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CANTORIAN MODELS OF PREDICATIVE NF

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Abstract. Tangled Type Theory was introduced by Randall Holmes in [3] as a new way of approaching the consistency problem for NF. Although the task of finding models for this theory is far from trivial (considering it is equiconsistent with NF), ways of constructing models for certain fragments of it have been discovered. In this article, we present a simpler way of constructing models of predicative Tangled Type Theory and consequently of predicative NF. In these new models of predicative NF, the universe is well-orderable and equinumerous to the set of singletons.

§1. Introduction.

1.1. Tangled type theory. The language $\mathcal{L}_{\text{TST}}$ of Simple Type Theory is the many-sorted language of set theory with one binary relation symbol ε and countably many types (or sorts) indexed by $\mathbb{N}$. Each variable of $\mathcal{L}_{\text{TST}}$ is assigned a unique type, which we indicate by a superscript. The $\mathcal{L}_{\text{TST}}$-formulas are built inductively from the atomic formulas $x^i \varepsilon y^{i+1}$ and $x^i = y^i$ in the usual way.

Simple Type Theory (TST) is axiomatized by two sets of axioms. The *Axiom of Extensionality* (Ext) is the set of all the following sentences for each type $i \in \mathbb{N}$:

$$\forall x^{i+1}, y^{i+1}(x^{i+1} = y^{i+1} \leftrightarrow \forall z^i(z^i \varepsilon x^{i+1} \leftrightarrow z^i \varepsilon y^{i+1})). \qquad (\text{Ext}^{i+1})$$

The *Axiom of Comprehension* (Co) is the set of all the following sentences for each type $i \in \mathbb{N}$ and formula ϕ of $\mathcal{L}_{\text{TST}}$:

$$\forall \bar{u} \exists y^{i+1} \forall x^i (x^i \varepsilon y^{i+1} \leftrightarrow \phi(x^i, \bar{u})), \qquad (\text{Co}^{i+1})$$

where y^{i+1} is not free in ϕ. We define TST = Ext + Co. For each $i \in \mathbb{N}$, we let CoP^{i+1} be the axiom we get from Co^{i+1} if we restrict the types of the bound variables in ϕ to not exceed i and the types of the free variables in ϕ to not exceed $i+1$. The *Axiom of predicative Comprehension* (CoP) is the set of all CoP^{i+1} for $i \in \mathbb{N}$. We define *predicative Simple Type Theory* (TSTP) as TSTP = Ext + CoP.

Now, the language $\mathcal{L}_{\text{TTT}}$ of Tangled Type Theory is the same as $\mathcal{L}_{\text{TST}}$, but its formulas are built inductively from the atomic formulas $x^i = y^i$ and $x^i \varepsilon y^j$ for $i < j$. For each function $s \colon \mathbb{N} \to \mathbb{N}$ and each $\mathcal{L}_{\text{TST}}$-formula ϕ, we denote by ϕ^s the

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$\mathcal{L}_{\text{TTT}}$-formula that we get if we replace each type index i of a variable in ϕ with $s(i)$. For each set of $\mathcal{L}_{\text{TST}}$-sentences T, we let

$$T^\circ = \{\sigma^s : \sigma \in T \text{ and } s : \mathbb{N} \to \mathbb{N} \text{ strictly increasing}\}.$$

Tangled Type Theory (TTT) is defined as TTT $=$ TST$^\circ$, whereas *predicative Tangled Type Theory* (TTTP) as TTTP $=$ TSTP$^\circ$.

A *structure $\mathcal{A}$ for the language* $\mathcal{L}_{\text{TST}}$ is a sequence $(A_0, A_1, \dots, \{\varepsilon^{\mathcal{A}}_{i,i+1}\}_{i\in\mathbb{N}})$, where $A_0, A_1, \dots$ are non-empty sets interpreting the countably many types of $\mathcal{L}_{\text{TST}}$, and each $\varepsilon^{\mathcal{A}}_{i,i+1} \subseteq A_i \times A_{i+1}$ is a binary relation interpreting ε for type $i \in \mathbb{N}$. Similarly, a *structure $\mathcal{A}$ for the language* $\mathcal{L}_{\text{TTT}}$ is a sequence $(A_0, A_1, \dots, \{\varepsilon^{\mathcal{A}}_{i,j})\}_{i<j}$, where $A_0, A_1, \dots$ are non-empty sets, and $\varepsilon^{\mathcal{A}}_{i,j} \subseteq A_i \times A_j$, for all $i, j \in \mathbb{N}$ such that $i < j$.

