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# Enumerative geometry for plane cubic curves in characteristic 2

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**Abstract.** Consider the plane cubic curves over an algebraically closed field of characteristic 2. By blowing up the parameter space $\mathbf{P}^9$ twice we obtain a variety $B$ of complete cubics. We then compute the characteristic numbers for various families of cubics by intersecting cycles on $B$.

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## 1. Introduction

One of the major objects of enumerative geometry is to determine the characteristic numbers for families of plane curves. A characteristic number counts the number of curves in a family that pass through $\alpha$ given points and touch $\beta$ given lines, where $\alpha + \beta$ equals the dimension of the family. For families of plane cubic curves these numbers were first found by Maillard and Zeuthen in the early 1870's, but their methods were based on assumptions that were not rigorously justified.

More than a century went by before these numbers were confirmed. Kleiman and Speiser [8, 9, 10] and Aluffi [1, 2] both compute the characteristic numbers for smooth, nodal and cuspidal cubics, but by very different means. Kleiman and Speiser's works are based on the classical degeneration method of Maillard and Zeuthen. They specialize their family to more degenerate ones and then use the numbers already obtained for the special families. In this way the characteristic numbers for smooth cubics depend on the numbers for nodal cubics, which in turn depend on the numbers for cuspidal cubics.

Aluffi's method is more direct. By a sequence of five blow-ups of $\mathbf{P}^9$ he constructs a variety of complete cubics. The characteristic numbers for smooth cubics are then obtained by intersecting certain divisors on this variety. By a closer examination of the space parametrizing the singular cubics, Aluffi also obtains the characteristic numbers for nodal and cuspidal cubics.

These papers, like most other papers on enumerative geometry, assume that the characteristic is different from 2 and 3. One exception is Vainsencher's *Conics* *in characteristic 2* [11] determining the number 51 of conics tangent to 5 other, assuming the characteristic is 2.

In this paper we will apply the method used by Aluffi [1] and Vainsencher [11, 12] to construct a variety of complete plane cubics in characteristic 2. The strategy is to blow up the parameter space along smooth centers until the proper transforms of the line conditions no longer intersect. The technical difficulties of this method is at each blow-up to determine the intersection of the line conditions and to compute certain Segre classes. In characteristic 2, the intersection of the line conditions has nonsingular support and is reduced at the general point. This makes our case significantly less demanding than the characteristic 0 case, where the intersection of the line conditions has more structure. In our case, two blow-ups will suffice, giving a smooth space, $B$, of complete cubics which is relatively easy to handle. The computation of the characteristic numbers follows the lines of Aluffi and Vainsencher. In particular we will rely on Aluffi's blow-up formula [1, Thm. II], which relates intersection numbers before and after taking proper transforms.

## 2. Generalities about characteristic numbers

In this section, which is independent of the characteristic of the base field $k$, we define the characteristic numbers and give their basic properties.

Intuitively, a characteristic number $N_{\alpha,\beta}$ for a family $R$ of plane curves is the number of curves passing through $\alpha$ given points and properly tangent to $\beta$ given lines where $\alpha+\beta = \dim R$. (We call a tangent proper if it is tangent at a nonsingular point. By just tangent we mean a line intersecting the curve with multiplicity at least 2 at a point.) To determine these numbers it is convenient to work in the $\mathbf{P}^n$ $(n = \frac{1}{2}d(d+3))$ parametrizing all plane curves of degree $d$.

DEFINITION 2.1. A *point condition* in $\mathbf{P}^n$ is a hyperplane $H$ parametrizing the curves containing a given point. A *line condition* is a hypersurface $M$ parametrizing the curves tangent to a given line.

The following definition of characteristic numbers differs slightly from the intuitive one in that it may count curves with multiplicity greater than one. But as we shall see, this need not be a big problem.

DEFINITION 2.2. Suppose $R \subset \mathbf{P}^n$ is an irreducible, $r$-dimensional subvariety parametrizing a family of curves such that the generic curve is reduced and irreducible. Suppose we have $\alpha$ points and $\beta$ lines in general position, with $\alpha+\beta = r$. Let $H_i$ and $M_j$ be the corresponding point and line conditions in $\mathbf{P}^n$. We define the *characteristic numbers* for $R$ to be

$$N_{\alpha,\beta} = \sum_{x\in Q} m(x\,, R\cdot H_1\cdot\,\cdots\,\cdot H_\alpha\cdot M_1\cdot\,\cdots\,\cdot M_\beta),$$

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where $Q = \{x \in \mathbf{P}^n : C_x$ intersects the $\beta$ lines only at smooth points$\}$ ($C_x$ is the curve corresponding to $x$), and $m$ is the usual intersection multiplicity. $N_{\alpha,\beta}$ counts the weighted number of curves in $R$ passing through the $\alpha$ points and properly tangent to the $\beta$ lines.

The following theorem shows that these numbers are well defined, and that the curves counted by a given characteristic number all appear with the same multiplicity.

THEOREM 2.3. *Let $R \subset \mathbf{P}^n$ be an irreducible, $r$-dimensional family of generically reduced and irreducible curves, and let $\alpha$ and $\beta$ be nonnegative integers such that $\alpha + \beta = r$.*

(i) *There exists an open dense subset $U \subset (\mathbf{P}^2)^\alpha \times (\check{\mathbf{P}}^2)^\beta$ and nonnegative integers $N$ and $e$ such that for each configuration of points and lines $(p_1, \dots, p_\alpha, l_1, \dots, l_\beta) \in U$ there are exactly $N$ different curves from $R$ passing through the $\alpha$ points and properly tangent to the $\beta$ lines, and such that the multiplicity $m$ (in the sense of Definition 2.2) at each of the $N$ curves is $p^e$ when the characteristic is $p$ and 1 when the characteristic is 0.*

(ii) *The multiplicity $m = m(\beta)$ is a non-decreasing function of $\beta$.*

*Proof of* (i). The existence of $U$ and $N$ is well known and follows from [5, Sect. 2]. It is also clear (same reference) that the number $N$ remains the same when $R$ is replaced by any open subset of $R$. We may then assume that all curves in $R$ are irreducible.

Let $T \subset U \times R$ be defined by

$$T = \{(p_1, \dots, p_\alpha, l_1, \dots, l_\beta; x) : C_x \text{ contains } p_i \text{ and is properly tangent to } l_j\}$$

and let $p$ and $q$ be the projections from $T$ to $U$ and $R$ respectively. Let $x \in R$ be any point. Then $q^{-1}(x)$ is an open subset of $(C_x)^\alpha \times (C_x^\vee)^\beta \subset (\mathbf{P}^2)^\alpha \times (\check{\mathbf{P}}^2)^\beta$. Since $C_x$ is irreducible, so is the dual $C_x^\vee$. This shows that the fibre $q^{-1}(x)$ is irreducible and it follows that $T$ is irreducible (since $R$ is).

We know that $p$ is a generically finite surjective map of integral varieties. Let $s$ and $m$ be the separable and inseparable degree of $p$. Then it is well known ([11, Sect. 7], is one reference) that the general fibre of $p$ has $s$ distinct points, and the multiplicity at each is $m$. Shrinking $U$ if necessary we have this statement for all the fibres.

Finally, the argument given in the remark in [11, Sect. 7] shows that the multiplicity in the last paragraph coincides with the intersection theoretic one. (The intersection-theoretic multiplicity can be obtained from an alternating sum of Tor's. Since the line conditions are smooth, and in particular Cohen-Macaulay, at the points of intersection, all the higher Tor's vanish.) $\square$

The following two lemmas are used in [1] to prove the characteristic 0 version of the above theorem. We will need the lemmas to prove the second part of the theorem.

LEMMA 2.4. *Suppose $S \subset \mathbf{P}^n$ is a curve parametrizing generically reduced and irreducible curves, and let $x \in S$ be a general point. Then there exist at most finitely many point conditions $H_p$ tangent to $S$ at $x$.*

*Proof*. Let $T_x$ be the tangent line to $S$ at $x$, and suppose $T_x \subset H_p$. Then $p$ is contained in all the curves parametrized by $T_x$. Clearly (since the curve parametrized by $x$ is irreducible), only a finite number of such $p$ can exist. □

LEMMA 2.5. *Suppose $R \subset \mathbf{P}^n$ is an irreducible family of generically reduced and irreducible curves. Then a general point condition will intersect $R$ transversally* (*by transversal we always mean that the scheme theoretical intersection has no nonreduced components*).

*Proof*. Since the set of points $p$ such that $H_p$ does not intersect $R$ transversally is closed, it is enough to show the existence of one $H_p$ that does. Suppose that all point conditions intersect $R$ in a nonreduced component. Then the union of these components will cover $R$. Let $x \in R$ be a general point, and let $S \subset R$ be any curve having $x$ as a smooth point. Since the set of point conditions is 2-dimensional there will be infinitely many point conditions tangent to $R$ at $x$. These will also be tangent to $S$, contradicting Lemma 2.4. □

*Proof of Theorem* 2.3 (ii). Let $H_1, \dots, H_\alpha$ and $M_1, \dots, M_\beta$ be general point and line conditions in $\mathbf{P}^n$. We know that the points in $R \cap H_1 \cap \cdots \cap H_\alpha \cap M_1 \cap \cdots \cap M_\beta$ counted by $N_{\alpha,\beta}$ all appear with the same multiplicity $m$. If we remove one of the point conditions, then by (2.5) the components containing these points will also have multiplicity $m$. When these components are intersected with a line condition, all the points in the new intersection must have multiplicity at least $m$. □

*Remark*. Note that the first part of this theorem seems to be a special case of [5, Thm. 2]. The important difference is the definition of the characteristic numbers. While we intersect in $\mathbf{P}^n$, the characteristic numbers in [5] are defined by intersections in $I^r \times R$ where $I \subset \mathbf{P}^2 \times \check{\mathbf{P}}^2$ is the incidence variety.

