interesting to compare densities. To date, we have been unable to devise a non-trivial packing procedure for which these lemmas would both hold.

Empirical evidence from computer simulations (Blaisdell and Solomon (1970), Akeda and Hori (1976), Jodrey and Tory (1979)) indicates that the conjecture itself is false and hence that x, y-coordinates are dependent, but it is not easy to see the effect of this dependence. If X is a random variable representing the x-coordinate of the centre of a rectangle and Y represents the corresponding y-coordinate (in the time-sequence mode of labelling), then symmetry implies that $\mathrm{Cov}(X, Y) = 0$. Though the simulated densities differ from the conjectured by eleven standard deviations, the actual difference in densities (0.0032) is small (Jodrey and Tory (1979)). Figure 1 suggests that filling B before A and C creates a slight tendency for the new rectangles to line up with those already in place. We speculate that this causes the small increase in packing density over that conjectured.

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Yours sincerely,

Mount Allison University — ELMER M. TORY
Harvard University — DAVID K. PICKARD

Dear Editor,

On Weiner's proof of the Palásti conjecture

In a recent paper Weiner (1978) claims to have proved the Palásti conjecture (see Palásti (1960)) respecting the asymptotic mean density of random sequential packing in the plane or the higher-dimensional space. This conjecture has previously been tested by the use of Monte Carlo simulation techniques. Earlier

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results of computer simulations (Solomon (1967), Akeda and Hori (1975)) are apparently consistent with the Palásti conjecture but do not attain satisfactory accuracy. The latest simulation data obtained by Akeda and Hori (1976) show that her conjecture is incorrect although the error is small both in two and three dimensions. Blaisdell and Solomon (1970) have also arrived at the same conclusion for the two-dimensional case. Therefore the question arises whether Weiner's argument is valid or not. The purpose of the present note is to point out fundamental errors creeping into his proof; I should like to add that I do not accept the arguments given in Weiner's reply.

Weiner treats three models called Model I, Model II, and the random car size model. In particular, the packing procedure referred to as Model I is nothing but the higher-dimensional analogue of the one-dimensional car parking due to Rényi (1958). Since the essential feature is common to all the three models, we shall hereafter restrict our attention to Model I in two dimensions. The procedure of random sequential packing in this case is as follows: Consider a rectangular boundary with corners at $(0,0)$, $(0,b)$, $(a,0)$, (a,b). The first car of size $\alpha \times \beta$ is parked in the space given by the corners (X, Y), $(X, Y+\beta)$, $(X+\alpha, Y)$, $(X+\alpha, Y+\beta)$ where (X, Y) is chosen uniformly at random in the subrectangle $(0,0)$, $(0, b-\beta)$, $(a-\alpha, 0)$, $(a-\alpha, b-\beta)$. Succeeding cars of the same size and orientation are placed independently and uniformly; they are parked if there is no overlap with a car already parked, and otherwise discarded.

Let $M_{\alpha\beta}(a,b)$ indicate the expectation of the total number of $\alpha \times \beta$ size cars parked in the $a \times b$ rectangle and $M_\alpha(a)$ represent the corresponding quantity for the one-dimensional parking of cars of length α in the interval $[0, a]$. The Palásti conjecture asserts that the limit of the mean packing density in two dimensions is equal to the square of that in one dimension. As described in Theorem 1 of Weiner, this statement implies

$$(1) \qquad \lim_{a,b\to\infty} \frac{\alpha\beta M_{\alpha\beta}(a,b)}{ab} = \eta^2 \simeq 0.5589,$$

where η stands for

$$(2) \qquad \begin{aligned} \eta &\equiv \lim_{a\to\infty} \frac{\alpha M_\alpha(a)}{a} \\ &= \int_0^\infty \exp\left\{-2\int_0^t (1-e^{-u})u^{-1}du\right\}dt \simeq 0.7476 \end{aligned}$$

(Rényi (1958)).

To prove Theorem 1 Weiner has used four lemmas. First we check the validity of Lemma 2, which he regards as a key lemma. Denote by l the line segment from $(0, b-\beta)$ to $(a, b-\beta)$. Lemma 2 states that the $\alpha \times \beta$ cars parked in the

$a \times b$ rectangle intersect the line segment l in segments of length α in accord with the one-dimensional law for cars of length α parked on a segment of length a. The row of parked cars which intersect l is named row 1. Below row 1, the immediately adjacent cars form row 2 from one end of the $a \times b$ rectangle to the other, and so on, until rows are exhausted, and afterwards partial rows form. Although Weiner adduces no evidence, he assumed that Lemma 2 is applicable not only to row 1 but also to any lower row and partial row.

In the proof of Lemma 2, Weiner argues that the horizontal placement and parking of cars on the line segment l is independent of all other parked cars and depends only on the x-coordinate. However, parked cars whose upper side has the y-coordinate ranging from $b - 2\beta$ to $b - \beta$ do not intersect l but influence the parking of subsequent cars on l. Consequently, it is not assured that the horizontal parking on l is equivalent to the one-dimensional parking on a line segment of length a. The situation becomes serious in the case of lower rows.