Let us now introduce the notion of standard transitive $\mathcal{L}_{\text{TTT}}$-structure. First, we fix some notation about tuples in our metatheory. For all n-tuples $u = (u_1, \dots, u_n)$ and $0 \leq i \leq n$, where $n > 0$, we let

$$(u)_i = \begin{cases} u, & \text{if } i = 0, \\ u_i, & \text{o.w.} \end{cases}$$

Definition 1.1. An $\mathcal{L}_{\text{TTT}}$-structure $\mathcal{A} = (A_0, A_1, \dots, \{\varepsilon^{\mathcal{A}}_{i,j}\}_{i<j})$ is *standard transitive* if:

(i) for all $n \in \mathbb{N}$,

$$A_{n+1} \subseteq \prod_{i=0}^{n} \mathcal{P}(\{(u)_i : u \in A_i\}),$$

(ii) for all $i < j$, $u \in A_i$, and $v \in A_j$,

$$u\varepsilon^{\mathcal{A}}_{i,j} v \Leftrightarrow (u)_i \in (v)_{i+1}.$$

To simplify notation, we will denote $\mathcal{A}$ as $(A_0, A_1, \dots, \in)$.

Note. *Let us make the definition above a bit less confusing. First of all, notice that in Tangled Type Theory, every set of type n has n extensions (one for each type below n). The elements of A_n are basically n-tuples that code this fact. More precisely, it follows by the definition of $\varepsilon^{\mathcal{A}}_{i,n}$ that the extension of a set $v \in A_n$ over type i is its $(i+1)$-th projection $(v)_{i+1}$. It is important to note that the extension of v over type i is a set of i-th projections of elements of A_i and not a set of elements of A_i. So, in the sense we just described, an element of A_n is a tuple of its n extensions over types $0, \dots, n-1$. Keep in mind that there is nothing mysterious about the tuples $v = ((v)_1, \dots, (v)_n)$ in A_n; each $(v)_i$ is simply an element of $\mathcal{P}^i(A_0)$. It is also worth noting that we imposed no restrictions on A_0, which means that A_0 can be any set.*

It is always easier to work with standard transitive structures, and as we show below we may always assume that extensional $\mathcal{L}_{\text{TTT}}$-structures (i.e., structures that satisfy Ext$^\circ$) are standard transitive.

Definition 1.2. Let $\mathcal{A}$, $\mathcal{B}$ be two $\mathcal{L}_{\text{TTT}}$-structures. We say that f is an $\mathcal{L}_{\text{TTT}}$-*isomorphism from $\mathcal{A}$ to $\mathcal{B}$*, if f is a sequence $(f_0, f_1, \dots)$ of functions such that:

(i) for all $i \in \mathbb{N}$, $f_i : A_i \to B_i$ is a bijection,

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(ii) for all $i, j \in \mathbb{N}$ such that $i < j$, and for all $u \in A_i$ and $v \in A_j$,

$$u \varepsilon_{i,j}^{\mathcal{A}} v \Leftrightarrow f_i(u) \varepsilon_{i,j}^{\mathcal{B}} f_j(v).$$

When such an $\mathcal{L}_{\mathrm{TTT}}$-isomorphism exists, we say that $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{L}_{\mathrm{TTT}}$ -*isomorphic* (or just *isomorphic*).

The next proposition follows easily by induction on the complexity of ϕ.

PROPOSITION 1.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{L}_{\mathrm{TTT}}$-structures. If $f : \mathcal{A} \to \mathcal{B}$ is an $\mathcal{L}_{\mathrm{TTT}}$-isomorphism, then for every formula $\phi(x_1^{i_1}, \dots, x_n^{i_n})$ of $\mathcal{L}_{\mathrm{TTT}}$ and $a_1 \in A_{i_1}, \dots, a_n \in A_{i_n}$, we have that*

$$\mathcal{A} \models \phi(a_1, \dots, a_n) \Leftrightarrow \mathcal{B} \models \phi(f_{i_1}(a_1), \dots, f_{i_n}(a_n)).$$

LEMMA 1.4. *Every extensional $\mathcal{L}_{\mathrm{TTT}}$-structure is isomorphic to a standard transitive $\mathcal{L}_{\mathrm{TTT}}$-structure.*

PROOF. Let $\mathcal{A} = (A_0, A_1, \dots, \{\varepsilon_{i,j}^{\mathcal{A}}\}_{i<j})$ be an extensional $\mathcal{L}_{\mathrm{TTT}}$-structure. We define a standard transitive $\mathcal{L}_{\mathrm{TTT}}$-structure $\mathcal{B} = (B_0, B_1, \dots, \in)$. Let $B_0 = A_0$ and f_0 be the identity function on A_0. For $n \in \mathbb{N}$, we define $B_{n+1} = \mathrm{ran}(f_{n+1})$, where f_{n+1} is defined such that for all $u \in A_{n+1}$,

$$f_{n+1}(u) = (\{(f_0(v))_0 : v \varepsilon_{0,n+1}^{\mathcal{A}} u\}, \dots, \{(f_n(v))_n : v \varepsilon_{n,n+1}^{\mathcal{A}} u\}).$$

It is easy to verify that $\mathcal{B}$ is standard transitive and that f is an $\mathcal{L}_{\mathrm{TTT}}$-isomorphism from $\mathcal{A}$ to $\mathcal{B}$. ⊣

The following lemma establishes a practical criterion for extensionality.