DEFINITION 2.6. With the same hypotheses as in Definition 2.2 we define the *total characteristic numbers* for $R$ to be

$$\Gamma_{\alpha,\beta} = \sum_{x \in \mathbf{P}^N \setminus L} m(x, R \cdot H_1 \cdot \cdots \cdot H_\alpha \cdot M_1 \cdot \cdots \cdot M_\beta),$$

where $L$ is the locus of the nonreduced curves. $\Gamma_{\alpha,\beta}$ is the weighted number of reduced curves passing through the $\alpha$ points and tangent to the $\beta$ lines (but not necessarily at smooth points).

In order to compute the total characteristic numbers $\Gamma_{\alpha,\beta}$ (from which the characteristic numbers will be deduced), we shall need the concept of a variety of complete curves, which is defined as follows.

DEFINITION 2.7. A variety $B$ together with a surjective morphism $\pi\colon B \to \mathbf{P}^n$ is called a *variety of complete curves* if:

(1) $\pi$ restricts to an isomorphism outside the locus, $L$, of nonreduced curves.
(2) The proper transforms of the line conditions in $\mathbf{P}^n$ do not have a common intersection in $B$.

PROPOSITION 2.8. *Suppose $B$ is a complete nonsingular variety of complete curves of degree $d$. Suppose $R \subset \mathbf{P}^n$ is a subvariety parametrizing a family of generically reduced and irreducible curves, and let $\tilde{R}$ be its proper transform in $B$. Also, denote by $\tilde{H}$ and $\tilde{M}$ the proper transforms of point and line conditions respectively. Then the total characteristic numbers for $R$ are given by*

$$\Gamma_{\alpha,\beta} = \int_B [\tilde{R}][\tilde{H}]^\alpha[\tilde{M}]^\beta \quad \textit{with } \alpha+\beta = r = \dim R.$$

*Proof.* Let $H_1, \dots, H_\alpha$ and $M_1, \dots, M_\beta$ be general point and line conditions in $\mathbf{P}^n$, and let $E = \pi^{-1}(L)$. Since $\pi\colon B \to \mathbf{P}^n$ restricts to an isomorphism $B \backslash E \tilde{\to} \mathbf{P}^n \backslash L$ it is sufficient to show that $\tilde{R} \cap \tilde{H}_1 \cap \cdots \cap \tilde{H}_\alpha \cap \tilde{M}_1 \cap \cdots \cap \tilde{M}_\beta$ does not intersect $E$. Since the general curve in $R$ is reduced we can assume that $\dim(\tilde{R} \cap E) \leqslant r \Leftrightarrow 1$. The result follows if we can show that $\tilde{H}_i$ and $\tilde{M}_j$ intersect a given irreducible subvariety $V \subset B$ properly. ($V$ and $W$ intersect properly if $\operatorname{codim}(V) + \operatorname{codim}(W) = \operatorname{codim}(V \cap W)$.) The set $\{l \in \check{\mathbf{P}}^2 : V \subset \tilde{M}_l\}$ is closed, and since $B$ is a variety of complete curves this set is not all of $\check{\mathbf{P}}^2$. It follows that the general line condition does not contain $V$, so the intersection is proper. Similarly, we can show that $\tilde{H}_i$ intersects a given $V \subset B$ properly. □

## 3. Generalities about cubics in characteristic 2

In this section we will discuss some elementary facts about plane cubic curves in characteristic 2. It is essential to get information about the subvarieties of $\mathbf{P}^9$ parametrizing special families of cubics.

From now on, and for the rest of the paper, we will assume that the characteristic of the ground field is 2. The defining polynomial of a plane cubic curve will be written in the following form

$$\begin{aligned}F(x,y,z) &= ax^3+by^3+cz^3+dx^2y+ex^2z\\ &\quad +fxy^2+gy^2z+hxz^2+iyz^2+jxyz.\end{aligned}$$

Let $V=\{(p,l)\colon l \text{ is tangent to } C_p\}\subset \mathbf{P}^9\times\check{\mathbf{P}}^2$, and let $\pi_1$ and $\pi_2$ be the two projections. Then the fibre $\pi_1^{-1}(p)$ is the 'total dual' of $C_p$ (the union of $C_p^\vee$ and the possible multiple lines corresponding to each singularity of $C$), and $\pi_2^{-1}(l)$ is the line condition $M_l$.

LEMMA 3.1. *If we use $x, y$ and $z$ as coordinates on $\check{\mathbf{P}}^2$, we have the following equation for $V$*

$$\begin{aligned}&(bc+gi)x^3+(ac+eh)y^3+(ab+df)z^3+(cf+gh+ij)x^2y\\ &\quad+(fi+bh+gj)x^2z+(cd+ei+hj)xy^2+(ai+dh+ej)y^2z\\ &\quad+(dg+be+fj)xz^2+(ag+ef+dj)yz^2+j^2xyz=0.\end{aligned}$$

*Proof.* Assume $\check{\mathbf{A}}^2=\check{\mathbf{P}}^2\backslash\{y=0\}$ has affine coordinates $m=x/y$ and $t=z/y$ and let $W=V\cap(\mathbf{P}^9\times\check{\mathbf{A}}^2)$. Let $l\in\check{\mathbf{A}}^2$ be the line in $\check{\mathbf{P}}^2$ given by $y=mx+tz$, and let $F(x,y,z)=0$ be the equation for a cubic $C_p$. Now $(p,l)\in W$ if and only if $g(x,z)=F(x,mx+tz,z)$ has multiple factors, and this happens exactly when the discriminant $\Delta g(x,1)=0$. In characteristic 2 the discriminant of a cubic polynomial $ax^3+bx^2+cx+d$ is $ad+bc$. Letting $F(x,y,z)=ax^3+by^3+\cdots+jxyz$, and computing $\Delta F(x,mx+t,1)$ we easily obtain (after some elementary, but tedious computations) the equation for $W$, and the equation for $V$ follows. □

It should be well known that in characteristic 0 any nonsingular plane cubic can be linearly transformed into one with equation $x^3+y^3+z^3+txyz=0$. This is also true in characteristic 2. See [4, Sect. 7.3] for a proof that works in all characteristics different from 3. Note that in characteristic 2 the curve $C_t$ given by $x^3+y^3+z^3+txyz=0$ is singular if and only if $t^3=1$.

LEMMA 3.2. *The following holds for plane cubic curves in characteristic* 2:

(1) *The dual of a nonsingular cubic is a nonsingular cubic.*
(2) *The dual of a nodal cubic is a nonsingular conic.*
(3) *The dual of a cuspidal cubic is a line. In particular, a cuspidal cubic is strange* (*there is a point common to all the proper tangents*).

*Proof.* (1) This can be checked on $C_t$ given by $x^3+y^3+z^3+txyz=0$ ($t^3\neq 1$). By Lemma 3.1 we see that $C_t^\vee$ is given by $x^3+y^3+z^3+t^2xyz=0$ which is nonsingular.

(2) It is an easy exercise to check that all nodal cubics are projectively equivalent, so we only need to consider the nodal cubic given by $x^3+y^3+xyz=0$. By

(3.1) the dual curve is $z^2 + xy = 0$ (we ignore the line $z = 0$ in $\check{\mathbf{P}}^2$ corresponding to the node) which is a nonsingular conic.

(3) The dual of the cuspidal curve $x^3 + y^2z = 0$ is the line $y = 0$. The arguments are similar to the nodal case. □

*Remark.* In characteristic 0, it is well known that the dual of a nonsingular plane curve $C$ has degree $d(d \Leftrightarrow 1)$, where $d = \deg C$. What happens in characteristic 2 is that the defining polynomial of $C^\vee$ reduces to the square of a polynomial of degree $\frac{1}{2}d(d \Leftrightarrow 1)$.

If $C_t$ is a nonsingular cubic given by $x^3 + y^3 + z^3 + txyz = 0$ we see from the proof of the first part of the lemma that $(C_t^\vee)^\vee = C_{t^4}$ so that in general biduality does not hold. We have biduality only for a special class of cubics characterized by the following proposition.

PROPOSITION 3.3. *The following are equivalent for a nonsingular cubic* $C \subset \mathbf{P}^2$:

(1) $C = (C^\vee)^\vee$.
(2) $C$ *is projectively equivalent to the curve with equation* $x^3 + y^3 + z^3 = 0$.
(3) $j = 0$ *in the equation for C.*
(4) $C$ *has Hasse-invariant* 0.
(5) $C$ *has* $j$*-invariant* 0.

