle \widehat{f} \otimes \widehat{e}, \widehat{x} \otimes \widehat{x} \rangle = \langle \widehat{f}, \widehat{x} \rangle \langle \widehat{e}, \widehat{x} \rangle$$

for all $\widehat{f} \in L^1(\widehat{\mathbb{G}})$. So $\langle \widehat{e}, \widehat{x} \rangle = 1$ for all $\widehat{x} \in \widehat{\mathbb{G}}$ and since $\text{Gr}(\widehat{\mathbb{G}}) = \text{sp}(L^1(\widehat{\mathbb{G}}))$ by Theorem 3.9, this implies that the constant function $\widehat{e}|_{\text{Gr}(\widehat{\mathbb{G}})} \equiv 1$ lies in $C_0(\text{Gr}(\widehat{\mathbb{G}}))$. Therefore $\text{Gr}(\widehat{\mathbb{G}})$ is compact.

Now let \mathbb{G} be discrete. Then $\widehat{\mathbb{G}}$ is compact, and again by Theorem 3.12 we need to show that $\text{Gr}(\widehat{\mathbb{G}})$ is discrete. Let $\widehat{x} \in \text{Gr}(\widehat{\mathbb{G}})$ and $\widehat{\varphi} \in L^1(\widehat{\mathbb{G}})$ be the Haar state. Then we have

$$\langle \widehat{f}, 1 \rangle \langle \widehat{\varphi}, \widehat{x} \rangle = \langle \widehat{f} \star \widehat{\varphi}, \widehat{x} \rangle = \langle \widehat{f} \otimes \widehat{\varphi}, \widehat{x} \otimes \widehat{x} \rangle = \langle \widehat{f}, \widehat{x} \rangle \langle \widehat{\varphi}, \widehat{x} \rangle$$

for all $\widehat{f} \in L^1(\widehat{\mathbb{G}})$. So if $\widehat{x} \neq 1$, we must have $\langle \widehat{\varphi}, \widehat{x} \rangle = 0$, and since $\langle \widehat{\varphi}, 1 \rangle = 1$, we see that $\widehat{\varphi}$, as a function on $\text{Gr}(\widehat{\mathbb{G}})$ is the characteristic function of $\{1\}$. But since $\widehat{\varphi}$ is continuous on $\text{Gr}(\widehat{\mathbb{G}})$, the latter must be discrete. ■

In the following (Theorem 3.16), we shall investigate the relation between the operations $\mathbb{G} \rightarrow \tilde{\mathbb{G}}$ and $\mathbb{G} \rightarrow \widehat{\mathbb{G}}$.

Lemma 3.15 *Let G and H be two locally compact groups in duality, i.e., there exists a continuous bi-homomorphism $\langle \cdot, \cdot \rangle: G \times H \rightarrow \mathbb{T}$, where \mathbb{T} denotes the unit circle. Define the sets*

$$G_1 := \{g \in G : \langle g, H \rangle = 1\},$$

$$H_1 := \{h \in H : \langle G, h \rangle = 1\}.$$

Then G_1 and H_1 are closed normal subgroups of G and H , containing the commutator subgroups, and we have

$$\frac{G}{G_1} \cong \widehat{\left(\frac{H}{H_1}\right)}.$$

Proof For all $g \in G, g_1 \in G_1$ and $h \in H$ we have

$$\langle g^{-1}g_1g, h \rangle = \langle g^{-1}, h \rangle \langle g_1, h \rangle \langle g, h \rangle = \langle g^{-1}, h \rangle \langle g, h \rangle = \langle e, h \rangle = 1.$$

Therefore, G_1 is normal in G . For all $g_1, g_2 \in G$ and $h \in H$ we have

$$\langle g_1g_2g_1^{-1}g_2^{-1}, h \rangle = \langle g_1, h \rangle \langle g_2, h \rangle \langle g_1^{-1}, h \rangle \langle g_2^{-1}, h \rangle = 1.$$

Thus, $[G, G] \subseteq G_1$; similarly, we see that H_1 is normal in H , and $[H, H] \subseteq H_1$. Now it just remains to show the last assertion. Define

$$\phi: G \rightarrow \widehat{\left(\frac{H}{H_1}\right)}, \phi(g)(\bar{h}) = \langle g, h \rangle.$$

The definition of H_1 implies that $\phi(g)$ is well-defined for each $g \in G$. Obviously, ϕ is a group homomorphism, and we have $\text{Ker}(\phi) = G_1$. Hence we have an injective group homomorphism

$$\bar{\phi}: \frac{G}{G_1} \hookrightarrow \widehat{\left(\frac{H}{H_1}\right)}.$$

Similarly, by exchanging the roles of G and H we obtain

$$\frac{H}{H_1} \hookrightarrow \widehat{\left(\frac{G}{G_1}\right)}$$

whence

$$\widehat{\left(\frac{G}{G_1}\right)} \twoheadrightarrow \widehat{\left(\frac{H}{H_1}\right)}.$$

If we compose the last surjection with the identification of $\frac{G}{G_1}$ with its second dual, we get $\bar{\phi}$. Hence $\bar{\phi}$ is onto, and

$$\frac{G}{G_1} \cong \widehat{\left(\frac{H}{H_1}\right)}. \quad \blacksquare$$

It follows from Theorem 3.8 and Lemma 3.15 that $\text{Gr}(\mathbb{G}) \cap \text{Gr}(\widehat{\mathbb{G}})'$ is a normal subgroup of $\text{Gr}(\mathbb{G})$, and

$$\frac{\text{Gr}(\mathbb{G})}{\text{Gr}(\mathbb{G}) \cap \text{Gr}(\widehat{\mathbb{G}})'}$$

is an abelian group. In the following we denote this group by $\widetilde{\mathbb{G}}_1$.

