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LEBESGUE DECOMPOSITION FOR REPRESENTABLE FUNCTIONALS ON *-ALGEBRAS

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Abstract. We offer a Lebesgue-type decomposition of a representable functional on a *-algebra into absolutely continuous and singular parts with respect to another. Such a result was proved by Zs. Szűcs due to a general Lebesgue decomposition theorem of S. Hassi, H.S.V. de Snoo, and Z. Sebestyén concerning non-negative Hermitian forms. In this paper, we provide a self-contained proof of Szűcs' result and in addition we prove that the corresponding absolutely continuous parts are absolutely continuous with respect to each other.

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1. Introduction. S. P. Gudder in [**3**] presented a Lebesgue-type decomposition theorem for positive functionals on a unital Banach *-algebra $\mathscr{A}$. In fact, he proved that for given two positive functionals f, g, there exist two positive functionals g_a, g_s such that $g = g_a + g_s$ where g_a is absolutely continuous with respect to f and g_s is f-semi-singular. Here, the concepts of absolute continuity and semi-singularity read as follows: g is called f-absolutely continuous ($g \ll f$) if the properties $f(a_n^* a_n) \to 0$ and $g((a_n - a_m)^*(a_n - a_m)) \to 0$ imply $g(a_n^* a_n) \to 0$. Furthermore, g is called f-semi-singular ($g \perp f$) if there exists a sequence $(a_n)_{n\in\mathbb{N}}$ such that $f(a_n^* a_n) \to 0$, $g((a_n - a_m)^*(a_n - a_m)) \to 0$ and $g(a) = \lim_{n\to\infty} g(a_n^* a)$ for any $a \in \mathscr{A}$. Zs. Szűcs [**13**] proved that the concept of semi-singularity is symmetric in the sense that $g \perp f$ if and only if $f \perp g$. Moreover, $f \perp g$ holds if and only if $h = 0$ is the unique positive functional which satisfies $h \le f$ and $h \le g$. This latter property is expressed by saying that f and g are mutually singular.

Generalizing Gudder's results, Szűcs in [**15**] developed a Lebesgue decomposition theory for representable forms over a complex algebra, and as a particular case, he also considered representable functionals of a *-algebra ([**15**, Theorem 3.1]). His treatment however makes essentially use of a general Lebesgue decomposition theorem due to Hassi, de Snoo and Sebestyén [**6**] concerning non-negative Hermitian forms, cf. also [**12**]. The aim of this paper is to provide a self-contained proof of Szűcs' decomposition theorem. Our treatment is similar to that of [**16**] in which the Lebesgue decomposition theory of positive operators on a Hilbert space is discussed. As a new result, we shall also show that the corresponding absolutely continuous parts f_a and g_a, arising by decomposing the representable positive functional f with respect to g, and g with respect to f, respectively, are absolutely continuous with respect to each other. A similar statement was proved by T. Titkos [**18**] in the context of non-negative Hermitian forms. Finally, we apply our results to obtain two classical results of the Lebesgue decomposition theory: the Lebesgue decomposition of measures (see [**4**]) and the Lebesgue–Darst decomposition of finitely additive set functions (see [**1**]).

2. Preliminaries. To begin with, we recall briefly the classical Gelfand–Neumark–Segal (GNS) construction which we shall use as a basic tool in our paper. The procedure presented below is slightly different from what can be find in the literature, see e.g. [**2**, **8**, **9**], or [**10**]. Why we use this modified version is because we want to point out the close analogy with the Lebesgue decomposition theory of positive operators in Hilbert spaces, see [**16**]. To this aim, we introduce first the concept of a positive operator from a vector space into its antidual, cf. [**11**]. Let $\mathscr{A}$ be a (not necessarily unital) *-algebra, and denote by $\mathscr{A}^*$ and $\bar{\mathscr{A}}^*$ the algebraic dual and antidual of $\mathscr{A}$, respectively. Here, the latter one is understood as the vector space of all mappings $\varphi : \mathscr{A} \to \mathbb{C}$ satisfying

$$\varphi(a+b) = \varphi(a) + \varphi(b), \qquad \varphi(\lambda a) = \bar{\lambda}\varphi(a), \qquad a, b \in \mathscr{A}, \lambda \in \mathbb{C}.$$