*Proof.* (1) $\Leftrightarrow$ (2) is a trivial consequence of the fact that $(C_t^\vee)^\vee = C_{t^4}$ when $C_t$ given by $x^3 + y^3 + z^3 + txyz = 0$. If $C \sim D$ (projective equivalence) and $j_D = 0$ then it is easy to verify that $j_C = 0$, so we have (2) $\Leftrightarrow$ (3). (3) $\Leftrightarrow$ (4) is a special case of [7, IV Prop. 4.21] (4) $\Leftrightarrow$ (5) follows from [7, IV 4.23] (note to corollary). □

*Remark.* It can be shown that the $j$-invariant of $C_t$ equals $t^{12}/(t^3 + 1)^3$, which gives another proof of (3) $\Leftrightarrow$ (5). When we use the notation $j_C$ we do not mean the $j$-invariant, but simply the coefficient $j$ in an equation for $C$.

The cubics described in Proposition 3.3 we call $j$-curves. We next show that the cuspidal cubics are degenerate $j$-curves. First we need some lemmas.

LEMMA 3.4. *Let* $C$ *be a cubic with* $j_C = 0$*, and let* $H$ *be the matrix*

$$\begin{pmatrix} a & f & h \\ d & b & i \\ e & g & c \end{pmatrix}.$$

*Then:*

(1) $C$ *is nonsingular* $\Leftrightarrow \mathrm{rk}(H) = 3$

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(2) $C$ *is singular and reduced* $\Leftrightarrow \mathrm{rk}(H) = 2$
(3) $C$ *is nonreduced* $\Leftrightarrow \mathrm{rk}(H) = 1$

*Proof.* The singular locus is precisely the set of points $(x, y, z)$ such that all the partial derivatives are zero, or equivalently: $(x^2, y^2, z^2)$ belongs to the nullspace of $H$. The lemma now follows by elementary linear algebra. □

LEMMA 3.5. *Let C be a singular cubic. Then C is cuspidal* (*possibly degenerate*) *if and only if* $j = 0$ *in the equation for C.*

*Proof.* Choose a $B \in \mathrm{PGL}(3)$ mapping a singularity of $C$ to $(0, 0, 1)$. Let $D = B(C)$ and introduce affine coordinates $x' = x/z$, $y' = y/z$. The affine equation of $D$ can be written as $f(x', y') = 0$. Let $f = f_3 + f_2 + f_1 + f_0$ where $f_i$ is homogeneous of degree $i$. Since $D$ is singular at $(0, 0)$ we have $f_1 = f_0 = 0$, and $f_2 = e_D {x'}^2 + g_D {y'}^2 + j_D x'y'$ is the equation of the tangent cone. $D$ is cuspidal exactly when the tangent cone is a double line, and that happens exactly when $j_D = 0$. Since by (3.3) $j_C = 0 \Leftrightarrow j_D = 0$ the lemma follows. □

PROPOSITION 3.6. *A plane cubic curve* $C$ *is cuspidal* (*possibly degenerate*) *if and only if* $\det(H) = j = 0$.

*Proof.* If $C$ is cuspidal, then $j = 0$ by (3.5) and $\det(H) = 0$ by (3.4). $\det(H) = j = 0$ implies that $C$ is singular by (3.4) and cuspidal by (3.5). □

Let $C$ be a non-degenerate cuspidal cubic given by $F(x, y, z) = ax^3 + by^3 + \cdots + jxyz$. By (3.4) we have that $\mathrm{rk}(H) = 2$. It follows that the cofactor matrix, $\mathrm{cof}(H)$, has rank 1, so that nonzero rows (resp. columns) of $\mathrm{cof}(H)$ define the same point in $\mathbf{P}^2$.

$$\mathrm{cof}(H) = \begin{pmatrix} bc + gi & cd + ei & dg + be \\ cf + gh & ac + eh & ag + ef \\ fi + bh & ai + dh & ab + df \end{pmatrix}.$$

Let $P$ be the point defined by the columns, and let $Q$ be the point defined by the square root of the rows: If $(\alpha, \beta, \gamma) \neq (0, 0, 0)$ is a row, then $Q = (\sqrt{\alpha}, \sqrt{\beta}, \sqrt{\gamma})$. This is well defined since there is only one square root in characteristic 2.

PROPOSITION 3.7. *Let* $C$, $P$ *and* $Q$ *be as above. Then* $Q$ *is the cusp of* $C$, *and* $P$ *is the only flex of C. Also,* $P$ *is a strange point, that is: every proper tangent of* $C$ *contains* $P$.

*Proof.* Suppose $Q = (\sqrt{bc + gi}, \sqrt{cd + ei}, \sqrt{dg + be})$ is given by the first row of $\mathrm{cof}(H)$. Using that $\det(H) = 0$ we easily see that $F_x(Q) = F_y(Q) = F_z(Q) = 0$, so $Q$ must be the cusp of $C$. Now the last part: The tangent at $(u_0, u_1, u_2) \in C$ is given by

$$(au_0^2 + fu_1^2 + hu_2^2)x + (du_0^2 + bu_1^2 + iu_2^2)y + (eu_0^2 + gu_1^2 + cu_2^2)z = 0$$

or equivalently $(x\ y\ z)H(u_0^2\ u_1^2\ u_2^2)^t = 0$. We must show that this equation is satisfied when $P = (x\ y\ z)$ is a column in $\mathrm{cof}(H)$ or a row in $\mathrm{adj}(H)$. But from the identity $(\mathrm{adj}\, H)H = I\det(H) = 0$ we have that $(x\ y\ z)H = 0$, and the result follows.

To prove that $P \in C$ just note that $F = (x\ y\ z)H(x^2\ y^2\ z^2)^t$ and use the same argument. If the tangent at $P$ meets the curve at another point $S$, then (since $P$ is a strange point) this tangent would be a bitangent which is impossible for cubics. This proves that $P$ is a flex. The tangents at other nonsingular points all contain $P$ so there cannot be more flexes. □

The set of $j$-curves (including degenerate curves) is parametrized by $\mathbf{P}^8$ when we to a point $(a, b, \ldots, i)$ in $\mathbf{P}^8$ associate the curve with equation $ax^3 + by^3 + \cdots + iyz^2$. We have seen that the cuspidal cubics are parametrized by a hypersurface of degree 3 in $\mathbf{P}^8$.

PROPOSITION 3.8. *The cuspidal cubics with cusp* (*resp. flex*) *on a given line are parametrized by a* 6-*fold of degree* 3 *in* $\mathbf{P}^8$.

*Proof.* Assume the line is given by $x = 0$. By the first part of (3.7) we find the desired locus to be given by $bc + gi = cf + gh = fi + bh = 0$ which the computer program *Macaulay* tells us has degree 3 and codimension 2 in $\mathbf{P}^8$. The case with the flex is similar. □

Let $\phi\colon \check{\mathbf{P}}^2 \times \check{\mathbf{P}}^2 \to \mathbf{P}^9$ be the map given by

$$\phi((a, b, c), (d, e, f)) = (ad, be, cf, ae, af, bd, bf, cd, ce, 0),$$

($\phi$ is the Segre embedding on the $\mathbf{P}^8$ given by $j = 0$). We claim that the image of $\phi$ is exactly $L$, the locus of the nonreduced curves. Indeed, the curve $l^2m \in L$ can easily be seen to be the image of $((x_0^2, y_0^2, z_0^2), (x_1, y_1, z_1))$ where $l = (x_0, y_0, z_0) \in \check{\mathbf{P}}^2$ and $m = (x_1, y_1, z_1) \in \check{\mathbf{P}}^2$. This shows that we have an isomorphism $L \simeq \check{\mathbf{P}}^2 \times \check{\mathbf{P}}^2$. Now, $T \subset L$, the locus of the triple lines, is isomorphic to $\check{\mathbf{P}}^2$ ($l \in \check{\mathbf{P}}^2$ corresponds to the triple line $l^3$), and the map $i\colon T \to L$ is via these isomorphisms given by $i(l) = (l^2, l)$ (the coordinates of $l^2 \in \check{\mathbf{P}}^2$ are the squares of the coordinates of $l$). Let $j$ be the embedding $L \to \mathbf{P}^9$. We record for later use:

LEMMA 3.9. *Let* $h_1$ *and* $h_2$ *denote the pullbacks of the hyperplane classes on the factors of* $L \simeq \check{\mathbf{P}}^2 \times \check{\mathbf{P}}^2$, *and let* $h$ *and* $t$ *be the hyperplane classes on* $\mathbf{P}^9$ *and* $T \simeq \check{\mathbf{P}}^2$ *respectively. Then* $j^*h = h_1 + h_2$, $i^*h_1 = 2t$ *and* $i^*h_2 = t$. □

## 4. The two blow-ups

We now follow the strategy of blowing up the parameter space, $\mathbf{P}^9$, along nonsingular varieties supported on the intersection of the line conditions, until we have a variety of complete curves. This strategy was successfully employed by Aluffi in [1], where five blow-ups were needed. We shall only need two blow-ups, and the varieties and maps involved in this process appear in the following diagram

$$\begin{tikzcd}
D \arrow[d] & & & B \arrow[d, "\pi_2"] \\
S \arrow[r] \arrow[d, "\sim"] & F \arrow[r, "k"] \arrow[d, "q"] & E \arrow[r] \arrow[d, "p"] & B_1 \arrow[d, "\pi_1"] \\
\check{\mathbf{P}}^2 \arrow[r, "\sim"'] & T \arrow[r, "i"'] & L \arrow[r, "j"'] & \mathbf{P}^9 .
\end{tikzcd}$$

By Section 3 we know that $L$ is nonsingular, so that will be the centre for our first blow-up.