Theorem 3.16 *Let \mathbb{G} be a locally compact quantum group. Then we have a group homeomorphism*

$$\widehat{\mathbb{G}}_1 \cong \widetilde{\mathbb{G}}_1.$$

Proof By Theorem 3.8 we have a duality between $\widetilde{\mathbb{G}}$ and $\widehat{\mathbb{G}}$. Hence, the theorem follows from Lemma 3.15. ■

In the above, to a locally compact quantum group \mathbb{G} , we have assigned the locally compact group $\widetilde{\mathbb{G}}$, which is easily seen to be an invariant for \mathbb{G} . We shall now assign another invariant to \mathbb{G} .

Let \mathbb{G} be a locally compact quantum group, $\nu \in \text{Gr}(\mathbb{G})$ and $\widehat{\nu} \in \text{Gr}(\widehat{\mathbb{G}})$. By Theorem 3.8, there exists $\lambda_{\nu, \widehat{\nu}} \in \mathbb{T}$ such that $\widehat{\nu\widehat{\nu}} = \lambda_{\nu, \widehat{\nu}}\widehat{\nu\nu}$. It is easy to see that $(\nu, \widehat{\nu}) \mapsto \lambda_{\nu, \widehat{\nu}}$ defines a bi-homomorphism $\gamma: \text{Gr}(\mathbb{G}) \times \text{Gr}(\widehat{\mathbb{G}}) \rightarrow \mathbb{T}$.

Proposition 3.17 *Let \mathbb{G} be a locally compact quantum group. Then $\text{Im}(\gamma)$, the image of γ , is a subgroup of \mathbb{T} , and an invariant for \mathbb{G} .*

Proof Using the above notation, we have a bi-character $\gamma_1: \widetilde{\mathbb{G}}_1 \times \widetilde{\mathbb{G}}_1 \rightarrow \mathbb{T}$, induced by γ , with $\text{Im}(\gamma_1) = \text{Im}(\gamma)$ (see the proof of Lemma 3.15). Since $\widetilde{\mathbb{G}}_1$ and $\widehat{\mathbb{G}}_1$ are abelian, by the universal property of the tensor product, there exists a homomorphism $\gamma_2: \widetilde{\mathbb{G}}_1 \otimes_{\mathbb{Z}} \widehat{\mathbb{G}}_1 \rightarrow \mathbb{T}$, with $\text{Im}(\gamma_2) = \text{Im}(\gamma_1)$. But $\text{Im}(\gamma_2)$ is a subgroup of \mathbb{T} since γ_2 is a group homomorphism. ■

Since there is a good classification of subgroups of \mathbb{T} (cf. [11, Theorem 25.13]), this invariant may be helpful towards some sort of classification of locally compact quantum groups.

4 Examples

In this section we calculate the locally compact group $\widetilde{\mathbb{G}}$ for some of the most interesting and well-known examples of non-classical, non-Kac, locally compact quantum groups.

4.1 Woronowicz’s Compact Matrix Pseudogroups

Let A be a C^* -algebra with unit, $U_N = [u_{ij}]$ an $N \times N$ ($N \in \mathbb{N}$) matrix with entries belonging to A , and \mathcal{A} be the $*$ -subalgebra of A generated by the entries of U_N . Then $\mathbb{G} = (A, U_N)$ is called a *compact matrix pseudogroup* [28, Definition 1.1] if the following hold:

- (i) \mathcal{A} is dense in A ;
- (ii) there exists a C^* -homomorphism Γ from A to $A \otimes_{\min} A$ such that

$$\Gamma(u_{ij}) = \sum_{k=1}^N u_{ik} \otimes u_{kj} \quad (i, j = 1, 2, \dots, N);$$

(iii) there exists a linear anti-multiplicative map $\kappa: \mathcal{A} \rightarrow \mathcal{A}$ such that

$$\kappa(\kappa(a^*)^*) = a$$

for all $a \in A$, and

$$\sum_k \kappa(u_{ik})u_{kj} = \delta_{ij}1,$$

$$\sum_k u_{ik}\kappa(u_{kj}) = \delta_{ij}1,$$

for all $i, j = 1, 2, \dots, N$.