The elements of $\bar{\mathscr{A}}^*$ are referred to as antilinear functionals of $\mathscr{A}$. For $\varphi \in \bar{\mathscr{A}}^*$ and $a \in \mathscr{A}$, we shall use the notation

$$\langle \varphi, a \rangle := \varphi(a).$$

In the centre of our attention, there are those antilinear functionals which derive from a given positive functional f ($f(a^*a) \geq 0$, $a \in \mathscr{A}$), defined by the correspondence

$$\mathscr{A} \to \mathbb{C}, \quad x \mapsto f(x^*a). \tag{2.1}$$

The mapping $(a, b) \mapsto f(b^*a)$ defines obviously a semi-inner product on $\mathscr{A}$, hence the Cauchy–Schwarz inequality implies

$$|f(b^*a)|^2 \leq f(a^*a)f(b^*b), \qquad a, b \in \mathscr{A}. \tag{2.2}$$

We associate now a *positive operator* A with the positive functional f: For fixed $a \in \mathscr{A}$, let Aa denote the functional (2.1). Then, A is a linear operator of $\mathscr{A}$ into $\bar{\mathscr{A}}^*$ which is non-negative definite in the sense that

$$\langle Aa, a \rangle = f(a^*a) \geq 0, \qquad a \in \mathscr{A}.$$

Observe immediately that A is symmetric:

$$\langle Aa, b \rangle = \overline{\langle Ab, a \rangle}, \qquad a, b \in \mathscr{A}.$$

Hereinafter, we make two additional assumptions on f: suppose that

$$|f(a)|^2 \leq C \cdot f(a^*a), \qquad a \in \mathscr{A}, \tag{2.3}$$

holds for a non-negative constant C and furthermore that

$$f(b^*a^*ab) \leq \lambda_a \cdot f(b^*b), \qquad b \in \mathscr{A}. \tag{2.4}$$

holds for any $a \in \mathscr{A}$ with some $\lambda_a \geq 0$. We notice here that (2.4) holds automatically in Banach *-algebras, namely by $\lambda_a = r(a^*a)$, where r stands for the spectral radius. As it is well known, assumptions (2.3) and (2.4) express the representability of the positive functional f. That means that there exist a Hilbert space H, a cyclic vector $\zeta \in H$, and a *-representation π of $\mathscr{A}$ into $\mathscr{B}(H)$ such that

$$f(a) = (\pi(a)\zeta \mid \zeta), \qquad a \in \mathscr{A}.$$

Such a triplet (H, π, ζ) is obtained due to the well-known GNS construction (see [**11**]): Consider the range space ran A of the linear operator A in $\bar{\mathscr{A}}^*$. This becomes a pre-Hilbert space endowed by the inner product

$$(Aa \mid Ab)_{_A} := f(b^*a), \qquad a \in \mathscr{A}. \tag{2.5}$$

(Note that the Cauchy–Schwarz inequality (2.2) shows that $(Aa \mid Aa)_{_A} = 0$ implies $Aa = 0$ for $a \in \mathscr{A}$ and hence that $(\,\cdot \mid \cdot\,)_{_A}$ defines an inner product on ran A, indeed.) The completion H_A is then a Hilbert space in which we introduce a densely defined continuous operator $\pi_A(x)$ for any fixed $x \in \mathscr{A}$ by letting

$$\pi_A(x)(Aa) := A(xa), \qquad a \in \mathscr{A}. \tag{2.6}$$

The continuity of $\pi_A(x)$ is due to (2.4):

$$(A(xa) \mid A(xa))_{_A} = f(a^*x^*xa) \leq \lambda_x \cdot f(a^*a) = \lambda_x \cdot (Aa \mid Aa)_{_A}.$$

If we continue to write $\pi_A(x)$ for its unique norm preserving extension, then it is easy to verify that π_A is a *-representation of $\mathscr{A}$ in $\mathscr{B}(H_A)$. The cyclic vector of π_A is obtained by considering the linear functional $Aa \mapsto f(a)$ from H_A into $\mathbb{C}$ whose continuity is guaranteed by (2.3). The Riesz representation theorem yields then a unique vector $\zeta_A \in H_A$ satisfying

$$f(a) = (Aa \mid \zeta_A)_{_A}, \qquad a \in \mathscr{A}. \tag{2.7}$$

It is again easy to verify identity

$$\pi_A(a)\zeta_A = Aa, \qquad a \in \mathscr{A}, \tag{2.8}$$

whence we infer that

$$f(a) = (\pi_A(a)\zeta_A \mid \zeta_A)_{_A}, \qquad a \in \mathscr{A}. \tag{2.9}$$

That ζ_A is a cyclic vector of π_A follows from identity (2.8).