4.1. *The first blow-up*. Let $B_1$ be the blow-up of $\mathbf{P}^9$ along $L$, let $N$ be the normal bundle of $L$ in $\mathbf{P}^9$, and let $E = \mathbf{P}(N)$ be the exceptional divisor with maps as shown in the diagram.

If $H$ and $M$ are point and line conditions on $\mathbf{P}^9$, denote by $\tilde{H}_1$ and $\tilde{M}_1$ their proper transforms in $B_1$. (We reserve the notation $\tilde{H}$ and $\tilde{M}$ for the proper transforms in $B$.) We call $\tilde{H}_1$ and $\tilde{M}_1$ point and line conditions in $B_1$.

To determine the intersection of the line conditions in $B_1$ we need to examine the tangent hyperplanes of the line conditions in $\mathbf{P}^9$.

LEMMA 4.1. *Let $M_l$ be the line condition in $\mathbf{P}^9$ corresponding to the line $l$, and let $r \in M_l$ be a cubic not containing $l$. Suppose $C_r$ is tangent to $l$ at $p$ and that $q$ is the other point of intersection.*

(1) *If $p \neq q$ the tangent hyperplane of $M_l$ at $r$ equals the linear span of $X_p$ and $Y_q$, where $X_p$ is the cubics tangent to $l$ at $p$ and $Y_q$ is the cubics through $q$ and tangent to $l$ at another point.*
(2) *If $p = q$ the tangent hyperplane of $M_l$ at $r$ equals the point condition $H_p$.*

*Proof.* Assume that $l$ is given by $x = 0$, and that $C_r$ is tangent to $l$ at $q = (0, 1, 0)$ and also meets $l$ at $p = (0, 0, 1)$. This means that $b = c = g = 0$ in the equation for $C_r$. We know from the beginning of Section 3 that $M_l$ is given by $bc + gi = 0$, and a simple computation shows that the tangent hyperplane of $M_l$ at $C_r$ is given by $g = 0$. $X_p$ $(b = g = 0)$ and $Y_q$ $(c = g = 0)$ are both contained in this hyperplane, and (1) follows. (2) follows by a similar argument ($H_p$ is given by $b = 0$). □

LEMMA 4.2.

(1) *Suppose $x \in T$ corresponds to the line $l$. The intersection of the tangent hyperplanes of the line conditions at $x$ is 5-dimensional and consists of the cubics having $l$ as a component.*
(2) *If $x \in L\backslash T$, the intersection of the tangent hyperplanes of the line conditions at $x$ is 4-dimensional and thus equals the tangent space of $L$ at $x$.*
(3) *The tangent space of $L$ at a triple line $l^3$ consists of the cubics having $l$ and a touching conic as components.*

*Proof.* (1) follows directly from the second part of Lemma 4.1. To prove (2) one needs to compute (using the first part of 4.1) tangent hyperplanes to 5 sufficiently general line conditions at a point $x \in L\backslash T$. For (3) assume $l$ is given by $x = 0$. Then the triple line $l^3$ corresponds to the point $(1, 0, \ldots, 0) \in \mathbf{P}^9$. By (3.4) we know that $L \subset \mathbf{P}^9$ is given by $j = 0$ and rk $H = 1$. A simple computation shows that the tangent space of $L$ at $l^3$ is given by $b = c = g = i = j = 0$ and the result follows. □

PROPOSITION 4.3. *The intersection $S$ of the line conditions in $B_1$ is a 2-dimensional subvariety of $E$. More precisely, $S = \mathbf{P}(\mathcal{L})$, where $\mathcal{L}$ is a sub line bundle of $i^*N$.*

*Proof.* Obviously, $S$ must be contained in the exceptional divisor $E$. Also, since the intersection of the line conditions is 'sufficiently transversal' at points in $L$ outside $T$ (the second part of Lemma 4.2) the line conditions $\tilde{M}$ in $B_1$ can only intersect in the fibres over $T$.

Let $v_3\colon \check{\mathbf{P}}^2 \to \mathbf{P}^9$ be the composition $j \circ i$, sending a line $l$ to the point corresponding to the triple line $l^3$. (This is the third Veronese embedding of $\check{\mathbf{P}}^2$ projected into the hyperplane $j = 0$.) Similarly, we let $v_2\colon \check{\mathbf{P}}^2 \to \mathbf{P}^5$ be the map sending $l$ to $l^2$, where $\mathbf{P}^5$ parametrizes the conics.

We have the following exact sequences of vector bundles on $T \simeq \check{\mathbf{P}}^2 = \mathbf{P}(\check{Q})$

$$0 \to \mathcal{O}_{\mathbf{P}^2} \to \operatorname{Sym}^3 \check{Q} \otimes \mathcal{O}(3) \to v_3^* T_{\mathbf{P}^9} \to 0,$$

$$0 \to \mathcal{O}_{\mathbf{P}^2} \to \operatorname{Sym}^2 \check{Q} \otimes \mathcal{O}(2) \to v_2^* T_{\mathbf{P}^5} \to 0,$$

$$0 \to i^* T_L \xrightarrow{f} v_3^* T_{\mathbf{P}^9} \to i^* N \to 0.$$

The first two are the pullbacks of the Euler sequences on $\mathbf{P}(\operatorname{Sym}^3 \check{Q})$ and $\mathbf{P}(\operatorname{Sym}^2 \check{Q})$, and the last is the pullback from $L$ of the standard sequence relating the normal bundle with the tangent bundles. From the composition

$$\operatorname{Sym}^2 \check{Q} \otimes \mathcal{O}(-1) \to \operatorname{Sym}^2 \check{Q} \otimes \check{Q} \to \operatorname{Sym}^3 \check{Q},$$

we get (by tensoring with $\mathcal{O}(3)$) an induced map $v_2^* T_{\mathbf{P}^5} \to v_3^* T_{\mathbf{P}^9}$. In the fiber over $l^3 \in T$ the image of this map can be identified with the cubics containing $l$. By the third part of (4.2) we see that the map $f$ in the third sequence above factors through $v_2^* T_{\mathbf{P}^5}$. The quotient $\mathcal{L} = v_2^* T_{\mathbf{P}^5} / i^* T_L$ is then a sub line bundle of $i^* N$. That $S = \mathbf{P}(\mathcal{L})$ can now be checked at each fibre using Lemma 4.2 (1). □

4.2. *The second blow-up*. Let $B$ be the blow-up of $B_1$ along $S$, and let $D$ be the exceptional divisor. To show that the intersection of the line conditions on $B$ is empty, we will introduce local coordinates on $B_1$ and compute the tangent hyperplanes of the line conditions there.

Let $U \subset \mathbf{P}^9$ be the open set where $a = 1$. Affine coordinates for $U$ are then $(b, c, d, e, f, g, h, i, j)$, and affine equations for $L$ are

$$b + df = 0, \quad c + eh = 0, \quad g + ef = 0, \quad i + dh = 0, \quad j = 0.$$

We may now choose affine coordinates $(\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}, \bar{j})$ on an open $V \subset B_1$ such that

$$\begin{aligned}
\bar{b}\bar{j} &= b + df, \quad \bar{d} = d, \quad \bar{j} = j.\\
\bar{g}\bar{j} &= g + ef, \quad \bar{e} = e,\\
\bar{c}\bar{j} &= c + eh, \quad \bar{f} = f,\\
\bar{i}\bar{j} &= i + dh, \quad \bar{h} = h,
\end{aligned}$$

Now $\bar{j} = 0$ is the exceptional divisor, and $(\bar{b}, \bar{c}, \bar{g}, \bar{i})$ are coordinates for a fiber in $E$. Let $l$ be the line given by $x + \alpha y + \beta z = 0$, and let $M_l$ be the corresponding line condition on $\mathbf{P}^9$. One may now calculate the equation of $\tilde{M}_l$ in $V$, which turns out to be

$$\det \begin{pmatrix} \bar{j} & \bar{f} + \alpha^2 & \bar{h} + \beta^2 \\ \bar{d} + \alpha & \bar{b} & \bar{i} + \beta \\ \bar{e} + \beta & \bar{g} + \alpha & \bar{c} \end{pmatrix} = 0.$$

Since $O = (0, 0, 0, 0, 0, 0, 0, 0, 0) \in V$ obviously lies on all the line conditions, we have $O \in S$. We need to show that the intersection of the tangent hyperplanes of the line conditions at $O$ equals the tangent plane of $S$ at $O$. By expanding the above determinant we find that $T_O \tilde{M}_l$ is given by

$$\alpha\beta\bar{j} + \beta^2 \bar{f} + \alpha^2 \bar{h} + \alpha\beta^2(\bar{d} + \bar{g}) + \beta^3 \bar{b} + \alpha^2\beta(\bar{e} + \bar{i}) + \alpha^3 \bar{c} = 0.$$

The intersection of these planes as $\alpha$ and $\beta$ varies $((\alpha, \beta) \neq (0, 0))$ is given by

$$\bar{b} = \bar{c} = \bar{f} = \bar{h} = \bar{j} = 0, \quad \bar{d} = \bar{g}, \quad \bar{e} = \bar{i},$$

which is 2-dimensional and thus equal to $T_O S$. This proves that the intersection of the line conditions on $B$ is empty. In other words

THEOREM 4.4. *The variety $B$, obtained by the sequence of two blow-ups of $\mathbf{P}^9$, is a variety of complete cubics.* □

## 5. The intersection ring of $B$

The first aim of this section is to give a complete multiplication table of the divisor classes on $B$. Our main tool will be the following intersection formula.