Theorem 4.1 *Let $\mathbf{G} = (A, U_N)$ be a compact matrix pseudogroup. Then $\tilde{\mathbb{G}}$ is homeomorphic to a compact subgroup of $GL_N(\mathbb{C})$, hence a compact Lie group.*

Proof Define the map

$$\Phi: \text{sp}(A) \rightarrow M_N(\mathbb{C}), \quad f \mapsto [f(u_{ij})]_{ij}.$$

Then Φ is injective since $\{u_{ij}\}$ generates A . For all $f, g \in \text{sp}(A)$, we have

$$\begin{aligned} \Phi(f \star g) &= [f \star g(u_{ij})]_{ij} = [(f \otimes g)\Gamma(u_{ij})]_{ij} = \left[(f \otimes g) \sum_{k=1}^N (u_{ik} \otimes u_{kj}) \right]_{ij} \\ &= \left[\sum_{k=1}^N f(u_{ik})g(u_{kj}) \right]_{ij} = [f(u_{ij})]_{ij}[g(u_{ij})]_{ij} = \Phi(f)\Phi(g). \end{aligned}$$

So, Φ is an injective group homomorphism. Obviously $\text{Im}(\Phi) \subseteq GL_N(\mathbb{C})$.

Since each of the maps $f \mapsto f(u_{ij})$ is continuous, Φ is also continuous. By Theorem 3.14, $\tilde{\mathbb{G}}$ is compact, and therefore Φ is a homeomorphism onto its image. ■

4.2 $SU_\mu(2)$

In the following, we consider Woronowicz’s twisted $SU_\mu(2)$ quantum group for $\mu \in (-1, 1)$ and $\mu \neq 0$ (cf. [31]). The quantum group $SU_\mu(2)$ is a co-amenable compact matrix pseudogroup with the quantum group C^* -algebra $C(SU_\mu(2)) = C_u(SU_\mu(2))$ generated by two operators u and v such that the matrix

$$U = \begin{bmatrix} u & -\mu v^* \\ v & u^* \end{bmatrix}$$

is a unitary matrix in $M_2(C(SU_\mu(2)))$.

Theorem 4.2 *Let $\mathbb{G} = SU_\mu(2)$. Then $\tilde{\mathbb{G}} = \mathbb{T}$.*

Proof Let $\Phi: \text{sp}(C(\text{SU}_\mu(2))) \rightarrow \text{GL}_2(\mathbb{C})$ be as in the proof of Theorem 4.1, and $f \in \text{sp}(C(\text{SU}_\mu(2)))$. It is easy to verify that $\mathbb{T} \subseteq \text{Im}(\Phi)$, under the identification of \mathbb{T} with the matrices of the form

$$\Phi(f) = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$$

where $\lambda \in \mathbb{T}$. To see that any element in $\text{Im}(\Phi)$ is of this form, note that $\Phi(f)$ is a unitary matrix in $M_2(\mathbb{C})$ for all $f \in \text{sp}(C(\text{SU}_\mu(2)))$. But since

$$\Phi(f) = \begin{pmatrix} f(u) & -\mu \overline{f(v)} \\ f(v) & \overline{f(u)} \end{pmatrix},$$

it follows that $f(v) = 0$, and the theorem follows. ■

4.3 $E_\mu(2)$ and its Dual

Let $(e_{k,l})_{k,l \in \mathbb{Z}}$ be the canonical basis for $\ell^2(\mathbb{Z} \times \mathbb{Z})$. Define operators v and n on $\ell^2(\mathbb{Z} \times \mathbb{Z})$ as follows:

$$\begin{cases} ve_{k,l} = e_{k-1,l} \\ ne_{k,l} = \mu^k e_{k,l+1}. \end{cases}$$

Then v is a unitary and n is a normal operator with $\text{sp}(n) \subseteq \overline{\mathbb{C}}^\mu$, where

$$\overline{\mathbb{C}}^\mu := \{z \in \mathbb{C} : z = 0 \text{ or } |z| \in \mu^{\mathbb{Z}}\}.$$

The (C^* -algebraic) locally compact quantum group $E_\mu(2)$ is defined (cf. [30, Section 1]) to be the non-unital C^* -algebra generated by the operators of the form $\Sigma v^k f_k(n)$, where k runs over a finite set of integers, and $f_k \in C_0(\overline{\mathbb{C}}^\mu)$. The comultiplication Γ is defined on $E_\mu(2)$ in the following way:

$$\begin{cases} \Gamma(v) = v \otimes v \\ \Gamma(n) = v \otimes n + n \otimes v. \end{cases}$$

Using [30, Theorem 1.1], we can calculate $\widetilde{\mathbb{G}}$ for the quantum group $E_\mu(2)$.

Theorem 4.3 *Let $\mathbb{G} = E_\mu(2)$. Then $\widetilde{\mathbb{G}} \cong \mathbb{T}$.*

Proof In view of [30, Theorem 1.1], for any $z \in \mathbb{T}$ we can define $f_z \in \text{sp}(C_0(E_\mu(2)))$ such that $f_z(v) = z$ and $f_z(n) = 0$. Conversely, for each $f \in \text{sp}(C_0(E_\mu(2)))$, by [30, Theorem 1.1] again, $\overline{f(v)}$ is a unitary and $f(n)$ is a normal operator on \mathbb{C} , and we have $f(n) = f(v)f(n)\overline{f(v)} = \mu f(n)$. Since $\mu \neq 1$, we must have $f(n) = 0$. Put $z := f(v) \in \mathbb{T}$. So $f = f_z$. Moreover, for $z, z' \in \mathbb{T}$ we have

$$\begin{aligned} f_z \star f_{z'}(v) &= (f_z \otimes f_{z'})\Gamma(v) = (f_z \otimes f_{z'})(v \otimes v) \\ &= f_z(v)f_{z'}(v) = zz' = f_{zz'}(v), \\ f_z \star f_{z'}(n) &= (f_z \otimes f_{z'})\Gamma(n) = (f_z \otimes f_{z'})(v \otimes n + n \otimes v) \\ &= f_z(v)f_{z'}(n) + f_z(n)f_{z'}(v) = 0 = f_{zz'}(n). \end{aligned}$$