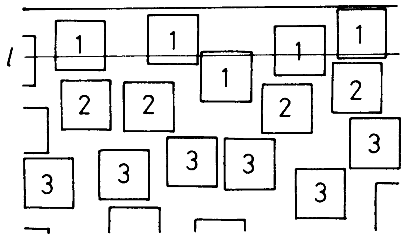

Figure 1

Figure 1 illustrates an example of a pattern of row 1 to row 3. In contrast to the parking in row 1, it is possible that the horizontal breadth of a gap between adjacent cars in row 2 is greater than α. This means that car parking in row 2 does not obey the same law as in the one-dimensional case, because linear cars of length α parked in the interval $[0, a]$ generate no gaps wider than α. Hence it is seen that Lemma 2 does not hold for lower rows.

Next we consider Lemma 3 of Weiner, which states that for $a \geqq 2\alpha$ or $b \geqq 2\beta$

(3) $$M_{\alpha\beta}(a, b+\beta) \geqq M_{\alpha\beta}(a, b),$$

(4) $$\frac{M_{\alpha\beta}(a, b+\beta)}{a(b+\beta)} \leqq \frac{M_{\alpha\beta}(a, b)}{ab},$$

(5) $$M_{\alpha\beta}(a, b) + M_{\alpha}(a) \geqq M_{\alpha\beta}(a, b+\beta),$$

(6) $$M_{\alpha\beta}(a, b) + M_{\alpha}(a) \leqq M_{\alpha\beta}(a, b+2\beta).$$

The inequalities (3) and (4) are special cases of

(7) $$M_{\alpha\beta}(a, b) \geqq M_{\alpha\beta}(c, d) \quad \text{for} \quad a \geqq c, b \geqq d,$$

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and

$$\text{(8)} \qquad \frac{M_{\alpha\beta}(a,b)}{ab} \leqq \frac{M_{\alpha\beta}(c,d)}{cd} \quad \text{for} \quad a \geqq c, b \geqq d,$$

respectively. It should be noted that (8) contradicts previous simulation results (Solomon (1967), Akeda and Hori (1975), (1976)), all of which seem to satisfy the reversed inequality.

Besides Lemma 2, Weiner founds his proof of (8) upon the assertion that for $a > 2\alpha$

$$\text{(9)} \qquad a^{-1}M_\alpha(a) \quad \text{is monotone decreasing.}$$

Nevertheless this is false; we cannot derive (9) by his prescription, that is, by taking the derivative of $a^{-1}M_\alpha(a)$ and checking its sign. Rényi (1958) and Ney (1962) have established that for any positive integer n

$$\text{(10)} \qquad \frac{\alpha M_\alpha(a)}{a} = \eta - (1-\eta)\frac{\alpha}{a} + O\left(\frac{1}{a^{n+1}}\right) \quad \text{as} \quad a \to \infty,$$

which is strengthened by Dvoretzky and Robbins (1964) to

$$\text{(11)} \qquad \frac{\alpha M_\alpha(a)}{a} = \eta - (1-\eta)\frac{\alpha}{a} + O\left(\frac{(2e)^a}{a^{a-1/2}}\right) \quad \text{as} \quad a \to \infty.$$

In direct opposition to (9), accordingly, $a^{-1}M_\alpha(a)$ increases monotonically for sufficiently large a. For this reason, we cannot affirm that the inequalities (4), (5), (6), (8) together with (9) are true.

The Palásti conjecture written in the form of Weiner's Theorem 1 is an immediate consequence of Lemma 4, which he has deduced from (5) and (6). Now that Lemmas 2 and 3 do not hold right, his proof of Theorem 1 turns out to be erroneous. Similar reasoning applies equally well to Theorem 2 for Model II in the plane, to Theorem 3 for Models I and II in higher dimensions, and to Theorem 4 for the random car size model. It is thus concluded that the Palásti conjecture has not yet been verified.

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Yours sincerely,
MOTOO HORI

Dear Editor,

Reply to letters of
M. Tanemura and E. M. Tory and D. K. Pickard

I regard the arguments of [2] to be heuristic, non-rigorous and hence incomplete.

M. Tanemura's comments on Lemma 3 of [2] are well taken. In fact, it may be shown by comparison of $M(x)$ with linear solutions to the basic integral equation for $M(x)$ in the one-dimensional Rényi model ([2], p. 803, Equation (1.2)) that for $a \geqq 3\alpha$, (2.7a) of [2] should read

$$\left.\begin{aligned}&(a-\alpha)^{-1}M_\alpha(a)\\&(b-\beta)^{-1}M_\beta(b)\end{aligned}\right\}\text{ are decreasing.}\tag{1}$$

Similarly, the other ratio results of Lemma 3 of [2], p. 806 should be correspondingly changed. For example, (2.8a) should read

$$\frac{M(a,b)}{(a-\alpha)(b-\beta)} \leqq \frac{M(c,d)}{(c-\alpha)(d-\beta)} \quad \text{for} \quad a \geqq c \gg \alpha,\ b \geqq d \gg \beta.\tag{2}$$

These changes do not alter the results of [2].

The phrase ([2], p. 806, above (2.8a)), 'From the independence of x, y coordinates...', refers to attempted placements, as Tory and Pickard indicate. Their examples in their Figures 1–3 and Tanemura's 'strip' example refer to particular configurations. To get the marginal density or likelihood of a 'staggered row', the conditional density for each type of configuration must be averaged with respect to its relative density of occurrence. Their comments apply only to certain configurations, not to the average over configurations. In addition, I disagree with Tanemura's remark that 'the probability of car

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