LEMMA 5.1. *Suppose $L \subset V$ are nonsingular varieties of dimensions $l$ and $n$. Let $\tilde{V}$ be the blowup of $V$ along $L$, and let $E$ be the exceptional divisor with maps as shown in the diagram*

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "p"'] & \tilde{V} \arrow[d, "\pi"] \\ L \arrow[r, "i"'] & V. \end{tikzcd}$$

*Suppose further that $N$ is the normal bundle of $L$ in $V$ with total Segre class $s(N)$. Assume $\beta \geqslant 1$, and let $x \in A_\beta(V)$ be the class of a $\beta$-dimensional cycle on $V$. Then the following formula holds.*

$$\int_{\tilde{V}} [E]^\beta \cdot \pi^* x = (\Leftrightarrow 1)^{\beta-1} \int_L s(N) \cdot i^* x.$$

*Proof*. Recall that $j^*[E] = c_1 \mathcal{O}_E(\Leftrightarrow 1)$ and that $s(N) = \Sigma_{r \geqslant 0} p_* c_1 \mathcal{O}_E(1)^r$ by definition. Now we have

$$\begin{aligned} \int_{\tilde{V}} [E]^\beta \cdot \pi^* x &= \int_E (j^*[E])^{\beta-1} \cdot j^* \pi^* x = \int_E (c_1 \mathcal{O}(\Leftrightarrow 1))^{\beta-1} \cdot p^* i^* x \\ &= (\Leftrightarrow 1)^{\beta-1} \int_L s(N) \cdot i^* x \end{aligned}$$

where the last equality is by the projection formula. □

In view of this lemma, what we need in order to compute intersection numbers on $B$ are the total Segre classes of the normal bundles $N_{L/\mathbf{P}^9}$ and $N_{S/B_1}$. Let $h_1$ and $h_2$ be the pullbacks of the hyperplane classes of each of the factors of $L \simeq \check{\mathbf{P}}^2 \times \check{\mathbf{P}}^2$, and let $t$ be the class of a line on $S \simeq \check{\mathbf{P}}^2$.

First we will examine the embedding $f$ of $S$ in $B_1$. Let $e = [E]$ be the class of the exceptional divisor in $B_1$. We would like to know the pullback of $e$ by $f$. We know that $e$ pulls back to $c_1\mathcal{O}(-1)$ on $E = \mathbf{P}(N_{L/\mathbf{P}^9})$ and then also to $c_1\mathcal{O}(-1)$ on $S = \mathbf{P}(\mathcal{L})$. But since $\mathcal{L}$ is a line bundle, $\mathcal{O}_S(-1) = \mathcal{L}$. It follows that

$$\begin{aligned} f^*e &= c_1\mathcal{L} = c_1(v_2^*T_{\mathbf{P}^5}/i^*T_L) = v_2^*c_1(T_{\mathbf{P}^5}) - i^*c_1(T_L) \\ &= v_2^*(6h) - i^*(3h_1 + 3h_2) = 12t - 9t = 3t. \end{aligned}$$

PROPOSITION 5.2.

$$\begin{aligned} (N_{L/\mathbf{P}^9}) &= 1 - 7h_1 - 7h_2 + 28h_1^2 + 28h_2^2 + 59h_1h_2 \\ &\qquad - 276h_1^2h_2 - 276h_1h_2^2 + 1479h_1^2h_2^2, \\ s(N_{S/B_1}) &= 1 - 15t + 120t^2. \end{aligned}$$

*Proof*. Recall that $L$ is the image of a Segre embedding, $r$, of $\check{\mathbf{P}}^2 \times \check{\mathbf{P}}^2$ in a hyperplane in $\mathbf{P}^9$. Then we have

$$c(N_{L/\mathbf{P}^9}) = \frac{r^*c(T_{\mathbf{P}^9})}{c(T_{\check{\mathbf{P}}^2 \times \check{\mathbf{P}}^2})} = \frac{(1 + h_1 + h_2)^{10}}{(1 + h_1)^3(1 + h_2)^3},$$

$s(N_{L/\mathbf{P}^9}) = c(N_{L/\mathbf{P}^9})^{-1}$ can now be obtained by expanding and inverting the above expression.

The second part is more complicated. The following exact sequence of vector bundles on $E$ is well known [6, Lem. 15.4]

$$0 \to \mathcal{O}_E \to p^*N_{L/\mathbf{P}^9} \otimes \mathcal{O}(1) \to T_E \to p^*T_L \to 0.$$

The total Chern class of the tensor product is given by (see [6, Remark 3.2.3])

$$\begin{aligned} c(p^*N_{L/\mathbf{P}^9} \otimes \mathcal{O}(1)) &= \sum_{i=0}^{5} c(\mathcal{O}(1))^{5-i} c_i(N_{L/\mathbf{P}^9}) \\ &= (1 - e)^5 + (1 - e)^4(7h_1 + 7h_2) \\ &\quad + (1 - e)^3(21h_1^2 + 21h_2^2 + 39h_1h_2) + \cdots. \end{aligned}$$

Restricting to $F = \pi^{-1}(T)$ we get (recall that $h_1$ and $h_2$ pull back to $2t$ and $t$ respectively)

$$\begin{aligned} k^*c(T_E) &= c(q^*i^*T_L)c(q^*i^*N_{L/\mathbf{P}^9} \otimes \mathcal{O}(1)) \\ &= (1 + 2t)^3(1 + t)^3(1 + 21t - 5e + 183t^2 - 84te + 10e^2 + \cdots) \\ &= 1 + 30t - 5e + 405t^2 - 129te + 10e^2 + \cdots, \end{aligned}$$

which pulls back on $S$ to $1+15t+108t^2$ (remember that $e$ pulls back to $3t$). Finally we have (we omit the pullbacks)

$$\begin{aligned} c(N_{S/B_1}) &= \frac{c(T_{B_1})}{c(T_S)} = \frac{c(N_{E/B_1})c(T_E)}{c(T_S)} \\ &= \frac{(1+e)(1+15t+108t^2)}{(1+t)^3} = 1+15t+105t^2 \end{aligned}$$

and the result follows by inverting this expression. □

Denote by $d = [D]$ the class of the exceptional divisor of the second blow-up. Also, denote by $e$ and $h$ the classes of the pullbacks to $B$ of $[E]$ and $[H]$ respectively. We omit pullback and integral signs when no confusion is likely to occur.

PROPOSITION 5.3. *The group of divisor classes on $B$ is generated by $d$, $e$ and $h$, and the multiplication table is as follows* (*all other terms are zero*)

$$\begin{aligned} &e^9 = 1479, && d^9 = 120, \\ &he^8 = 552, && d^8e = d^8h = 45, \\ &h^2e^7 = 174, && d^7e^2 = d^7eh = d^7h^2 = 9, \\ &h^3e^6 = 42, && \\ &h^4e^5 = 6, && h^9 = 1. \end{aligned}$$

*Proof.* The first assertion follows from the general theory of blowing-up [6, Sect. 6.7], and the numbers can be computed using (5.1) and (5.2) above. For example

$$\begin{aligned} h^2e^7 &= \int_B \pi_2^*\pi_1^*[H]^2\pi_2^*[E]^7 = \int_{B_1} \pi_1^*[H]^2[E]^7 \\ &= \int_L s(N_{L/\mathbf{P}^9})(h_1+h_2)^2 \\ &= \int_L (28h_1^2+28h_2^2+59h_1h_2)(h_1^2+h_2^2+2h_1h_2) \\ &= 28+28+118 = 174, \end{aligned}$$

where the last 'integral' was evaluated using that $h_1^2h_2^2 = 1$ and $h_i^3 = 0$ in $A(L)$. □

Suppose $W \subset \mathbf{P}^9$ is a hypersurface. The proper transforms of $W$ in $B$ and $B_1$ will be denoted by $\tilde{W}$ and $\tilde{W}_1$ respectively. We shall need a formula for computing $[\tilde{W}] \in A^1(B)$ in terms of $h$, $e$ and $d$. The following lemma follows directly from

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the general theory of blowing-up [6, Sect. 6.7].

LEMMA 5.4. *Let $m_1$ be the multiplicity of $W$ along $L$, and let $m_2$ be the multiplicity of $\tilde{W}_1$ along $S$. Then*

$$[\tilde{W}] = (\deg W)h \Leftrightarrow m_1 e \Leftrightarrow m_2 d$$

*in* $A^1(B)$. □

We shall also need to compute the classes of proper transforms of subvarieties of higher codimension. This can be done with Fulton's blow-up formula [6, Th. 6.7], but for our purposes it is more convenient to use a different version (Theorem 5.5 below).