Hence $f_z \star f_{z'} = f_{zz'}$. ■

In [30] Woronowicz has also described $\widehat{E}_\mu(2)$, the dual quantum group of $E_\mu(2)$. Similarly to $E_\mu(2)$, this quantum group is determined by two operators N and b , with co-multiplication determined by $\widehat{\Gamma}(N) = N \otimes 1 + 1 \otimes N$, $\widehat{\Gamma}(b) = b \otimes \mu^{\frac{N}{2}} + \mu^{-\frac{N}{2}} \otimes b$. The dual quantum group $\widehat{E}_\mu(2)$ has a universal property as well, which makes it easy to calculate our group.

Theorem 4.4 *Let $\mathbb{G} = \widehat{E}_\mu(2)$. Then $\widetilde{\mathbb{G}} \cong \mathbb{Z}$.*

Proof For any $s \in \mathbb{Z}$ we define $\widehat{f}_s \in \text{sp}(C_0(\widehat{E}_\mu(2)), \mathbb{C})$ by $\widehat{f}_s(b) = 0$, $\widehat{f}_s(N) = s$. Then it is clear from [30, Theorem 3.1] that the map $s \mapsto \widehat{f}_s \in \text{sp}(C_0(\widehat{E}_\mu(2)))$ is a bi-continuous bijection. For $s, s' \in \mathbb{Z}$ we have

$$\begin{aligned} \widehat{f}_s \star \widehat{f}_{s'}(b) &= (\widehat{f}_s \otimes \widehat{f}_{s'})\widehat{\Gamma}(b) = (\widehat{f}_s \otimes \widehat{f}_{s'})(b \otimes \mu^{\frac{N}{2}} + \mu^{-\frac{N}{2}} \otimes b) \\ &= \widehat{f}_s(b)\widehat{f}_{s'}(\mu^{\frac{N}{2}}) + \widehat{f}_s(\mu^{-\frac{N}{2}})\widehat{f}_{s'}(b) = 0 = \widehat{f}_{s+s'}(b), \\ \widehat{f}_s \star \widehat{f}_{s'}(N) &= (\widehat{f}_s \otimes \widehat{f}_{s'})\widehat{\Gamma}(N) = (\widehat{f}_s \otimes \widehat{f}_{s'})(N \otimes 1 + 1 \otimes N) \\ &= \widehat{f}_s(N)\widehat{f}_{s'}(1) + \widehat{f}_s(1)\widehat{f}_{s'}(N) = s + s' = \widehat{f}_{s+s'}(N). \end{aligned}$$

So $\widehat{f}_s \star \widehat{f}_{s'} = \widehat{f}_{s+s'}$. ■

5 Structural Properties of \mathbb{G} encoded by $\widetilde{\mathbb{G}}$

In this section, we investigate the relation between the structure of \mathbb{G} and that of $\widetilde{\mathbb{G}}$.

5.1 Unimodularity of \mathbb{G}

Let \mathbb{G} be a locally compact quantum group. The main goal of this section (Theorem 5.8) is to show that if both $\widetilde{\mathbb{G}}$ and $\widehat{\widetilde{\mathbb{G}}}$ are small, then \mathbb{G} is of a very specific type, namely a unimodular Kac algebra.

In the sequel, for a locally compact group G , we denote by $\mathbf{Z}(G)$ the center of the group G .

Proposition 5.1 *Let \mathbb{G} be a locally compact quantum group. If $\mathbf{Z}(\text{Gr}(\mathbb{G}))$ is discrete, then \mathbb{G} is unimodular.*

Proof By [2, Proposition 4.2.] we have $\delta^{it} \in \mathbf{Z}(\text{Gr}(\mathbb{G}))$ for all $t \in \mathbb{R}$, where δ is the modular element of \mathbb{G} , and $\mathbf{Z}(\text{Gr}(\mathbb{G}))$ is the center of the intrinsic group $\text{Gr}(\mathbb{G})$. Since the map

$$\mathbb{R} \ni t \mapsto \delta^{it} \in \mathbf{Z}(\text{Gr}(\mathbb{G}))$$

is continuous, its range must be connected. But since $\mathbf{Z}(\text{Gr}(\mathbb{G}))$ is discrete, the range must be a single point. Therefore, we obtain $\delta^{it} = 1$ for all $t \in \mathbb{R}$, which implies $\delta = 1$. ■

Combining Proposition 5.1 with Theorem 4.4, we obtain the following.