LEMMA 1.5. *A standard transitive $\mathcal{L}_{\mathrm{TTT}}$-structure $\mathcal{A} = (A_0, A_1, \dots, \in)$ is extensional iff for all $0 \leq i < n$, and $u, v \in A_n$,*

$$(u)_{i+1} = (v)_{i+1} \Rightarrow u = v.$$

PROOF. Just notice that $\mathcal{A}$ is extensional iff for all $0 \leq i < n$,

$$\begin{aligned}
&\mathcal{A} \models \forall u^n, v^n (u^n = v^n \leftrightarrow \forall w^i (w^i \varepsilon u^n \leftrightarrow w^i \varepsilon v^n)) \\
&\Leftrightarrow \forall u, v \in A_n (u = v \leftrightarrow \forall w \in A_i ((w)_i \in (u)_{i+1} \leftrightarrow (w)_i \in (v)_{i+1})) \\
&\Leftrightarrow \forall u, v \in A_n (u = v \leftrightarrow (u)_{i+1} = (v)_{i+1}),
\end{aligned}$$

where the second equivalence holds because $(u)_{i+1}, (v)_{i+1} \in \mathcal{P}(\{(w)_i : w \in A_i\})$. ⊣

NOTE. *Notice that by the previous lemma, if $\mathcal{A} = (A_0, A_1, \dots, \in)$ is extensional, then for all $0 \leq i \leq n$, and $u, v \in A_n$,*

$$(u)_{i+1} = (v)_{i+1} \Leftrightarrow u = v.$$

1.2. New foundations. The language $\mathcal{L}_{\mathrm{NF}}$ of New Foundations is the usual one-sorted language of set theory, $\{\varepsilon\}$, where ε is a binary relation symbol. New Foundations (NF) is axiomatized by the axioms of TST if we erase all type superscripts. Similarly, by erasing all type superscripts from the axioms of TSTP, we get the axioms of *predicative* NF (NFP).

What do we know about NFP? First of all, we know that it is a rather weak subtheory of NF since for example its consistency can be proved in PA (see [1]). We also know that it is finitely axiomatizable (see [4]), and that if the Union axiom

$$\forall z \exists y \forall x (x \varepsilon y \leftrightarrow \exists v (v \varepsilon z \wedge x \varepsilon v)) \tag{Union}$$

is added to it, we get full NF, i.e., NFP + Union = NF (see [1]). The most interesting fact about NFP though is that it is consistent with properties (like choice principles; see [1, 3]) that fail in NF (see [6] or [2]). Below, we describe a way of constructing such models of NFP.

§2. Models of predicative NF. We work in $\mathrm{ZF} + (V = L)$. We are going to construct a standard transitive model $\mathcal{A}$ of TTTP. Let κ be an infinite cardinal. For all $n \in \mathbb{N}$, we define recursively a set X_n, an ordinal $\kappa \leq \alpha_n < \kappa^+$, and a bijection $f_n \colon \kappa \to X_n$ such that $f_n \in L_{\alpha_{n+1}}$. Let $\alpha_0 = \kappa$, $X_0 = L_{\alpha_0}$, and let $f_0 \colon \kappa \to X_0$ be a bijection. For $n > 0$, let

$$X_n = L_{\alpha_n} \cap \mathcal{P}(X_{n-1}),$$

where α_n is the least limit ordinal that is greater than α_{n-1} and for which $f_{n-1} \in L_{\alpha_n}$. We know that $\alpha_n < \kappa^+$ and $|X_n| = \kappa$, so there exists some bijection $f_n \colon \kappa \to X_n$.

Let $A_0 = X_0$ and for all $n > 0$, let

$$A_n = \{(f_1(\alpha), \dots, f_n(\alpha)) : \alpha < \kappa\}.$$

We define $\mathcal{A} = (A_0, A_1, \dots, \{\varepsilon^{\mathcal{A}}_{i,j}\}_{i<j})$, where for all $i < j$, $u \in A_i$, and $v \in A_j$,

$$u \varepsilon^{\mathcal{A}}_{i,j} v \Leftrightarrow (u)_i \in (v)_{i+1}.$$

Notice that for all $n \in \mathbb{N}$, since $f_0, \dots, f_n \in L_{\alpha_{n+1}}$ and α_{n+1} is a limit ordinal, we have

$$X_0, \dots, X_n, A_0, \dots, A_n \in L_{\alpha_{n+1}}.$$

Lemma 2.1. *$\mathcal{A}$ is a standard transitive model of* TTTP.

Proof. We have that for all $i \in \mathbb{N}$,

$$\{(u)_i : u \in A_i\} = \mathrm{ran}(f_i) = X_i.$$

Therefore, for all $n \in \mathbb{N}$,

$$A_{n+1} \subseteq \prod_{i=0}^{n} \mathrm{ran}(f_{i+1}) = \prod_{i=0}^{n} X_{i+1} \subseteq \prod_{i=0}^{n} \mathcal{P}(X_i) = \prod_{i=0}^{n} \mathcal{P}(\{(u)_i : u \in A_i\}),$$

i.e., $\mathcal{A}$ is standard transitive.