Let $V$ be a nonsingular variety, $L$ a nonsingular closed subvariety, and let $X$ be any pure-dimensional subscheme of $V$. Define the full intersection class of $X$ by $L$ in $V$ by

$$L \circ X = c(N_{L/V}) \cap s(L \cap X, X).$$

Note that if $X$ has codimension 1 in $V$, $L \circ X = \mu_X(L) + i^*[X]$ where $\mu_X(L)$ is the multiplicity of $X$ along $L$ (see [1, Sect. 2]). Applied to our first blow-up we have for example

$$L \circ H = h_1 + h_2,$$

$$L \circ M = 1 + 2h_1 + 2h_2.$$

Another convenient result is that $L \circ X = X \circ L$ when both $L$ and $X$ are nonsingular [2, Lem. A.1].

THEOREM 5.5. [1, Thm. II]. *Suppose that $\tilde{V}$ is the blowup of $V$ along $L$ as in* (5.1), *and that $X_1, \ldots, X_r$ are pure-dimensional subschemes of $V$ whose codimensions add to the dimension of $V$. Then the following formula holds.*

$$\int_{\tilde{V}} [\tilde{X}_1] \cdot \cdots \cdot [\tilde{X}_r] = \int_V [X_1] \cdot \cdots \cdot [X_r] \Leftrightarrow \int_L \frac{\prod (L \circ X_v)}{c(N_{L/V})}.$$

*Proof.* Details may be found in [1]. □

## 6. Characteristic numbers of nonsingular cubics

We will now determine the characteristic numbers $N_{\alpha,\beta}$ for the family of all nonsingular cubics. By (2.8) this amounts to compute the intersections $[\tilde{H}]^\alpha [\tilde{M}]^\beta$ in $A(B)$. The following lemma is an application of (5.4).

LEMMA 6.1. *In the intersection ring $A(B)$ we have the following relations*:

(1) $[\tilde{H}] = h$.

(2) $[\tilde{M}] = 2h - e - d$.

*Proof.* (1) is obvious as $L$ is not contained in any point condition. From (3.1) we know that $bc + gi = 0$ is the equation for the line condition corresponding to the line $x = 0$. It follows that $M$ has degree 2 and is generically smooth along $L$. We also have that $\tilde{M}_1$ is generically smooth along $S$ (in Section 4.2 we computed the tangent spaces of $\tilde{M}_1$ at points in $S$). Hence (2) follows from Lemma 5.4. □

By this lemma the characteristic numbers are given by

$$N_{\alpha,\beta} = h^{\alpha}(2h - e - d)^{\beta}.$$

This can be evaluated by (5.3), and the result is:

THEOREM 6.2. *The characteristic numbers for nonsingular cubic curves in characteristic* 2 *are*

$$N_{9,0}, N_{8,1}, \ldots, N_{0,9} = 1, 2, 4, 8, 16, 26, 34, 29, 13, 2,$$

*where the last number, $N_{0,9}$, counts one curve with multiplicity* 2, *and the other numbers count each curve once.*

*Proof.* We only need to justify the multiplicities. Since by (2.3) all multiplicities must be powers of 2, the second last number, $N_{1,8} = 13$, must count each curve with multiplicity one. By the second part of (2.3) so must the 8 preceding numbers.

We will now show that $N_{0,9}$ only counts one curve. Assume that $C_1$ and $C_2$ are 2 different nonsingular cubics tangent to 9 given lines in general position. The dual curves, $C_1^{\vee}$ and $C_2^{\vee}$, then contain the 9 corresponding points in $\check{\mathbf{P}}^2$. Since there is only one cubic $D$ passing through the 9 given points, we must have that $C_1^{\vee} = C_2^{\vee}$. Since $N_{0,9} = 2$, $C_1$ and $C_2$ are the only cubic curves having $D$ as dual.

If we assume that the equation of $D$ is in normal form, $x^3+y^3+z^3+t^2xyz = 0$, we have from Section 3 that the curve given by $x^3 + y^3 + z^3 + txyz = 0$ is the only cubic in normal form having $D$ as dual. We may then assume that $C_1$ is in normal form, and that $C_2$, given by $F(x, y, z) = 0$, is not. Obviously, the 6 curves given by $F$ and permutations of the variables must all have $D$ as dual. It follows that these 6 curves must be equal, so we may assume that $F(x, y, z)$ is a symmetric polynomial (with $d \neq 0$)

$$F(x, y, z) = a(x^3 + y^3 + z^3) + d(x^2y + \cdots + yz^2) + jxyz.$$

By (3.1) the dual of $C_2$ is given by

$$(a^2 + d^2)(x^3 + y^3 + z^3) + d(a + d + j)(x^2y + \cdots + yz^2) + j^2xyz.$$

Since this polynomial is assumed to be in normal form we must have that $a+d=j$. Then $a^2+d^2=j^2$, and the equation for $C_2^\vee$ reduces to $x^3+y^3+z^3+xyz=0$, which implies that $C_2^\vee$ is singular. This is a contradiction since $C_2$ was assumed nonsingular. $\square$

*Remark*. Let $\delta\colon \mathbf{P}^9 \cdots \to \check{\mathbf{P}}^9$ be the rational map associating to each nonsingular cubic its dual. Then we obviously have that $\overline{\delta(M_l)} = H_{l^\vee}$, but since biduality does not hold we cannot expect $\overline{\delta(H_p)}$ to be a line condition. In fact, the degree of $\overline{\delta(H_p)} \subset \check{\mathbf{P}}^9$ must equal $N_{1,8}$. (The duals of the curves counted by $N_{1,8}$ is the intersection of $\delta(H_p)$ and 8 point conditions in $\check{\mathbf{P}}^9$.) Computing the degree of the image of a variety by a rational map can be done with the help of *Macaulay*. Doing this, we find that $\overline{\delta(H_p)} \subset \check{\mathbf{P}}^9$ is given by a polynomial of degree 13 (with 303 terms), confirming our computation of $N_{1,8}$. (By the same argument, *Macaulay* should in principle be able to compute $N_{2,7}=29$, $N_{3,6}=34$ and $N_{4,5}=26$ as well. These numbers are the degrees of the images of lower-dimensional linear spaces. The complexity of these computations seems to be more than our installation of *Macaulay* can handle within reasonable time.)

## 7. Characteristic numbers of nodal cubics

The computation of the characteristic numbers for nodal cubics is considerably more difficult than in the nonsingular case. Here we will take advantage of Aluffi's results and methods in [1], [2] and [3]; in particular Theorem 5.5 and the results about the full intersection classes. Many of our intermediate results are similar to Aluffi's and some of the proofs carry over.

Suppose that $R$ is an $r$-dimensional family of singular curves where the generic curve is reduced and irreducible. Denote by $R^l$ the curves with a singularity on a given line $l$, and $R^p$ those with singularity at a given point $p$. The following definition will be useful when we consider nodal and cuspidal curves.

DEFINITION 7.1. Suppose $\alpha+\beta = r \Leftrightarrow 1$. Define the following numbers associated to the family $R$

$$\Gamma^l_{\alpha,\beta} = \text{the total characteristic numbers for } R^l,$$

$$N^l_{\alpha,\beta} = \text{the characteristic numbers for } R^l.$$

Suppose $\alpha+\beta = r \Leftrightarrow 2$. Then define

$$\Gamma^p_{\alpha,\beta} = \text{the total characteristic numbers for } R^p,$$

$$N^p_{\alpha,\beta} = \text{the characteristic numbers for } R^p.$$

Note that when the generic curve in $R$ has exactly one singularity we clearly have $N^p_{\alpha,\beta} = \Gamma^p_{\alpha,\beta}$.

Letting $R$ in the above definition be the nodal cubics, the numbers $\Gamma_{\alpha,\beta}$, $\Gamma^l_{\alpha,\beta}$ and $N^p_{\alpha,\beta}$ are the total characteristic numbers for the three families $N$, $G$ and $P$ where

$$\begin{aligned} N &= \text{nodal cubics}, \\ G &= \text{cubics with singularity on a given line } l, \\ P &= \text{cubics with singularity at a given point } p. \end{aligned}$$

Also, let $F$ be the nodal cubics properly tangent to a given line $l$. The characteristic numbers $N_{\alpha,\beta}$ follow from the total characteristic numbers by the following lemma. (Compare with [3, Thm. I].)

LEMMA 7.2. *We have the following relations between the characteristic numbers and the total characteristic numbers for the families $N$, $G$ and $P$*

$$\begin{aligned} \Gamma_{\alpha,\beta} &= N_{\alpha,\beta} + \beta N^l_{\alpha,\beta-1} + \binom{\beta}{2} N^p_{\alpha,\beta-2}, \\ \Gamma^l_{\alpha,\beta} &= N^l_{\alpha,\beta} + \beta N^p_{\alpha,\beta-1}. \end{aligned}$$

*Proof.* We will prove that $[N \cap M_l] = [F] + [G]$, in other words that $N$ and $M_l$ intersect transversally. Let $x$ be a general point of $N \cap M_l$. All we need to show is that the tangent hyperplanes $T_x N$ and $T_x M_l$ are different. It is sufficient to show this for a general line $l$ tangent to $C_x$. But (4.1) tells us that $T_x M_l$ varies as $l$ varies, so in general we must have that $T_x M_l \neq T_x N$. The lemma now follows by arguments similar to those following Lemma 1.3 in [3]. $\square$

*Remark.* That $N$ and $M$ intersect transversally is special for characteristic 2. In all other characteristics we have $[N \cap M_l] = [F] + 2[G]$ as shown in [3, Lem. 1.3]. The rest of Aluffi's proof is characteristic free. The only difference is that the multiplicity 2 appears as a coefficient in the formulas.