Corollary 5.2 *The quantum group $E_\mu(2)$ is unimodular.*

Lemma 5.3 *Let Φ and Ψ be weak* continuous linear maps on $L^\infty(\mathbb{G})$. If we have $(\iota \otimes \Phi) \circ \Gamma = (\iota \otimes \Psi) \circ \Gamma$ or $(\Phi \otimes \iota) \circ \Gamma = (\Psi \otimes \iota) \circ \Gamma$, then $\Phi = \Psi$.*

Proof Assume that $(\iota \otimes \Phi) \circ \Gamma = (\iota \otimes \Psi) \circ \Gamma$. Then, for all $x \in L^\infty(\mathbb{G})$ and $\omega \in L^1(\mathbb{G})$, we have $\Phi((\omega \otimes \iota)\Gamma(x)) = \Psi((\omega \otimes \iota)\Gamma(x))$. Since the set $\{(\omega \otimes \iota)\Gamma(x) : \omega \in L^1(\mathbb{G}), x \in L^\infty(\mathbb{G})\}$ is weak* dense in $L^\infty(\mathbb{G})$, the conclusion follows. The argument assuming the second relation is analogous. ■

Lemma 5.4 *If $\delta = 1$ and $\sigma_t^\varphi = \tau_t$ for all $t \in \mathbb{R}$, then $\tau_t = \sigma_t^\varphi = \iota$ for all $t \in \mathbb{R}$, and \mathbb{G} is a Kac algebra.*

Proof Since $\delta = 1$, we have $\sigma_t^\psi = \sigma_t^\varphi$ for all $t \in \mathbb{R}$. Moreover, since $\sigma_t^\varphi = \tau_t$, by the equations (2.2) and 2.3, we have

$$(\sigma_t^\varphi \otimes \sigma_{-t}^\psi) \circ \Gamma = \Gamma \circ \tau_t = (\tau_t \otimes \tau_t) \circ \Gamma = (\sigma_t^\varphi \otimes \sigma_t^\psi) \circ \Gamma,$$

which implies that $(\iota \otimes \sigma_{-t}^\psi) \circ \Gamma = (\iota \otimes \sigma_t^\psi) \circ \Gamma$ for all $t \in \mathbb{R}$. Now, Lemma 5.3 yields $\sigma_{-t}^\psi = \sigma_t^\psi$, i.e., $\sigma_{2t}^\psi = \iota$, for all $t \in \mathbb{R}$. Hence, $\tau_t = \sigma_t^\varphi = \sigma_t^\psi = \iota$ for all $t \in \mathbb{R}$, and therefore \mathbb{G} is a Kac algebra. ■

Proposition 5.5 *If \mathbb{G} and $\widehat{\mathbb{G}}$ are both unimodular, then \mathbb{G} is a Kac algebra.*

Proof Since $\widehat{\delta} = 1$, equations (2.1) imply that $P^{it} = \Delta_\varphi^{it}$, whence $\sigma_t^\varphi = \tau_t$ for all $t \in \mathbb{R}$. Since, in addition, $\delta = 1$, Lemma 5.4 yields the claim. ■

In particular, combining Propositions 5.1 and 5.5, we see that for a unimodular locally compact quantum group \mathbb{G} , the smallness of the group $\widetilde{\mathbb{G}}$ forces the quantum group \mathbb{G} to be of Kac type.

Corollary 5.6 *Let \mathbb{G} be a unimodular locally compact quantum group. If $\mathbf{Z}(\widetilde{\mathbb{G}})$ is discrete, then \mathbb{G} is a Kac algebra.*

Since every compact quantum group is unimodular, we obtain the following interesting result.

Corollary 5.7 *Let \mathbb{G} be a compact quantum group. If $\mathbf{Z}(\widetilde{\mathbb{G}})$ is discrete, then \mathbb{G} is a Kac algebra.*

The class of non-Kac compact quantum groups is one of the most important and well-known classes of non-classical quantum groups. Some important examples of such objects are deformations of compact Lie groups, such as Woronowicz’s famous $SU_\mu(2)$, which we have discussed in Section 4.2. This shows the significance of our Corollary 5.7: there is some richness of classical information in these classes of quantum structures!

Theorem 5.8 *Let \mathbb{G} be a locally compact quantum group. If $\mathbf{Z}(\widetilde{\mathbb{G}})$ and $\mathbf{Z}(\widehat{\mathbb{G}})$ are both discrete, then \mathbb{G} is a unimodular Kac algebra.*

Proof Since $\mathbf{Z}(\widetilde{\mathbb{G}})$ is discrete, $\widehat{\mathbb{G}}$ is unimodular by Proposition 5.1, and hence our assertion follows from Corollary 5.6, applied to $\widehat{\mathbb{G}}$. ■

5.2 Traciality of the Haar Weights

There is a strong connection between the group $\widetilde{\mathbb{G}}$, assigned to \mathbb{G} , and traciality of the Haar weights, especially in the Kac algebra case. Since the scaling group τ_t in this case is trivial, equation (2.2) implies

$$\Gamma\sigma_t^\varphi = (\iota \otimes \sigma_t^\varphi)\Gamma \quad \forall t \in \mathbb{R},$$

and so $\sigma_t^\varphi \in \mathcal{CB}_{\text{cov}}^\sigma(L^\infty(\mathbb{G}))$. Thus, we obtain $\Delta_\varphi^{it} \in \text{Gr}(\widehat{\mathbb{G}})$ for all $t \in \mathbb{R}$ (this was also proved in [5, Corollary 2.4]). Hence, we see that if \mathbb{G} is a Kac algebra such that $\widetilde{\mathbb{G}}$ is trivial, then φ is tracial. Similarly to Proposition 5.1, the fact that the map $t \mapsto \Delta_\varphi^{it}$ is continuous, allows us to further generalize this result.