3. Lebesgue decomposition for representable functionals. Throughout this section, we fix two representable positive functionals f and g on the *-algebra $\mathscr{A}$. Let A and B stand for the positive operators associated with f and g, respectively. The GNS-triplets (H_A, π_A, ζ_A) and (H_B, π_B, ζ_B), induced by f and g, respectively, are defined along the construction of the previous section. Let us recall the notions of absolute continuity and singularity regarding positive functionals (see [**3**] and [**14**]): g is called *absolutely continuous* with respect to f (shortly, g is f-absolutely continuous) if

$$f(a_n^*a_n) \to 0 \quad \text{and} \quad g((a_n - a_m)^*(a_n - a_m)) \to 0 \quad \text{imply} \quad g(a_n^*a_n) \to 0$$

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for any sequence $(a_n)_{n\in\mathbb{N}}$ of $\mathscr{A}$. On the other hand, f and g are mutually *singular* if the properties $h \leq f$ and $h \leq g$ imply $h = 0$ for any representable positive functional h.

Our aim in the rest of this section is to establish a Lebesgue decomposition theorem for representable positive functionals. More precisely, we shall show that g splits into a sum $g = g_a + g_s$ where both g_a and g_s are representable positive functionals with g_a f-absolutely continuous and g_s f-singular. Such a result was proved by Gudder [**3**] for positive functionals on a unital Banach *-algebra and by Szűcs [**15**] in a more general setting, namely for representable forms over a complex algebra.

Our treatment is based on the following observation: If $(a_n)_{n\in\mathbb{N}}$ is a sequence from $\mathscr{A}$ such that

$$(Aa_n \mid Aa_n)_{_A} \to 0 \qquad \text{and} \qquad (B(a_n - a_m) \mid B(a_n - a_m))_{_B} \to 0, \tag{3.1}$$

then $Ba_n \to \zeta$ for some $\zeta \in H$. If g is f-absolutely continuous, then ζ must be 0. We introduce therefore the following closed linear subspace of H_B (cf. also [**7**, **12**, **16**]):

$$\mathfrak{M} := \{\zeta \in H_B \mid \exists (a_n)_{n\in\mathbb{N}} \subseteq \mathscr{A}, (Aa_n \mid Aa_n)_{_A} \to 0, Ba_n \to \zeta \text{ in } H_B\}. \tag{3.2}$$

In fact, $\mathfrak{M}$ is nothing but the so called *multivalued part* of the closure of the following linear relation:

$$T := \{(Aa, Ba) \in H_A \times H_B \mid a \in \mathscr{A}\}, \tag{3.3}$$

cf. [**5**] and [**16**]. That is to say,

$$\mathfrak{M} = \operatorname{mul} \overline{T} := \{\zeta \in H_B \mid (0, \zeta) \in \overline{T}\}.$$

Furthermore, g is f-absolutely continuous precisely if $\mathfrak{M} = \{0\}$, i.e., when T is (the graph of) a closable operator. We will see below that $\mathfrak{M} = H_B$ holds if and only if g and f are mutually singular. In any other cases, $\mathfrak{M}$ is a proper closed π_B-invariant subspace of H_B (see Lemma 3.1 below).

Let P stand for the orthogonal projection of H_B onto $\mathfrak{M}$, and introduce the functionals g_a and g_s by setting:

$$g_a(a) := (\pi_B(a)(I - P)\zeta_B \mid (I - P)\zeta_B)_{_B}, \qquad g_s(a) := (\pi_B(a)P\zeta_B \mid P\zeta_B)_{_B}, \tag{3.4}$$

for $a \in \mathscr{A}$. Our main purpose is to prove that both g_a and g_s are representable positive functionals such that $g = g_a + g_s$ where g_a is f-absolutely continuous and g_s is singular with respect to f. Moreover, g_a is maximal in the sense that $h \leq g_a$ holds for each f-absolutely continuous representable functional h satisfying $h \leq g$.

LEMMA 3.1. *Let $\mathscr{A}$ be *-algebra and let f, g be representable functionals of $\mathscr{A}$. Then, $\mathfrak{M}$ and $\mathfrak{M}^{\perp}$ are both π_B-invariant subspaces of H_B, and the following identities hold:*

(a) $\pi_B(a)P\zeta_B = P\pi_B(a)\zeta_B = P(Ba)$, $a \in \mathscr{A}$,
(b) $\pi_B(a)(I - P)\zeta_B = (I - P)\pi_B(a)\zeta_B = (I - P)(Ba)$, $a \in \mathscr{A}$.