We show that $\mathcal{A}$ is an extensional structure. Let $0 \leq i < n$, and $u, v \in A_n$. We know that $u = (f_1(\alpha), \dots, f_n(\alpha))$ and $v = (f_1(\beta), \dots, f_n(\beta))$ for some $\alpha, \beta < \kappa$. If $(u)_{i+1} = (v)_{i+1}$, then $f_{i+1}(\alpha) = f_{i+1}(\beta)$, so since f_{i+1} is 1–1, we have that $\alpha = \beta$, i.e., $u = v$. Therefore, by Lemma 1.5, $\mathcal{A}$ is extensional.

It remains to show that $\mathcal{A} \models \mathrm{CoP}^\circ$. Let $s \colon \mathbb{N} \to \mathbb{N}$ be strictly increasing, and let $\phi(x^i, u_1^{i_1}, \dots, u_n^{i_n})$ be some $\mathcal{L}_{\mathrm{TST}}$-formula

$$\mathrm{Q}_1\, x_1^{j_1} \dots \mathrm{Q}_m\, x_m^{j_m}\, \psi(x_1^{j_1}, \dots, x_m^{j_m}, x^i, u_1^{i_1}, \dots, u_n^{i_n}),$$

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where $Q_1, \dots, Q_n$ are quantifiers, ψ is quantifier-free, $\max\{j_1, \dots, j_m\} \le i$, and $\max\{i_1, \dots, i_n\} \le i+1$. We show that $\mathcal{A} \models (\forall u_1^{i_1}, \dots, u_n^{i_n} \exists y^{i+1} \forall x^i (x^i \varepsilon y^{i+1} \leftrightarrow \phi(x^i, u_1^{i_1}, \dots, u_n^{i_n})))^s$, i.e., that

$$\mathcal{A} \models \forall u_1^{s(i_1)}, \dots, u_n^{s(i_n)} \exists y^{s(i+1)} \forall x^{s(i)} (x^{s(i)} \varepsilon y^{s(i+1)} \leftrightarrow \phi(x^{s(i)}, u_1^{s(i_1)}, \dots, u_n^{s(i_n)})).$$

Let $u_1 \in A_{s(i_1)}, \dots, u_n \in A_{s(i_n)}$. Let

$$y' = \{(x)_{s(i)} : x \in A_{s(i)} \wedge Q_1\, x_1 \in A_{s(j_1)} \dots Q_m\, x_m \in A_{s(j_m)}$$
$$\mathcal{A} \models \psi(x_1, \dots, x_m, x, u_1, \dots, u_n)\}.$$

Now, by Remark 1.1 and the way ε is interpreted in standard transitive structures, it follows that the statement $\mathcal{A} \models \psi(x_1, \dots, x_m, x, u_1, \dots, u_n)$ is equivalent to a quantifier-free $\mathcal{L}_{\mathrm{ZF}}$-formula that (apart from $x_1, \dots, x_m, x$) has as parameters only the $s(i)+1$ first coordinates of $u_1, \dots, u_n$, where each such coordinate is in $L_{\alpha_{s(i)+1}}$. Moreover, we know that $A_{s(i)}, A_{s(j_1)}, \dots, A_{s(j_n)} \in L_{\alpha_{s(i)+1}}$ because $s(j_1), \dots, s(j_n) \le s(i)$. Therefore, since $\alpha_{s(i)+1}$ is a limit ordinal, we have that

$$y' \in L_{\alpha_{s(i)+1}} \cap \mathcal{P}(X_{s(i)}) = X_{s(i)+1}.$$

Let $y \in A_{s(i+1)}$ such that $(y)_{s(i)+1} = y'$. Clearly, y witnesses that

$$\mathcal{A} \models \exists y^{s(i+1)} \forall x^{s(i)} (x^{s(i)} \varepsilon y^{s(i+1)} \leftrightarrow \phi(x^{s(i)}, u_1, \dots, u_n)). \quad \dashv$$

We now show that $\mathcal{A}$ inherits some interesting properties from L. Below, we present two such properties. We begin by proving that in $\mathcal{A}$ every universe is well-orderable. Before we proceed though, let us examine what it means for an element to be a Wiener–Kuratowski pair in $\mathcal{A}$. For each $i \in \mathbb{N}$, let $\mathrm{Pair}_i(u^i, v^i, z^{i+2})$ be the following $\mathcal{L}_{\mathrm{TST}}$-formula:

$$\exists x_1^{i+1}, x_2^{i+1} (\forall w^i (w^i \varepsilon x_1^{i+1} \leftrightarrow w^i = u^i) \wedge \forall w^i (w^i \varepsilon x_2^{i+1} \leftrightarrow w^i = u^i \vee w^i = v^i)$$
$$\wedge \forall x^{i+1} (x^{i+1} \varepsilon z^{i+2} \leftrightarrow x^{i+1} = x_1^{i+1} \vee x^{i+1} = x_2^{i+1})),$$