Proposition 5.9 *Let \mathbb{G} be a Kac algebra. If $\widetilde{\mathbb{G}}$ is discrete, then φ is tracial.*

We have seen in Corollary 5.6 that for a unimodular locally compact quantum group \mathbb{G} , the smallness of $\widetilde{\mathbb{G}}$ forces the quantum group to be a Kac algebra. The situation is similar for traciality.

Proposition 5.10 *Let \mathbb{G} be a locally compact quantum group with tracial Haar weight. If $\text{Gr}(\mathbb{G})$ is discrete, then \mathbb{G} is a Kac algebra.*

Proof Since $\text{Gr}(\mathbb{G})$ is discrete, we have $\delta = 1$. Also, traciality of φ implies that

$$\Gamma = \Gamma\sigma_t^\varphi = (\tau_t \otimes \sigma_t^\varphi)\Gamma = (\tau_t \otimes \iota)\Gamma,$$

for all $t \in \mathbb{R}$, which implies that $\tau = \iota$, by Lemma 5.3. Hence, \mathbb{G} is a Kac algebra. ■

Moreover, by combining [8, Proposition 6.1.2] with Theorem 5.8, we obtain a stronger version of the latter.

Theorem 5.11 *Let \mathbb{G} be a locally compact quantum group. If $\mathbf{Z}(\widetilde{\mathbb{G}})$ and $\mathbf{Z}(\widehat{\mathbb{G}})$ are both discrete, then \mathbb{G} is a unimodular Kac algebra with tracial Haar weight.*

5.3 Amenability

In the last part of this section we investigate the question of whether amenability passes from \mathbb{G} to $\widetilde{\mathbb{G}}$. In the following, for a locally compact quantum group \mathbb{G} , we denote by $N_{\mathbb{G}}$ the von Neumann subalgebra of $L^\infty(\mathbb{G})$ generated by the intrinsic group $\text{Gr}(\mathbb{G})$.

Proposition 5.12 *Let \mathbb{G} be a locally compact quantum group such that the restriction $\varphi|_{N_{\mathbb{G}}}$ is semi-finite. Then we have the identification*

$$N_{\mathbb{G}} \cong \text{VN}(\text{Gr}(\mathbb{G})).$$

Proof It is obvious that the restriction of Γ to $N_{\mathbb{G}}$ defines a co-multiplication on $N_{\mathbb{G}}$, and by our assumption, on $N_{\mathbb{G}}$, the restriction of φ to $N_{\mathbb{G}}$ is an n.s.f. left invariant weight on $N_{\mathbb{G}}$. Also, since $R(v) = v^*$ for all $v \in \text{Gr}(\mathbb{G})$, we have $R(N_{\mathbb{G}}) \subseteq N_{\mathbb{G}}$, which implies that $\varphi \circ R$ defines a right Haar weight on $N_{\mathbb{G}}$. Therefore, $N_{\mathbb{G}}$ can be given a locally compact quantum group structure. Obviously, it is co-commutative, and so $N_{\mathbb{G}} \cong \text{VN}(\text{Gr}(\mathbb{G}))$, by [24, Theorem 2]. ■

Corollary 5.13 *If \mathbb{G} is a compact quantum group, then $N_{\mathbb{G}} \cong \text{VN}(\text{Gr}(\mathbb{G}))$.*

Theorem 5.14 *Let \mathbb{G} be a discrete quantum group. If \mathbb{G} is amenable, then so is the group $\widehat{\mathbb{G}}$.*

Proof If $F \in L^\infty(\mathbb{G})^*$ is an invariant mean on \mathbb{G} , then one can see that the map $(F \otimes \iota) \circ \Gamma: \mathcal{B}(L^2(\mathbb{G})) \rightarrow \mathcal{B}(L^2(\mathbb{G}))$ is a conditional expectation on $L^\infty(\widehat{\mathbb{G}})$. Also, since $\sigma_t^{\widehat{\varphi}}(\widehat{x}) \in \mathbb{C}\widehat{x}$ for all $t \in \mathbb{R}$ and $\widehat{x} \in \text{Gr}(\widehat{\mathbb{G}})$, we have $\sigma_t^{\widehat{\varphi}}(N_{\widehat{\mathbb{G}}}) \subseteq N_{\widehat{\mathbb{G}}}$, and therefore, by [25, Theorem 4.2], there exists a conditional expectation $E: L^\infty(\widehat{\mathbb{G}}) \rightarrow N_{\widehat{\mathbb{G}}}$. Hence, $E \circ (F \otimes \iota) \circ \Gamma$ is a conditional expectation from $\mathcal{B}(L^2(\mathbb{G}))$ on $N_{\widehat{\mathbb{G}}} \cong \text{VN}(\widehat{\mathbb{G}})$, and so $\text{VN}(\widehat{\mathbb{G}})$ is injective. Since, by Theorem 3.14, $\widehat{\mathbb{G}}$ is discrete, it follows that $\widehat{\mathbb{G}}$ is amenable (cf. [4]). ■

Let $i: N_{\mathbb{G}} \hookrightarrow L^\infty(\mathbb{G})$ be the canonical injection. Obviously, i is weak* continuous, so we have the pre-adjoint map $i_*: L^1(\mathbb{G}) \rightarrow (N_{\mathbb{G}})_*$. Then direct calculation implies the following.