Proof. In order to prove the π_B-invariancy of $\mathfrak{M}$ fix $a \in \mathscr{A}$ and $\zeta \in \mathfrak{M}$, and consider a sequence $(a_n)_{n\in\mathbb{N}}$ from $\mathscr{A}$ satisfying

$$(Aa_n \mid Aa_n)_{_A} \to 0 \qquad \text{and} \qquad Ba_n \to \zeta \text{ in } H_B.$$

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Then, we have

$$A(aa_n) = \pi_A(a)(Aa_n) \to 0 \qquad \text{and} \qquad B(aa_n) = \pi_B(a)(Ba_n) \to \pi_B(a)\zeta \text{ in } H_B,$$

so that $\pi(a)\zeta \in \mathfrak{M}$, indeed. Consequently, $\pi_B(a)\langle\mathfrak{M}\rangle \subseteq \mathfrak{M}$ for all $a \in \mathscr{A}$, as claimed. That $\mathfrak{M}^\perp$ is also π_B-invariant follows immediately from the fact that π_B is a *-representation. We are going to prove now (a): for $a \in \mathscr{A}$ we have $\pi_B(a)\zeta_B = Ba$ by (2.8) so it suffices to show the first equality of (a). So fix $\zeta \in \mathfrak{M}$; by the π_B-invariancy of $\mathfrak{M}$, we have that

$$\begin{aligned}(P\pi_B(a)\zeta_B - \pi_B(a)P\zeta_B \mid \zeta)_B &= (\pi_B(a)\zeta_B - \pi_B(a)P\zeta_B \mid \zeta)_B \\ &= (\pi_B(a)(I-P)\zeta_B \mid \zeta)_B = 0,\end{aligned}$$

as $\pi_B(a)(I-P)\zeta_B \in \mathfrak{M}^\perp$ which yields (a). Assertion (b) is obtained easily from (a). □

As an immediate consequence, we have the following:

Corollary 3.2. *Both of the positive functionals g_a and g_s are representable and their sum satisfies*

$$g = g_a + g_s. \tag{3.5}$$

More precisely, $\pi_{B,a} := \pi_B(\cdot)(I-P)$ and $\pi_{B,s} := \pi_B(\cdot)P$ are both *-representations of $\mathscr{A}$ in the Hilbert spaces $\mathfrak{M}^\perp$ and $\mathfrak{M}$, respectively, with cyclic vectors $(I-P)\zeta_B$ and $P\zeta_B$, respectively, which satisfy

$$g_a(a) = (\pi_{B,a}(a)(I-P)\zeta_B \mid (I-P)\zeta_B)_B, \qquad g_s(a) = (\pi_{B,s}(a)P\zeta_B \mid P\zeta_B)_B, \tag{3.6}$$

for $a \in \mathscr{A}$.

We are now in position to state and prove the main result of the paper, the Lebesgue decomposition theorem of representable functionals ([**15**, Theorem 3.1]):

Theorem 3.3. *Let $\mathscr{A}$ be a *-algebra, f, g representable functionals on $\mathscr{A}$. Then*

$$g = g_a + g_s$$

is according to the Lebesgue decomposition, that is to say, both g_a and g_s are representable functionals such that g_a is absolutely continuous with respect to f and that g_s and f are mutually singular. Furthermore, g_a is maximal in the following sense: $h \leq g$ and $h \ll f$ imply $h \leq g_a$ for any representable positive functional h.

Proof. We start by proving that g_a is f-absolutely continuous. Consider therefore a sequence $(a_n)_{n\in\mathbb{N}}$ such that

$$f(a_n^*a_n) \to 0, \qquad \text{and} \qquad g_a((a_n - a_m)^*(a_n - a_m)) \to 0.$$

Then, by Lemma 3.1 we have

$$(Aa_n \mid Aa_n)_A \to 0, \qquad ((I-P)(B(a_n - a_m)) \mid (I-P)(B(a_n - a_m)))_B \to 0.$$

Nevertheless, the operator $H_A \supseteq \operatorname{ran} A \to H_B$, $Ax \mapsto (I-P)(Bx)$ coincides with the so-called regular part T_{reg} (see [**5**, (4.1)]) of the linear relation T of (3.3) hence it is

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closable in virtue of [**5**, Theorem 4.1]. Consequently,

$$g_a(a_n^*a_n) = ((I-P)(Ba_n) \,|\, (I-P)(Ba_n))_{_B} \to 0,$$

which proves the absolute continuity part of the statement.