expressing that z^{i+2} is the Wiener–Kuratowski pair of u^i, v^i. Notice that for $s\colon \mathbb{N} \to \mathbb{N}$ strictly increasing, $i \in \mathbb{N}$, $u, v \in A_{s(i)}$, and $z \in A_{s(i+2)}$, we have that $\mathcal{A} \models \mathrm{Pair}_i^s(u, v, z)$ is equivalent to

$$\exists x_1, x_2 \in A_{s(i+1)} (\forall w \in A_{s(i)} ((w)_{s(i)} \in (x_1)_{s(i)+1} \leftrightarrow (w)_{s(i)} = (u)_{s(i)})$$
$$\wedge (\forall w \in A_{s(i)} ((w)_{s(i)} \in (x_2)_{s(i)+1} \leftrightarrow (w)_{s(i)} = (u)_{s(i)} \vee (w)_{s(i)} = (v)_{s(i)})$$
$$\wedge \forall x \in A_{s(i+1)} ((x)_{s(i+1)} \in (z)_{s(i+1)+1}$$
$$\leftrightarrow (x)_{s(i+1)} = (x_1)_{s(i+1)} \vee (x)_{s(i+1)} = (x_2)_{s(i+1)})),$$

or in more compact notation iff

$$\exists x_1, x_2 \in A_{s(i+1)} ((z)_{s(i+1)+1} = \{(x_1)_{s(i+1)}, (x_2)_{s(i+1)}\}$$
$$\wedge (x_1)_{s(i)+1} = \{(u)_{s(i)}\} \wedge (x_2)_{s(i)+1} = \{(u)_{s(i)}, (v)_{s(i)}\}).$$

To simplify notation, let us denote by $\mathrm{Pair}_i^{\mathcal{A},s}(u, v, z)$ the above $\mathcal{L}_{\mathrm{ZF}}$-formula which is equivalent to $\mathcal{A} \models (\mathrm{Pair}_i(u, v, z))^s$.

LEMMA 2.2. $\mathcal{A} \models (\{\text{“}V^{i+1}$ *is well-orderable”*$\}_{i \in \mathbb{N}})^\circ$.

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PROOF. Let $s : \mathbb{N} \to \mathbb{N}$ be strictly increasing, $i \in \mathbb{N}$, and let $\phi(u^i, v^i, W^{i+2})$ be the following $\mathcal{L}_{\mathrm{TST}}$-formula:

$$\exists x^{i+1}(x^{i+1} \varepsilon W^{i+2} \wedge u^i \varepsilon x^{i+1} \wedge v^i \not\varepsilon x^{i+1}).$$

Let $\chi(W^{i+2})$ be the following $\mathcal{L}_{\mathrm{TST}}$-formula:

$$\begin{aligned} &\forall u^i, v^i(\phi(u^i, v^i, W^{i+2}) \vee \phi(v^i, u^i, W^{i+2}) \vee u^i = v^i) \\ &\quad \wedge \forall u^i, v^i, w^i(\phi(u^i, v^i, W^{i+2}) \wedge \phi(v^i, w^i, W^{i+1}) \to \phi(u^i, w^i, W^{i+2})) \\ &\quad \wedge \forall x^{i+1} \exists u^i(u^i \varepsilon x^{i+1} \wedge \forall v^i(v^i \varepsilon x^{i+1} \wedge u^i \neq v^i \to \phi(u^i, v^i, W^{i+2}))). \end{aligned}$$

For all $W \in A_{s(i+2)}$, we have $\mathcal{A} \models \chi^s(W)$ iff

$$\begin{aligned} &\forall u, v \in A_{s(i)}(\phi^{\mathcal{A},s}(u, v, W) \vee \phi^{\mathcal{A},s}(v, u, W) \vee (u)_{s(i)} = (v)_{s(i)}) \\ &\quad \wedge \forall u, v, w \in A_{s(i)}(\phi^{\mathcal{A},s}(u, v, W) \wedge \phi^{\mathcal{A},s}(v, w, W) \to \phi^{\mathcal{A},s}(u, w, W)) \\ &\quad \wedge \forall x \in A_{s(i+1)} \exists u \in A_{s(i)}((u)_{s(i)} \in (x)_{s(i)+1} \\ &\quad \wedge \forall v \in A_{s(i)}((v)_{s(i)} \in (x)_{s(i)+1} \wedge (u)_{s(i)} \neq (v)_{s(i)} \to \phi^{\mathcal{A},s}(u, v, W))), \end{aligned}$$

where $\phi^{\mathcal{A},s}(u, v, W)$ is the following $\mathcal{L}_{\mathrm{ZF}}$-formula:

$$\exists x \in A_{s(i+1)}((x)_{s(i+1)} \in (W)_{s(i+1)+1} \wedge (u)_{s(i)} \in (x)_{s(i)+1} \wedge (v)_{s(i)} \notin (x)_{s(i)+1}).$$