Lemma 5.15 *The map $i_*: L^1(\mathbb{G}) \rightarrow (N_{\mathbb{G}})_*$ is a completely bounded algebra homomorphism.*

If a locally compact quantum group \mathbb{G} satisfies the condition of Proposition 5.12, then $\text{VN}(\text{Gr}(\mathbb{G})) \cong \text{Gr}(\mathbb{G})'' \cap L^\infty(\mathbb{G})$, and by the above, we have a surjective continuous algebra homomorphism $i_*: L^1(\mathbb{G}) \rightarrow A(\text{Gr}(\mathbb{G}))$. Therefore, in this case, many of the algebraic properties of $L^1(\mathbb{G})$ will be satisfied by the Fourier algebra of the intrinsic group as well.

There are many different equivalent characterizations of amenability for a locally compact group. The question of whether the quantum counterpart of these conditions are equivalent as well, remains unsolved in many important instances.

In the following, we present a few of those equivalent characterizations in the group case which can be found for instance in [23].

Theorem 5.16 *For a locally compact group G , the following are equivalent:*

- (i) G is amenable;
- (ii) $L^1(G)$ is an amenable Banach algebra;
- (iii) $A(G)$ has a bounded approximate identity (BAI);
- (iv) $A(G)$ is operator amenable.

The equivalence (1) \Leftrightarrow (2) is due to Johnson (1972) [13], (1) \Leftrightarrow (3) is Leptin's theorem (1968) [19], and (1) \Leftrightarrow (4) is due to Ruan [22].

Proposition 5.17 *Let \mathbb{G} be a discrete quantum group. If $L^1(\widehat{\mathbb{G}})$ has a BAI, then $\widehat{\mathbb{G}}$ is amenable.*

Proof Since $\widehat{\mathbb{G}}$ is compact, by Corollary 5.13, we have $N_{\mathbb{G}} \cong \text{VN}(\text{Gr}(\mathbb{G}))$. If $(\widehat{\omega}_\alpha)$ is a BAI for $L^1(\widehat{\mathbb{G}})$, then as easily seen $(i_*(\widehat{\omega}_\alpha))$ is a BAI for $A(\text{Gr}(\mathbb{G}))$, whence $\text{Gr}(\mathbb{G})$ is amenable, by Theorem 5.16. ■

Proposition 5.18 *Let \mathbb{G} be a discrete quantum group. If $L^1(\mathbb{G})$ is operator amenable, then $\widetilde{\mathbb{G}}$ is amenable.*

Proof Since \mathbb{G} is discrete, we have, by Lemma 5.15, a completely bounded surjective algebra homomorphism from $L^1(\widehat{\mathbb{G}})$ onto $A(\widetilde{\mathbb{G}})$. Since $L^1(\widehat{\mathbb{G}})$ is operator amenable, then so is $A(\widetilde{\mathbb{G}})$. Hence, by Theorem 5.16, $\widetilde{\mathbb{G}}$ is amenable. ■

Proposition 5.19 *Let \mathbb{G} be a discrete quantum group. If $L^1(\widehat{\mathbb{G}})$ is an amenable Banach algebra, then $\widetilde{\mathbb{G}}$ is almost abelian.*

Proof The argument is analogous to the one given in Proposition 5.18, but instead of Theorem 5.16, we use [9, Theorem 2.3], stating that $A(G)$ is amenable if and only if G is almost abelian. ■