In the next step, we prove the extremal property of g_a. Consider a representable functional h on $\mathscr{A}$ such that $h \le g$ and that h is f-absolutely continuous. Then, we have by representability

$$|h(a)|^2 \le C \cdot h(a^*a) \le C \cdot g(a^*a) = C \cdot (Ba \,|\, Ba)_{_B}, \qquad a \in \mathscr{A},$$

hence the linear functional $Ba \mapsto h(a)$ is continuous on $\operatorname{ran} B$ of H_B. The Riesz representation theorem yields therefore a (unique) representing vector $\zeta_h \in H_B$ that fulfils

$$h(a) = (Ba \,|\, \zeta_h)_{_B}, \qquad a \in \mathscr{A}. \tag{3.7}$$

We state that $\zeta_h \in \mathfrak{M}^\perp$. Fix therefore $\zeta \in \mathfrak{M}$ and consider a sequence $(a_n)_{n\in\mathbb{N}}$ from $\mathscr{A}$ such that

$$(Aa_n \,|\, Aa_n)_{_A} \to 0 \qquad \text{and} \qquad Ba_n \to \zeta \text{ in } H_B.$$

In particular, $(Ba_n)_{n\in\mathbb{N}}$ is Cauchy in H_B, therefore $h((a_n - a_m)^*(a_n - a_m)) \to 0$ holds by $h \le g$ and thus $h(a_n^*a_n) \to 0$ as h is f-absolutely continuous. That implies that

$$|(\zeta \,|\, \zeta_h)_{_B}|^2 = \lim_{n\to\infty} |(Ba_n \,|\, \zeta_h)_{_B}|^2 = \lim_{n\to\infty} |h(a_n)|^2 \le C \cdot \lim_{n\to\infty} h(a_n^*a_n) = 0,$$

which yields the desired identity. Fix now $a \in \mathscr{A}$; by Lemma 3.1 and according to identity $(I-P)\zeta_h = \zeta_h$, we conclude that

$$\begin{aligned} h(a^*a) &= (B(a^*a) \,|\, \zeta_h)_{_B} = ((I-P)(Ba) \,|\, \pi_B(a)\zeta_h)_{_B} \\ &\le \|(I-P)(Ba)\|_{_B} \|\pi_B(a)\zeta_h\|_{_B} = \sqrt{g_a(a^*a)}\|\pi_B(a)\zeta_h\|_{_B}, \end{aligned}$$

thus $h \le g_a$ will be obtained once we prove that

$$(\pi_B(a)\zeta_h \,|\, \pi_B(a)\zeta_h)_{_B} \le h(a^*a), \qquad a \in \mathscr{A}. \tag{3.8}$$

By using the density of $\operatorname{ran} B$ in H_B, it follows that

$$\begin{aligned} (\pi_B(a)\zeta_h \,|\, \pi_B(a)\zeta_h)_{_B} &= \sup\{|(Bx \,|\, \pi_B(a)\zeta_h)_{_B}\|^2 \,|\, x \in \mathscr{A}, (Bx \,|\, Bx)_{_B} \le 1\} \\ &= \sup\{|(B(a^*x) \,|\, \zeta_h)_{_B}\|^2 \,|\, x \in \mathscr{A}, g(x^*x) \le 1\} \\ &= \sup\{|h(a^*x)|^2 \,|\, x \in \mathscr{A}, g(x^*x) \le 1\} \\ &\le \sup\{h(a^*a)h(x^*x) \,|\, x \in \mathscr{A}, g(x^*x) \le 1\} \\ &\le h(a^*a), \end{aligned}$$

as it is claimed.

There is nothing left but to prove that g_s and f are singular with respect to each other. Fix therefore a representable functional h of $\mathscr{A}$ such that $h \le f$ and $h \le g_s$. Then, clearly $h \le g$ and h is f-absolutely continuous. By the previous step, there exists

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$\zeta_h \in \mathfrak{M}^\perp$ with property (3.7). By density of ran B, we may choose $(a_n)_{n\in\mathbb{N}}$ from $\mathscr{A}$ such that $Ba_n \to \zeta_h$ in H_B. Then, we find that

$$\begin{aligned} |(\zeta_h \mid \zeta_h)_B|^2 &= \lim_{n\to\infty} |(Ba_n \mid \zeta_h)_B|^2 = \lim_{n\to\infty} |h(a_n)|^2 \leq C \cdot \limsup_{n\to\infty} h(a_n^* a_n) \\ &\leq C \cdot \limsup_{n\to\infty} g_s(a_n^* a_n) = C \cdot \limsup_{n\to\infty} (P(Ba_n) \mid P(Ba_n))_B \\ &= C \cdot (P\zeta_h \mid P\zeta_h)_B = 0, \end{aligned}$$

whence $h = 0$. The proof is therefore complete. □

4. Mutually absolute continuity of the absolutely continuous parts. Let f, g be representable positive functionals on the *-algebra $\mathscr{A}$ and consider the Lebesgue decompositions

$$f = f_a + f_s, \qquad g = g_a + g_s,$$

where f_a, f_s and g_a, g_s are obtained along the procedure presented in the previous section. In accordance with Theorem 3.3, $f_a \ll g$ and $g_a \ll f$. Our purpose in this section is to show that the absolutely continuous parts f_a and g_a are mutually absolutely continuous, that is that $f_a \ll g_a$ and $g_a \ll f_a$ hold true. The heart of the matter is in the following lemma which may be of interest on its own right.