Let $\psi(u, v)$ be an $\mathcal{L}_{\mathrm{ZF}}$-formula that defines some well-ordering of L. Let

$$\begin{aligned} W' = \{&(x)_{s(i+1)} : x \in A_{s(i+1)} \\ &\wedge \exists u \in A_{s(i)} \forall v \in A_{s(i)}((v)_{s(i)} \in (x)_{s(i)+1} \leftrightarrow \psi((u)_{s(i)}, (v)_{s(i)}))\}. \end{aligned}$$

We know that $W' \in L_{\alpha_{s(i+1)+1}} \cap \mathcal{P}(X_{s(i+1)}) = X_{s(i+1)+1}$, so there exists some $W \in A_{s(i+2)}$ such that $(W)_{s(i+1)+1} = W'$. Notice that W' is the set of all $(x)_{s(i+1)}$ for which $x \in A_{s(i+1)}$ and $(x)_{s(i)+1}$ is an initial segment of the well-ordering defined by ψ restricted to $X_{s(i)}$. Therefore, W witnesses that $\mathcal{A} \models \chi^s(W)$. Now, let

$$R' = \{(z)_{s(i+2)} : z \in A_{s(i+2)} \wedge \exists u, v \in A_{s(i)}(\mathrm{Pair}_i^{\mathcal{A},s}(u, v, z) \wedge \phi^{\mathcal{A},s}(u, v, W))\}.$$

We have that $R' \in L_{\alpha_{s(i+2)+1}} \cap \mathcal{P}(X_{s(i+2)}) = X_{s(i+2)+1}$, so there exists an $R \in A_{s(i+3)}$ such that $(R)_{s(i+2)+1} = R'$. It is easy to see that $\mathcal{A} \models \chi^s(W)$ implies $\mathcal{A} \models$ (“R is a well-ordering of V^{i+1}”)^s. Hence, $\mathcal{A} \models$ (“V^{i+1} is well-orderable”)^s. ⊣

Next, we show that in $\mathcal{A}$ every universe is cantorian (i.e., it is equinumerous to the set of singletons). This essentially follows from the fact that for all $i, j \in \mathbb{N}$, there are functions in $\mathcal{A}$ witnessing that $|X_i| = \kappa = |\iota``X_j|$.

LEMMA 2.3. $\mathcal{A} \models (\{$*“V^{i+2} is cantorian”*$\}_{i \in \mathbb{N}})^\circ$.

PROOF. Let $s : \mathbb{N} \to \mathbb{N}$ be strictly increasing, and let $i \in \mathbb{N}$. We show that $\mathcal{A} \models$ (“V^{i+2} is cantorian”)^s, i.e., that

$$\mathcal{A} \models (\exists g^{i+4}(\text{“}g \text{ is a bijection from } V^{i+2} \text{ to } \iota\text{``}V^{i+1}\text{''}))^s.$$

An $\mathcal{L}_{\mathrm{TST}}$-sentence that expresses the statement

$$\exists g^{i+4}(\text{“}g \text{ is a bijection from } V^{i+2} \text{ to } \iota\text{``}V^{i+1}\text{''})$$

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is the following:

$$\begin{aligned}&\exists g^{i+4}(\forall z^{i+3}(z^{i+3}\varepsilon g^{i+4}\to\exists u^{i+1},v^{i+1}(\mathrm{Pair}_{i+1}(u^{i+1},v^{i+1},z^{i+3})))\\&\wedge\forall u^{i+1},v^{i+1},w^{i+1},x^{i+3},y^{i+3}(\mathrm{Pair}_{i+1}(u^{i+1},v^{i+1},x^{i+3})\wedge\mathrm{Pair}_{i+1}(u^{i+1},w^{i+1},y^{i+3})\\&\wedge x^{i+3}\varepsilon g^{i+4}\wedge y^{i+3}\varepsilon g^{i+4}\to v^{i+1}=w^{i+1})\\&\wedge\forall u^{i+1},v^{i+1},w^{i+1},x^{i+3},y^{i+3}(\mathrm{Pair}_{i+1}(u^{i+1},w^{i+1},x^{i+3})\wedge\mathrm{Pair}_{i+1}(v^{i+1},w^{i+1},y^{i+3})\\&\wedge x^{i+3}\varepsilon g^{i+4}\wedge y^{i+3}\varepsilon g^{i+4}\to u^{i+1}=v^{i+1})\\&\wedge\forall u^{i+1}\exists v^{i+1},z^{i+3}(\mathrm{Pair}_{i+1}(u^{i+1},v^{i+1},z^{i+3})\wedge z^{i+3}\varepsilon g^{i+4}\\&\wedge\exists a^i\forall b^i(b^i\varepsilon v^{i+1}\leftrightarrow b^i=a^i))\\&\wedge\forall v^{i+1}(\exists a^i\forall b^i(b^i\varepsilon v^{i+1}\leftrightarrow b^i=a^i)\\&\to\exists u^{i+1},z^{i+3}(\mathrm{Pair}_{i+1}(u^{i+1},v^{i+1},z^{i+3})\wedge z^{i+3}\varepsilon g^{i+4}))).\end{aligned}$$