Consider the following blow-up diagram where $B_1$ is the blowup of $\mathbf{P}^9$ along $L$ as before.

$$\begin{tikzcd} \mathbf{P}^2 \times E \arrow[r, "j"] \arrow[d, "g"'] & \mathbf{P}^2 \times B_1 \arrow[d, "f"] \\ \mathbf{P}^2 \times L \arrow[r, "i"'] & \mathbf{P}^2 \times \mathbf{P}^9. \end{tikzcd}$$

Let $D \subset \mathbf{P}^2 \times \mathbf{P}^9$ be given by the vanishing of the three partial derivatives of $F = ax^3 + by^3 + \cdots + jxyz$. This means that $(p, t) \in D$ if and only if $C_t$ is singular at $p$. Let $k$ be the class of a hyperplane in $\mathbf{P}^2$.

LEMMA 7.3. *The full intersection class* $(\mathbf{P}^2 \times L) \circ D$ *(regarded as a class in* $\mathbf{P}^2 \times L$*) is a quadratic polynomial in* $k$*, and the coefficients are given by*

$$\begin{aligned} L \circ N &= \text{coefficient of } k^2, \\ L \circ G &= \text{coefficient of } k, \\ L \circ P &= \text{the constant term.} \end{aligned}$$

*Proof.* This follows from the birational invariance of Segre classes [5, Sect. 4.2]. See [2, Prop. 2.1] and [3, Lem. 2.2] for details of the proofs of similar results. □

PROPOSITION 7.4. *The full intersection classes of* $N, G$ *and* $P$ *by* $L$ *are*

$$\begin{aligned} L \circ N &= 8 + 12(h_1 + h_2), \\ L \circ G &= 2 + 8h_1 + 6h_2 + 6(h_1 + h_2)^2, \\ L \circ P &= h_1 + 2h_1^2 + 3h_1h_2 + (h_1 + h_2)^3. \end{aligned}$$

*Proof.* By (7.3) this amounts to computing $(\mathbf{P}^2 \times L) \circ D$. Let $W = (\mathbf{P}^2 \times L) \cap D$. (That the intersection is transversal can easily be checked by pulling the equations for $D$ back to $\mathbf{P}^2 \times L$.) This means that $W = \{(p, l^2m) : p \in l\} \subset \mathbf{P}^2 \times L$, so that $[W] = h_1 + 2k \in A(\mathbf{P}^2 \times L)$. Letting $p$ be the inclusion of $W$ in $\mathbf{P}^2 \times L$ we have

$$p_* s(W, \mathbf{P}^2 \times L) = p_* \left[ \frac{1}{c(N_{W/\mathbf{P}^2 \times L})} \right] = \frac{h_1 + 2k}{1 + h_1 + 2k}.$$

Since $D \subset \mathbf{P}^2 \times \mathbf{P}^9$ is given by 3 equations of bidegrees (2,1) we have $c(N_{D/\mathbf{P}^2 \times \mathbf{P}^9}) = (1 + h + 2k)^3$ which pulls back on $\mathbf{P}^2 \times L$ to $(1 + h_1 + h_2 + 2k)^3$.

$$\begin{array}{ccc} \mathbf{P}^2 \times L & \xrightarrow{i} & \mathbf{P}^2 \times \mathbf{P}^9 \\ {\scriptstyle p}\uparrow & & \uparrow \\ W & \longrightarrow & D. \end{array}$$

As all the varieties in the above diagram are nonsingular, the full intersection classes commute, and we get (as classes in $\mathbf{P}^2 \times L$)

$$\begin{aligned} & p_*[(\mathbf{P}^2 \times L) \circ D] \\ & \quad = p_*[D \circ (\mathbf{P}^2 \times L)] = p_*[c(N_{D/\mathbf{P}^2 \times \mathbf{P}^9}) s(W, \mathbf{P}^2 \times L)] \\ & \quad = p_* s(W, \mathbf{P}^2 \times L) i^* c(N_{D/\mathbf{P}^2 \times \mathbf{P}^9}) = \frac{(1 + h_1 + h_2 + 2k)^3 (h_1 + 2k)}{1 + h_1 + 2k} \end{aligned}$$

and the result follows by expanding the last expression. □

Since $L \circ N = 8 + 12(h_1 + h_2)$, the multiplicity, $m_1$, of $N$ along $L$ is 8. Let $m_2$ denote the multiplicity of $\tilde{N}_1$ along $S$. By (5.4) we have that $[\tilde{N}] = 12h \Leftrightarrow 8e \Leftrightarrow m_2 d$. To determine $m_2$ we will compute $\Gamma_{0,8}$ in two different ways. Since the dual of a nodal cubic is a conic, no nodal cubic can be tangent to more than 5 given lines in general position. Also, at most two of the lines can pass through the node, so we see that $\Gamma_{0,8} = 0$. On the other hand by (2.8) and (5.3) we have

$$\Gamma_{0,8} = \int_B (12h \Leftrightarrow 8e \Leftrightarrow m_2 d)(2h \Leftrightarrow e \Leftrightarrow d)^8 = 60 \Leftrightarrow 12m_2,$$

so we must have $m_2 = 5$.

THEOREM 7.5. *The following table gives the complete list of characteristic numbers and total characteristic numbers for the families $N$, $G$ and $P$*

| $\alpha, \beta$ | $\Gamma_{\alpha,\beta}$ | $N_{\alpha,\beta}$ | $\Gamma^l_{\alpha,\beta-1}$ | $N^l_{\alpha,\beta-1}$ | $N^p_{\alpha,\beta-2}$ |
|---|---|---|---|---|---|
| 8, 0 | 12 | 12 | | | |
| 7, 1 | 24 | 18 | 6 | 6 | |
| 6, 2 | 48 | 25 | 12 | 11 | 1 |
| 5, 3 | 96 | 30 | 24 | 20 | 2 |
| 4, 4 | 144 | 24 | 36 | 24 | 4 |
| 3, 5 | 168 | 8 | 42 | 22 | 5 |
| 2, 6 | 123 | 0 | 33 | 8 | 5 |
| 1, 7 | 42 | 0 | 12 | 0 | 2 |
| 0, 8 | 0 | 0 | 0 | 0 | 0 |

*Proof*. The total characteristic numbers for $N$ are given by

$$\Gamma_{\alpha,\beta} = \int_B [\tilde{N}][\tilde{H}]^\alpha [\tilde{M}]^\beta = h^\alpha (2h \Leftrightarrow e \Leftrightarrow d)^\beta (12h \Leftrightarrow 8e \Leftrightarrow 5d),$$

which can be evaluated by (5.3).

The numbers $\Gamma^l_{\alpha,\beta}$ and $N^p_{\alpha,\beta}$ with $\beta \leqslant 4$ follow by applying (5.5) and (7.4) to the first blow-up. For example

$$\Gamma^l_{3,4} = \int_B [\tilde{G}][\tilde{H}]^3[\tilde{M}]^4 = \int_{B_1} [\tilde{G}_1][\tilde{H}_1]^3[\tilde{M}_1]^4$$

(since the intersection of 3 general point conditions in $B_1$ does not meet $S$)

$$= \int_{\mathbf{P}^9} [G][H]^3[M]^4 \Leftrightarrow \int_L (L \circ G)(L \circ H)^3 (L \circ M)^4 s(N_{L/\mathbf{P}^9})$$

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$$
\begin{aligned}
&= 96 \Leftrightarrow (2 + 8h_1 + 6h_2 + \cdots)(h_1 + h_2)^3 \\
&\quad \times (1 + 2h_1 + 2h_2)^4 (1 \Leftrightarrow 7h_1 \Leftrightarrow 7h_2 + \cdots) \\
&= 96 \Leftrightarrow (h_1 + h_2)^3 (2 + 10h_1 + 8h_2 + \cdots) = 96 \Leftrightarrow 54 = 42.
\end{aligned}
$$

All the zeros follow from the fact (3.2) that the dual of a nodal cubic is a nonsingular conic, so that a nodal cubic can be properly tangent to at most 5 lines in general position. The remaining numbers can now be computed using the relations in (7.2) □

*Remark*. Some of the arguments above (in particular the 'zero-arguments') could have been replaced by the computation of the full intersection classes

$$
\begin{aligned}
S \circ \tilde{N}_1 &= 5 + 12t, \\
S \circ \tilde{G}_1 &= 1 + 7t + 6t^2 \\
S \circ \tilde{P}_1 &= t + 2t^2,
\end{aligned}
$$

combined with another application of Theorem 5.5. See [2, Sect. 2.1] for a similar computation in characteristic 0.

## 8. Other characteristic numbers

In Section 3 we mentioned the very special class of curves called $j$-curves. These curves are parametrized by an open subset of the hypersurface $J \subset \mathbf{P}^9$ given by $j = 0$. The computation of the characteristic numbers for this family is now an easy consequence of our previous results.