LEMMA 4.1. *Let T be a linear relation between two Hilbert spaces H and K. Let $\overline{T}$ stand for the closure of T and let P, Q be the orthogonal projections onto* $\ker \overline{T}$ *and* $\operatorname{mul} \overline{T}$*, respectively. Then*

$$S_0 := \{((I-P)h, (I-Q)k) \mid (h,k) \in T\}$$

is (the graph of) a closable linear operator whose closure $\overline{S_0}$ is one-to-one.

Proof. We shall show that

$$S := \{((I-P)h, (I-Q)k) \mid (h,k) \in \overline{T}\}$$

is an invertible closed operator. As $S_0 \subseteq S$, this contains our original assertion. Consider first the so-called regular part $(\overline{T})_{\mathrm{reg}}$ of $\overline{T}$, which is defined by

$$(\overline{T})_{\mathrm{reg}} := \{(h, (I-Q)k) \mid (h,k) \in \overline{T}\},$$

cf. [**5**]. Let us denote it by R for the sake of brevity. We claim first that R is a closed linear operator such that $R \subseteq \overline{T}$. The proof of this statement can be found in [**5**], we include here a short proof however, for the sake of the reader. It is seen easily that $\{0\} \times \operatorname{mul} \overline{T} \subseteq \overline{T}$, and that $\overline{T} - (\{0\} \times \operatorname{mul} \overline{T}) = R$. Consequently, $R \subseteq \overline{T}$ and, as $\{0\} \times \operatorname{mul} \overline{T}$ and R are orthogonal to each other, we infer that $R = \overline{T} \ominus (\{0\} \times \operatorname{mul} \overline{T})$, hence R is closed. To see that R is an operator, assume that $(0, (I-Q)k) \in R$ where $(0,k) \in \overline{T}$. This implies that $k \in \operatorname{mul} \overline{T}$ whence $(I-Q)k = 0$, that is, R is an operator, indeed. Observe furthermore that $\ker R = \ker \overline{T}$: indeed, by the definition of R it is clear that $\ker \overline{T} \subseteq \ker R$, and the converse inclusion is due to $R \subseteq \overline{T}$.

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Consider now the relation $(R^{-1})_{\mathrm{reg}}$. Then, $(R^{-1})_{\mathrm{reg}}$ is a a closed linear operator due to the above reasoning, such that $(R^{-1})_{\mathrm{reg}} \subseteq R^{-1}$. At the same time,

$$\operatorname{mul} R^{-1} = \ker R = \ker \overline{T},$$

whence

$$\begin{aligned}(R^{-1})_{\mathrm{reg}} &= \{(k', (I-P)h') \mid (h', k') \in R\} \\ &= \{((I-Q)k, (I-P)h) \mid (h,k) \in \overline{T}\} = S^{-1}.\end{aligned}$$

We conclude therefore that S^{-1} is a closed operator, such that $S^{-1} \subseteq R^{-1}$, or equivalently, $S \subseteq R$. Hence, S is an operator as well. □

THEOREM 4.2. *Let f and g be representable positive functionals on the *-algebra $\mathscr{A}$. Denote by f_a and g_a the g-absolutely continuous and the f-absolutely continuous parts of f and g, respectively. Then, f_a and g_a are absolutely continuous with respect to each other: $f_a \ll g_a$ and $g_a \ll f_a$.*

Proof. Consider the linear relation T of (3.3) and let P, Q be the orthogonal projections onto $\ker \overline{T}$ and $\operatorname{mul} \overline{T}$, respectively. By Theorem 3.3, the corresponding absolutely continuous parts satisfy

$$f_a(a^*a) = \|(I-P)Aa\|_A^2, \qquad g_a(a^*a) = \|(I-Q)Ba\|_B^2, \qquad a \in \mathscr{A}.$$