References

- [1] S. Baaj and G. Skandalis, *Unitaires multiplicatifs et dualité pour les produits croisés de C^* -algèbres*. Ann. Sci. École Norm. Sup. **26**(1993), 425–488.
- [2] S. Baaj and S. Vaes, *Double crossed products of locally compact quantum groups*. J. Inst. Math. Jussieu **4**(2005), 135–173. <http://dx.doi.org/10.1017/S1474748005000034>
- [3] E. Bedos and L. Tuset, *Amenability and co-amenability for locally compact quantum groups*. Internat. J. Math. **14**(2003), 865–884. <http://dx.doi.org/10.1142/S0129167X03002046>
- [4] A. Connes, *Classification of injective factors*. Ann. of Math. (2) **104**(1976), 73–115. <http://dx.doi.org/10.2307/1971057>
- [5] J. de Cannière, *On the intrinsic group of a Kac algebra*. Proc. London Math. Soc. (3) **40**(1980), 1–20. <http://dx.doi.org/10.1112/plms/s3-40.1.1>
- [6] K. De Commer and A. Van Daele, *Algebraic quantum groups imbedded in C^* -algebraic quantum groups*. Rocky Mountain J. Math. **40**(2010), 1149–1182. <http://dx.doi.org/10.1216/RMJ-2010-40-4-1149>
- [7] E. G. Effros and Z.-J. Ruan, *Operator Spaces*. London Math. Soc. Monogr. New Series **23**, The Clarendon Press, Oxford University Press, New York, 2000.
- [8] M. Enock and J. M. Schwartz, *Kac Algebras and Duality of Locally Compact Groups*. Springer-Verlag, Berlin, 1992.
- [9] B. Forrest and V. Runde, *Amenability and weak amenability of the Fourier algebra*. Math. Z. **250**(2005), 731–744. <http://dx.doi.org/10.1007/s00209-005-0772-2>
- [10] U. Haagerup, *The standard form of von Neumann algebras*. Math. Scand. **37**(1975), 271–283.
- [11] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis. Vol. I. Structure of Topological Groups, Integration Theory, Group Representations*. Second edition. Grundlehren Math. Wiss. **115**, Springer-Verlag, Berlin–New York, 1979.
- [12] Z. Hu, M. Neufang, and Z.-J. Ruan, *Completely bounded multipliers over locally compact quantum groups*. Proc. London Math. Soc., to appear. <http://dx.doi.org/10.1112/plms/pdq041>
- [13] B. E. Johnson, *Cohomology in Banach algebras*. Memoirs of the American Mathematical Society, **127**, American Mathematical Society, Providence, RI, 1972.
- [14] M. Junge, M. Neufang, Z.-J. Ruan, *A representation theorem for locally compact quantum groups*. Internat. J. Math. **20**(2009), 377–400. <http://dx.doi.org/10.1142/S0129167X09005285>
- [15] M. Kalantar, *Towards harmonic analysis on locally compact quantum groups: From Groups to Quantum Groups and Back*. Ph.D. thesis, Carleton University, Ottawa, 2010.
- [16] J. Kustermans, *Locally compact quantum groups in the universal setting*. Internat. J. Math. **12**(2001), 289–338.
- [17] J. Kustermans and S. Vaes, *Locally compact quantum groups*. Ann. Sci. Ecole Norm. Sup. **33**(2000), 837–934.

- [18] ———, *Locally compact quantum groups in the von Neumann algebraic setting*. Math. Scand. **92**(2003), 68–92.
- [19] H. Leptin, *Sur l'algèbre de Fourier d'un groupe localement compact*. C. R. Acad. Sci. Paris Sér. A–B **266**(1968), A1180–A1182.
- [20] A. Müller, *Classifying spaces for quantum principal bundles*. Comm. Math. Phys. **149**(1992), 495–512. <http://dx.doi.org/10.1007/BF02096940>
- [21] P. F. Renaud, *Centralizers of Fourier algebra of an amenable group*. Proc. Amer. Math. Soc. **32**(1972), 539–542.
- [22] Z.-J. Ruan, *The operator amenability of $A(G)$* . Amer. J. Math. **117**(1995), 1449–1474. <http://dx.doi.org/10.2307/2375026>
- [23] V. Runde, *Lectures on Amenability*. Lecture Notes in Math. **1774**, Springer-Verlag, Berlin, 2002.
- [24] M. Takesaki, *A characterization of group algebras as a converse of Tannaka–Stinespring–Tatsuuma duality theorem*. Amer. J. Math. **91**(1969), 529–564. <http://dx.doi.org/10.2307/2373525>
- [25] ———, *Theory of Operator Algebras. II*. Encyclopaedia Math. Sci. **125**, Operator Algebras and Non-commutative Geometry 6. Springer-Verlag, Berlin, 2003.
- [26] S. Vaes, *Locally Compact Quantum Groups*. Ph.D. thesis, K.U. Leuven, 2000.
- [27] J. G. Wendel, *Left centralizers and isomorphisms of group algebras*. Pacific J. Math. **2**(1952), 251–261.
- [28] S. L. Woronowicz, *Compact matrix pseudogroups*. Comm. Math. Phys. **111**(1987), 613–665. <http://dx.doi.org/10.1007/BF01219077>
- [29] ———, *Compact quantum groups*. In: Symétries quantiques (Les Houches, 1995), North-Holland, Amsterdam, 1998, 845–884.
- [30] ———, *Quantum $E(2)$ group and its Pontryagin dual*. Lett. Math. Phys. **23**(1991), 251–263. <http://dx.doi.org/10.1007/BF00398822>
- [31] ———, *Tannaka–Krein duality for compact matrix pseudogroups. Twisted $SU(N)$ groups*. Invent. Math. **93**(1998), 35–76. <http://dx.doi.org/10.1007/BF01393687>

School of Mathematics and Statistics, Carleton University, Ottawa, Ontario K1S 5B6
e-mail: mkalanta@math.carleton.ca

School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, K1S 5B6
e-mail: mneufang@math.carleton.ca

Université Lille 1 – Sciences et Technologies, UFR de Mathématiques, Laboratoire de Mathématiques Paul Painlevé (UMR CNRS 8524), 59655 Villeneuve d'Ascq Cédex, France
e-mail: Matthias.Neufang@math.univ-lille1.fr