LEMMA 8.1 $[\tilde{J}] = h \Leftrightarrow e$ *as a class in the intersection ring of* $B$.

*Proof*. We know that $\deg J = 1$, and that $L \subset J$ with multiplicity one, so by (5.4) we only need to prove that $S$ is not contained in $\tilde{J}_1$. From Section 4 we know that $S = \mathbf{P}(\mathcal{L})$ where the fiber of $\mathcal{L}$ over a triple line $l^3$ can be identified with $\cap T_{l^3} M_l / T_{l^3} L$. Now, by (4.2), $\cap T_{l^3} M_l$ is just the 5-dimensional space of cubics containing the line $l$. Obviously, $\cap T_{l^3} M_l \not\subset J$, and it follows that $S \not\subset \tilde{J}_1$. □

THEOREM 8.2. *The characteristic numbers for the family,* $J$, *of cubics with* $j$*-invariant* 0 *are*

$$N_{8,0}, N_{7,1}, \ldots, N_{0,8} = 1, 2, 4, 8, 10, 8, 4, 2, 1$$

*and all the numbers above count curves with multiplicity* 1.

*Proof*. We apply (2.8) with $R = J$ to obtain $N_{\alpha,\beta} = h^{\alpha}(2h \Leftrightarrow e \Leftrightarrow d)^{\beta}(h \Leftrightarrow e)$ which can be evaluated by (5.3). Since the last number, $N_{0,8} = 1$, clearly counts

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curves with multiplicity 1, it follows from (2.3) that all the other numbers will also count curves with multiplicity 1. $\square$

*Remark*. The symmetry of the numbers in (8.2) reflects the fact that the dual of a $j$-curve is also a $j$-curve, and that $(C^\vee)^\vee = C$ for a $j$-curve $C$. This is similar to the case of smooth conics and cuspidal cubics in characteristic 0.

We now proceed to the characteristic numbers for families of cuspidal cubics. Let $K, G_k$ and $P_k$ be the subvarieties of $\mathbf{P}^9$ defined by

$$\begin{aligned} K &= \text{cuspidal cubics,} \\ G_k &= \text{cubics with cusp on a given line } l \\ P_k &= \text{cubics with cusp at a given point } p. \end{aligned}$$

LEMMA 8.3. *We have the following relations in the intersection ring of $B$*

$$\begin{aligned} [\tilde{N}][\tilde{J}] &= 4[\tilde{K}], \\ [\tilde{P}][\tilde{J}] &= [\tilde{P}_k]. \end{aligned}$$

*Proof*. We know from Section 3 that $K$ has degree 3 and is contained in $J$. From the equation for $K$ we can easily check that $K$ is singular along $L$ of multiplicity 2. It follows that $[\tilde{K}] = 3h \Leftrightarrow 2e$ regarded as a class in $A(\tilde{J})$ (we omit the pullback signs). Note that the proof of (8.1) also shows that $S \cap \tilde{J}_1 = \emptyset$, so that $i^*d = 0$, where $i$ is the inclusion $\tilde{J} \subset B$. Now, we have (as classes in $A(B)$)

$$[\tilde{N}][\tilde{J}] = i_* i^* [\tilde{N}] = i_* i^* (12h \Leftrightarrow 8e \Leftrightarrow 5d) = i_*(12h \Leftrightarrow 8e) = 4[\tilde{K}].$$

That $[\tilde{P}][\tilde{J}] = [\tilde{P}_k]$ is clear since $P$ and $J$ are both linear subspaces of $\mathbf{P}^9$ intersecting transversally along $P_k$. $\square$

We now use the notation from Definition 7.1 with $R = K$, the cuspidal cubics. This means that $N_{\alpha,\beta}$, $N^l_{\alpha,\beta}$ and $N^p_{\alpha,\beta}$ are the characteristic numbers for the three families $K$, $G_k$ and $P_k$, and $\Gamma_{\alpha,\beta}$, $\Gamma^l_{\alpha,\beta}$ are corresponding total characteristic numbers. Also, let $F_k$ be the cuspidal cubics properly tangent to a given line $l$. From (3.7) we know that $F_k$ can also be described as the cuspidal cubics with a flex on the given line $l$. The following lemma is the cuspidal version of Lemma 7.2.

LEMMA 8.4. *We have the following relations between the characteristic numbers and the total characteristic numbers for the families $K$, $G_k$ and $P_k$*

$$\begin{aligned} \Gamma_{\alpha,\beta} &= N_{\alpha,\beta} + \beta N^l_{\alpha,\beta-1} + \binom{\beta}{2} N^p_{\alpha,\beta-2}, \\ \Gamma^l_{\alpha,\beta} &= N^l_{\alpha,\beta} + \beta N^p_{\alpha,\beta-1}. \end{aligned}$$

*Proof*. This is similar to the proof of (7.2). All we need to show is that $[K \cap M_l] = [F_k] + [G_k]$. Recall that $\deg(K) = 3$ and $\deg(M_l) = 2$. From (3.8) we have that $G_k$ and $F_k$ both have degree 3, and it follows that $K$ and $M_l$ intersect transversally. □

Before we prove our final result, we note that

$$\begin{aligned} L \circ J &= \mu_J(L) + j^*[J] = 1 + h_1 + h_2, \\ S \circ \tilde{J}_1 &= \mu_{\tilde{J}_1}(S) + f^*[\tilde{J}_1] = f^*(h \Leftrightarrow e) = 3t \Leftrightarrow 3t = 0, \end{aligned}$$

where $f$ is the embedding of $S$ in $B_1$. The last relation, $S \circ \tilde{J}_1 = 0$, basically tells us that the second blow-up is superfluous when our family is contained in $J$.

THEOREM 8.5. *The following table gives the complete list of characteristic numbers and total characteristic numbers for the families $K$, $G_k$ and $P_k$, and all the numbers count curves with multiplicity* 1.

| $\alpha, \beta$ | $\Gamma_{\alpha,\beta}$ | $N_{\alpha,\beta}$ | $\Gamma^l_{\alpha,\beta-1}$ | $N^l_{\alpha,\beta-1}$ | $N^p_{\alpha,\beta-2}$ |
|---|---|---|---|---|---|
| $7, 0$ | 3 | 3 | | | |
| $6, 1$ | 6 | 3 | 3 | 3 | |
| $5, 2$ | 12 | 1 | 6 | 5 | 1 |
| $4, 3$ | 12 | 0 | 6 | 2 | 2 |
| $3, 4$ | 6 | 0 | 3 | 0 | 1 |
| $2, 5$ | 0 | 0 | 0 | 0 | 0 |
| $1, 6$ | 0 | 0 | 0 | 0 | 0 |
| $0, 7$ | 0 | 0 | 0 | 0 | 0 |

*Proof*. The total characteristic numbers for the families $K$ and $P_k$ follow from (5.5) and (8.3). Recall that $S \circ \tilde{J}_1 = 0$ so that we only need to apply (5.5) on the first blow-up. For example

$$\begin{aligned} \Gamma_{3,4} &= \int_B [\tilde{K}][\tilde{H}]^3[\tilde{M}]^4 = \frac{1}{4}\int_B [\tilde{N}][\tilde{J}][\tilde{H}]^3[\tilde{M}]^4 \\ &= \frac{1}{4}\left(\int_{\mathbf{P}^9} [N][J][H]^3[M]^4 \right. \\ &\qquad \left. \Leftrightarrow \int_L (L \circ N)(L \circ J)(L \circ H)^3 (L \circ M)^4 s(N_{L/\mathbf{P}^9})\right) \\ &= \tfrac{1}{4}(192 \Leftrightarrow 168) = 6. \end{aligned}$$

Since $G_k$ has degree 3 we clearly have $\Gamma^l_{6,0} = 3$ and $\Gamma^l_{5,1} = 6$. Also, since a cuspidal cubic is strange, it can not be properly tangent to 3 general lines. It follows that

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$N_{\alpha,\beta} = 0$ when $\beta \geqslant 3$. The remaining numbers in the above table now follow by Lemma 8.4.

It remains to prove that the multiplicities are 1. By (2.3) this holds for $N_{\alpha,\beta}$ and $N^p_{\alpha,\beta}$. If we can show that $N^l_{4,2}$ counts two different curves, the result will be true for $N^l_{\alpha,\beta}$ and then by (8.4) also for $\Gamma_{\alpha,\beta}$ and $\Gamma^l_{\alpha,\beta}$.

$N^l_{4,2}$ counts the curves passing through 4 given points, with the flex at another given point $p$ (the intersection of the two lines) and with cusp on a given line $l$. The curves counted by $N^p_{4,1}$ have the same description with cusp and flex interchanged. Suppose $p = (0,0,1)$ and $l$ is given by $z = 0$. Let $C$ be a curve counted by $N^p_{4,1}$. Let $H$ be its matrix and let $C^t$ be the cuspidal cubic given by the transpose $H^t$. Since cof$(H^t) = (\text{cof } H)^t$ it follows from (3.7) that $C^t$ has a flex at $p$ and a cusp on $l$. So $C^t$ is counted by $N^l_{4,2}$. Since $N^p_{4,1}$ counts different curves then also $N^l_{4,2}$ must count different curves. □

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