According to Lemma 4.1, the relation

$$S := \{((I-P)Aa, (I-Q)Ba) \mid a \in \mathscr{A}\}$$

is (the graph of) a closable operator. Hence, if

$$f_a(a_n^*a_n) = \|(I-P)Aa_n\|_A^2 \to 0,$$

and

$$g_a((a_n - a_m)^*(a_n - a_m)) = \|(I-Q)B(a_n - a_m)\|_B^2 \to 0$$

hold for some $(a_n)_{n\in\mathbb{N}}$, then

$$g_a(a_n^*a_n) = \|(I-Q)Ba_n\|_B^2 \to 0,$$

whence we deduce that g_a is f_a-absolutely continuous. That f_a is g_a-absolutely continuous follows from the fact that S is one-to-one with closable inverse, according again to Lemma 4.1. □

5. Examples. We conclude the paper with some applications of Theorem 3.3. Namely, we prove two classical results of the Lebesgue decomposition theory: the Lebesgue decomposition of measures (see e.g. [**4**]) and the Lebesgue–Darst decomposition of finitely additive set functions (see [**1**], or [**17**] for a functional analytic approach).

EXAMPLE 5.1. Let T be a non-empty set with a σ-algebra $\mathscr{R}$ on it. Denote by $\mathscr{S}(T, \mathscr{R})$ the unital *-algebra of $T \to \mathbb{C}$ measurable step functions. If we consider a

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finite measure μ on $\mathscr{R}$, then the mapping

$$\varphi \mapsto \int \varphi \, d\mu$$

defines a positive linear functional on $\mathscr{S}(T, \mathscr{R})$. Observe that this functional is representable: indeed, for $\varphi, \psi \in \mathscr{S}(T, \mathscr{R})$ we have

$$\left| \int \varphi \, d\mu \right|^2 \leq \mu(T) \cdot \int |\varphi|^2 \, d\mu, \qquad \int |\varphi\psi|^2 \, d\mu \leq M_\varphi^2 \int |\psi|^2 \, d\mu,$$

where M_φ is the maximum of the step function $|\varphi|$. Let us denote this representable functional by f. Assume that we are given another finite measure ν on $\mathscr{R}$ and let g stand for the representable functional induced by ν. According to Theorem 3.3, we may consider the Lebesgue decomposition $g = g_a + g_s$ of g with respect to f. One can easily verify that the mappings

$$\nu_a(E) := g_a(\chi_E) \qquad \text{and} \qquad \nu_s(E) := g_s(\chi_E)$$

are non-negative valued additive set functions on $\mathscr{R}$. Here, χ_E denotes the characteristic function of the measurable set E. Moreover, inequalities $\nu_a, \nu_s \leq \nu$ imply that ν_a and ν_s are σ-additive. It is clear that

$$\nu = \nu_a + \nu_s. \tag{5.1}$$

We claim that the decomposition (5.1) is the Lebesgue decomposition of ν with respect to μ, that is, ν_a is μ-absolutely continuous and ν_s is μ-singular. Indeed, let $E \in \mathscr{R}$ such that $\mu(E) = 0$. By choosing $\varphi_n := \chi_E$ for any integer n, we see that $f(\varphi_n^* \varphi_n) \to 0$ and $g_a((\varphi_n - \varphi_m)^*(\varphi_n - \varphi_m)) \to 0$, whence we infer that $\nu_a(E) = g_a(\varphi_n) \to 0$, due to the f-absolute continuity of g_a. This means that ν_a is μ-absolutely continuous. To prove that μ and ν_s are mutually singular, consider a measure ϑ on $\mathscr{R}$ such that $\vartheta \leq \mu, \nu_s$. We claim that $\vartheta = 0$. Indeed, if h denotes the representable functional induced by ϑ, then by inequalities $h \leq f, g_s$ and due to f-singularity of g_s we conclude that $h = 0$. Hence, $\vartheta = 0$ as well, which means that μ and ν_s are mutually singular. Finally, if $\mu = \mu_a + \mu_s$ is the Lebesgue decomposition of μ with respect to ν then we obtain that $\mu_a \ll \nu_a$ and $\nu_a \ll \mu_a$, in the view of Theorem 4.2.

EXAMPLE 5.2. Let $\mathscr{R}$ be a ring of sets over the non-empty set T and denote by $\mathscr{S}(T, \mathscr{R})$ the (not necessarily unital) *-algebra of $T \to \mathbb{C}$ measurable step functions. Consider two non-negative valued (finitely) additive set functions α, β on $\mathscr{R}$ which we suppose to be bounded:

$$\sup_{E \in \mathscr{R}} \alpha(E) < +\infty, \qquad \sup_{E \in \mathscr{R}} \beta(E) < +\infty.$$