We therefore have that $\mathcal{A}\models$ (“V^{i+2} is cantorian”)^s iff

$$\begin{aligned}&\exists g\in A_{s(i+4)}(\forall z\in A_{s(i+3)}((z)_{s(i+3)}\in(g)_{s(i+3)+1}\\&\to\exists u,v\in A_{s(i+1)}(\mathrm{Pair}^{\mathcal{A},s}_{i+1}(u,v,z)))\\&\wedge\forall u,v,w\in A_{s(i+1)}\forall x,y\in A_{s(i+3)}(\mathrm{Pair}^{\mathcal{A},s}_{i+1}(u,v,x)\wedge\mathrm{Pair}^{\mathcal{A},s}_{i+1}(u,w,y)\\&\wedge(x)_{s(i+3)}\in(g)_{s(i+3)+1}\wedge(y)_{s(i+3)}\in(g)_{s(i+3)+1}\to(v)_{s(i+1)}=(w)_{s(i+1)})\\&\wedge\forall u,v,w\in A_{s(i+1)}\forall x,y\in A_{s(i+3)}(\mathrm{Pair}^{\mathcal{A},s}_{i+1}(u,w,x)\wedge\mathrm{Pair}^{\mathcal{A},s}_{i+1}(v,w,y)\\&\wedge(x)_{s(i+3)}\in(g)_{s(i+3)+1}\wedge(y)_{s(i+3)}\in(g)_{s(i+3)+1}\to(u)_{s(i+1)}=(v)_{s(i+1)})\\&\wedge\forall u\in A_{s(i+1)}\exists v\in A_{s(i+1)}\exists z\in A_{s(i+3)}(\mathrm{Pair}^{\mathcal{A},s}_{i+1}(u,v,z)\wedge(z)_{s(i+3)}\in(g)_{s(i+3)+1}\\&\wedge\exists a\in A_{s(i)}\forall b\in A_{s(i)}((b)_{s(i)}\in(v)_{s(i)+1}\leftrightarrow(b)_{s(i)}=(a)_{s(i)}))\\&\wedge\forall v\in A_{s(i+1)}(\exists a\in A_{s(i)}\forall b\in A_{s(i)}((b)_{s(i)}\in(v)_{s(i)+1}\leftrightarrow(b)_{s(i)}=(a)_{s(i)})\\&\to\exists u\in A_{s(i+1)}\exists z\in A_{s(i+3)}(\mathrm{Pair}^{\mathcal{A},s}_{i+1}(u,v,z)\wedge(z)_{s(i+3)}\in(g)_{s(i+3)+1}))).\end{aligned}$$

Let

$$\begin{aligned}g'=\{(z)_{s(i+3)}:z\in A_{s(i+3)}\wedge\exists u,v\in A_{s(i+1)}(\mathrm{Pair}^{\mathcal{A},s}_{i+1}(u,v,z)\\\wedge\exists\alpha<\kappa((u)_{s(i+1)}=f_{s(i+1)}(\alpha)\wedge(v)_{s(i)+1}=\{f_{s(i)}(\alpha)\}))\}.\end{aligned}$$

We have that $g'\in L_{s(i+3)+1}\cap\mathcal{P}(X_{s(i+3)})=X_{s(i+3)+1}$, so there is a $g\in A_{s(i+4)}$ such that $(g)_{s(i+3)+1}=g'$. Clearly, g witnesses that

$$\mathcal{A}\models(\exists g^{i+4}(\text{“}g\text{ is a bijection from }V^{i+2}\text{ to }\iota\text{“}V^{i+1}\text{”}))^s. \qquad\dashv$$

We have shown that

$$\mathrm{TTTP}+(\{\text{“}V^{i+1}\text{ is well-orderable”}\}_{i\in\mathbb{N}})^\circ+(\{\text{“}V^{i+2}\text{ is cantorian”}\}_{i\in\mathbb{N}})^\circ$$

is consistent. By slightly modifying Holmes’ proof for the equiconsistency of TTT and NF (see [3]), we can now prove the following proposition.

PROPOSITION 2.4. NFP + *“V is well-orderable”* + *“V is cantorian” is consistent*.