Recall that β is called α-absolutely continuous if for any $\varepsilon > 0$ there is $\delta > 0$ such that $\beta(E) < \varepsilon$ for any $E \in \mathscr{R}$ with $\alpha(E) < \delta$. Furthermore, α and β are called mutually singular if $\vartheta = 0$ is the unique non-negative additive set function such that $\vartheta \leq \alpha, \beta$. The Lebesgue–Darst decomposition theorem ([**17**, Theorem 4]) states that there exist two non-negative additive set functions β_a, β_s on $\mathscr{R}$ with β_a α-absolutely continuous and β_s α-singular such that $\beta = \beta_a + \beta_s$. Below, we are going to prove this result due to Theorem 3.3. To this aim, let us define first the representable positive functionals

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f, g on $\mathscr{S}(T, \mathscr{R})$ by setting

$$f(\varphi) := \int \varphi \, d\alpha, \qquad g(\varphi) := \int \varphi \, d\beta.$$

The representability of f, g follows similarly as in Example 5.1. Let us consider the Lebesgue decomposition $g = g_a + g_s$ of g with respect to f. If we set

$$\beta_a(E) := g_a(\chi_E) \qquad \text{and} \qquad \beta_s(E) := g_s(\chi_E),$$

then clearly, both β_a and β_s are non-negative valued additive set functions on $\mathscr{R}$ such that

$$\beta = \beta_a + \beta_s. \tag{5.2}$$

We claim that (5.2) is according to the Lebesgue–Darst decomposition. Indeed, by [**12**, Theorem 3.2 (a)], the f-absolute continuity of g_a implies α-absolutely continuity on β_a. That β_s and α are mutually singular is deduced by the argument used by proving the singularity of ν_s and μ in Example 5.1. Furthermore, if we consider the Lebesgue–Darst decomposition $\alpha = \alpha_a + \alpha_s$ of α with respect to β then, in the view of Theorem 4.2, we also obtain that α_a and β_a are absolutely continuous with respect to each other.

REFERENCES

1. R. B. Darst, A decomposition of finitely additive set functions, *J. Reine Angew. Math.* **210** (1962), 31–37.

2. J. Dixmier, *C*-algebras*, Translated from the French by F. Jellett, North-Holland Mathematical Library, vol. 15 (North-Holland publishing Co., Amsterdam–New York–Oxford, 1977).

3. S. P. Gudder, A Radon-Nikodym theorem for *-algebras, *Pacific J. Math.* **80** (1979), 141–149.

4. P. R. Halmos, *Measure theory* (Springer-Verlag, New York, 1974).

5. S. Hassi, Z. Sebestyén, H. S. V. de Snoo and F. H. Szafraniec, A canonical decomposition for linear operators and linear relations, *Acta Math. Hungar.* **115** (2007), 281–307.

6. S. Hassi, Z. Sebestyén and H. de Snoo, Lebesgue type decompositions for nonnegative forms, *J. Funct. Anal.* **257** (2009), 3858–3894.

7. H. Kosaki, Remarks on Lebesgue-type decomposition of positive operators, *J. Operator Theory* **11** (1984), 137–143.

8. T. W. Palmer, *Banach algebras and the general theory of *-algebras II*, (Cambridge University Press, Cambridge, 2001).

9. C. La Russa and S. Triolo, Radon–Nikodym theorem in quasi *-algebras, *J. Operator Theory* **69** (2013), 423–433.

10. Z. Sebestyén, On representability of linear functionals on *-algebras, *Period. Math. Hungar.* **15** (1984), 233–239.

11. Z. Sebestyén, Zs. Szűcs and Zs. Tarcsay, Extensions of positive operators and functionals, *Linear Algebra Appl.* **472** (2015), 54–80.

12. Z. Sebestyén, Zs. Tarcsay and T. Titkos, Lebesgue decomposition theorems, *Acta Sci. Math. (Szeged)* **79** (2013), 219–233.

13. Zs. Szűcs, The singularity of positive linear functionals, *Acta Math. Hungar.* **136** (2012), 138–155.

14. Zs. Szűcs, On the Lebesgue decomposition of positive linear functionals, *Proc. Amer. Math. Soc.* **141** (2013), 619–623.

15. Zs. Szűcs, The Lebesgue decomposition of representable forms over algebras, *J. Operator Theory* **70** (2013), 3–31.

16. Zs. Tarcsay, Lebesgue-type decomposition of positive operators, *Positivity* **17** (2013), 803–817.

17. Zs. Tarcsay, A functional analytic proof of the Lebesgue-Darst decomposition theorem, *Real Anal. Exchange* **39** (2013), 219–226.

18. T. Titkos, Lebesgue decomposition of contents via nonnegative forms, *Acta Math. Hungar.* **140** (2013), 151–161.