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PROOF. Let $\Delta = \{\sigma_1, \dots, \sigma_n\}$ be a finite subset of

$$\text{TTTP} + (\{\text{“}V^{i+1}\text{ is well-orderable”}\}_{i\in\mathbb{N}})^\circ + (\{\text{“}V^{i+2}\text{ is cantorian”}\}_{i\in\mathbb{N}})^\circ.$$

Let $m \in \mathbb{N}$ be such that all variables appearing in $\sigma_1, \dots, \sigma_n$ have types that are less or equal to m. For each $X \subseteq \mathbb{N}$ such that $|X| = m+1$, let s_X be the unique strictly increasing function from $\{0, \dots, m\}$ to X. Let $F: [\mathbb{N}]^{m+1} \to 2^n$ such that for all $X \in [\mathbb{N}]^{m+1}$, $F(X) = (\delta_1, \dots, \delta_n)$, where for each $1 \le i \le n$, $\delta_i = 1$ iff $\mathcal{A} \models (\sigma_i)^{s_X}$. By Ramsey's theorem, there exists some infinite $H \subseteq \mathbb{N}$ such that H is homogeneous for F. Let $H = \{h_0, h_1, \dots\}$, where $h_0 < h_1 < \cdots$, and let $\mathcal{B} = (A_{h_0}, A_{h_1}, \dots, \{\varepsilon^{\mathcal{A}}_{h_i, h_{i+1}}\}_{i\in\mathbb{N}})$. It is easy to see that $\mathcal{B}$ is a model of $\text{TSTP} + \{\text{“}V^{i+1}\text{ is well-orderable”}\}_{i\in\mathbb{N}} + \{\text{“}V\text{ is cantorian”}\}_{i\in\mathbb{N}} + \{\sigma \leftrightarrow \sigma^+ : \sigma \in \Delta\}$, where σ^+ is the sentence we get if we raise the type of every variable of σ by one. By Compactness, it follows that $\text{TSTP} + \{\text{“}V^{i+1}\text{ is well-orderable”}\}_{i\in\mathbb{N}} + \{\text{“}V^{i+2}\text{ is cantorian”}\}_{i\in\mathbb{N}} + \{\sigma \leftrightarrow \sigma^+ : \sigma \text{ is a sentence}\}$ is consistent. Therefore, by Specker's results on ambiguity (see [2] or [7]), it follows that there is a model of $\text{TSTP} + \{\text{“}V^{i+1}\text{ is well-orderable”}\}_{i\in\mathbb{N}} + \{\text{“}V^{i+2}\text{ is cantorian”}\}_{i\in\mathbb{N}}$ with a type shifting automorphism, which means that there is a model of NFP + “V is well-orderable” + “V is cantorian.” ⊣

Although NFP is consistent with a very strong choice principle like the one above, it is inconsistent with some other forms of choice. For example, let AC be the statement “for any set x of non-empty pairwise disjoint sets, there exists a choice set z, i.e., a set that has exactly one element in common with each element of x,” which can be can be expressed formally as the following $\mathcal{L}_{\text{NF}}$-sentence:

$$\forall x(\forall y_1 \forall y_2((y_1 \varepsilon x \wedge y_2 \varepsilon x \to \exists v(v \varepsilon y_1) \wedge (y_1 = y_2 \vee \forall u(u \not\varepsilon y_1 \vee u \not\varepsilon y_2))) \\ \to \exists z \forall y(y \varepsilon x \to \exists a(a \varepsilon z \wedge \forall b(b \varepsilon z \wedge b \varepsilon y \leftrightarrow a = b)))).$$

THEOREM 2.5. NFP $\vdash \neg$AC.

PROOF. As Crabbé observed in [1], $\text{NFP} + \forall x(x \subseteq \iota\text{“}V \to \exists y(x = \iota\text{“}y)) = \text{NF}$. But, in NFP, AC implies $\forall x(x \subseteq \iota\text{“}V \to \exists y(x = \iota\text{“}y))$ because if x is a set of singletons and y is a choice set for x, then $x = \iota\text{“}y$. So, NFP + AC = NF, which means that NFP $\vdash \neg$AC. ⊣

NOTE. *Notice of course that the above form of choice is impredicative, and therefore not really suitable for a predicative theory. A more sensible and unproblematic statement in this setting would be the following: “for any set x of non-empty pairwise disjoint sets, there exists a set of singletons z, where every element of a singleton in z belongs to exactly one element of x” (notice that in this version, z has the same relative type as x).*

§3. Conclusion. We described a simple way of constructing a model of NFP using L. We also showed that there are properties of L that can be transferred naturally to this model. We have chosen to present just two such properties that are inconsistent with NF, but there are others. It would be nice to have a more general result on what kind of properties can be transferred though.

QUESTION 1. *What properties of L can be transferred to our model of* NFP?

The construction of our model seems to be quite flexible. For example, we could modify the definition of α_n so that more sets are included in X_{α_n}. This could lead us to stronger consistent extensions of NFP. For example, the following question seems promising.

QUESTION 2. *Can this construction be modified so that we get a model of* NFP *satisfying some weak version of Union*?

Finding consistency proofs for subtheories of Tangled Type Theory is really important for understanding the problem of the consistency of NF. On the other hand, the opposite direction is equally interesting: if we assume the consistency of NF (see [5]), and therefore of Tangled Type Theory, we may get new results in other areas of set theory. For example, as we have shown, there seems to be a connection between models of Tangled Type Theory and L.

QUESTION 3. *Does the consistency of Tangled Type Theory have any implications for L*?

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