AN ALGEBRAIC PROOF OF COMPLETENESS FOR MONADIC FUZZY PREDICATE LOGIC **MMTL∀**

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Abstract. Monoidal t-norm based logic **MTL** is the weakest *t*-norm based residuated fuzzy logic, which is a [0, 1]-valued propositional logical system having a t-norm and its residuum as truth function for conjunction and implication. Monadic fuzzy predicate logic **mMTL∀** that consists of the formulas with unary predicates and just one object variable, is the monadic fragment of fuzzy predicate logic **MTL∀**, which is indeed the predicate version of monoidal t-norm based logic **MTL**. The main aim of this paper is to give an algebraic proof of the completeness theorem for monadic fuzzy predicate logic **mMTL∀** and some of its axiomatic extensions. Firstly, we survey the axiomatic system of monadic algebras for *t*-norm based residuated fuzzy logic and amend some of them, thus showing that the relationships for these monadic algebras completely inherit those for corresponding algebras. Subsequently, using the equivalence between monadic fuzzy predicate logic **mMTL∀** and S5-like fuzzy modal logic **S5**(**MTL**), we prove that the variety of monadic MTL-algebras is actually the equivalent algebraic semantics of the logic **mMTL∀**, giving an algebraic proof of the completeness theorem for this logic via functional monadic MTL-algebras. Finally, we further obtain the completeness theorem of some axiomatic extensions for the logic **mMTL∀**, and thus give a major application, namely, proving the strong completeness theorem for monadic fuzzy predicate logic based on involutive monoidal *t*-norm logic **mIMTL∀** via functional representation of finitely subdirectly irreducible monadic IMTL-algebras.

§1. Introduction. The reader is assumed to know basic facts on monadic fuzzy predicate logics and their Kripke semantics, including the most well-known fuzzy modal logic S5; on the other hand, he/she is also assumed to know *t*-norm based residuated fuzzy logic as presented in the most famous books [26, 37]. Fuzzy logic

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is more suitable than classical logic to handle uncertain and fuzzy information, has become a subject of increasing interest as logics of vagueness [56]. Algebraic semantics underlying fuzzy logic involve various binary operations on the unit real interval [0, 1], generalizing the classical Boolean truth functions on $\{0, 1\}$. For example, MV-algebras were introduced in [5] by Chang as algebraic semantics of the infinitely valued logic of Łukasiewicz Ł, while BL-algebras were introduced in [26] by Hájek as algebraic semantics of Basic fuzzy logic **BL**, a general framework in which tautologies of continuous t-norm and their residua can be captured [8]. Inspired by Hájek's famous work, Esteva and Godo proposed in [15] a new formal deductive system **MTL**, called monoidal t-norm based logic, and intended to cope with left-continuous t-norms and their residua [33]. These logical systems have been called t-norm based residuated fuzzy logics and can be suitably placed in a hierarchy of logic depending on their characteristic axioms, all of them being extensions of the logic **MTL**, which also shows that the logic **MTL** is the weakest t-norm based residuated fuzzy logic, and having classical logic as common extension [37]. In the past two decades, studies in **MTL** and its formal extensions as well as on the algebraic semantics side draw attention to and establish strong connections between the fields [2, 7, 9, 10, 15, 17, 20, 24, 27, 39, 58].

Monadic (Boolean) algebra $(L, \exists)$, in the sense of Halmos [32], is a Boolean algebra L equipped with a closure operator $\exists$, which abstracts algebraic properties of the standard existential quantifier "for some" [46]. The name "monadic" comes from the connection with predicate logics for languages having one placed predicates and just a single quantifier. Subsequently, monadic MV-algebras, the algebraic counterpart of monadic Łukasiewicz predicate logic, were introduced and studied in [11, 12, 31, 42, 45]. In order to algebraize the monadic basic fuzzy predicate logic **mBL**$\forall$, three kinds of monadic BL-algebras have been introduced and studied successively, one by Drăgulici [13], one by Grigolia [25] and another by Castaño [4]. Along with the same line of the above works, monadic residuated ℓ-monoids [44], residuated lattices [43], bounded hoops [49], Heyting algebras [14, 34, 38, 41] and other related monadic algebraic structures were introduced in [21, 48–54]. However, it should be pointed out here that some monadic algebras of t-norm based fuzzy residuated logic abandon the naturally containing relations between corresponding algebras. Indeed, readers please refer to Sections 2 and 3 for details,

$$\mathbb{RL}_\ell = \mathbb{RL} + \mathbf{DIV},\ \text{but}\ \mathbb{MRL}_\ell \neq \mathbb{MRL} + \mathbf{DIV}$$

and

$$\mathbb{MV} = \mathbb{BL} + \mathbf{INV},\ \text{but}\ \mathbb{MMV} \neq \text{Drăgulici's}\ \mathbb{MBL} + \mathbf{INV}.$$

Thus the first aim of this paper is to make up this important drawback. In Section 2, we survey monadic algebras of t-norm based fuzzy residuated logic and revise some of their axiomatic systems, then showing that the relationships between monadic algebras of t-norm based fuzzy residuated logic completely conserve those between corresponding algebras of t-norm based fuzzy residuated logic.

In order to generalize fuzzy predicate logic **BL**$\forall$ [26, 28–30], Esteva and Godo introduced a weaker fuzzy predicate logic **MTL**$\forall$, which was built up from variables, predicate symbols, connectives $\sqcap$ (additive conjunction), & (multiplicative conjunction), $\sqcup$ (additive disjunction), $\Rightarrow$ (multiplicative implication), the constant $\bar{0}$, and the quantifiers $\exists$ and $\forall$ [16, 18]. From the application point of view, fuzzy predicate logic is one of the most important techniques for the representation of knowledge.

A familiarity with fuzzy predicate logic is important for the following reasons. First of all, logic is a formal method for reasoning. Many concepts that can be verbalized can also be translated into symbolic representations that closely approximate the meaning of these concepts. These symbolic structures can then be manipulated in programs to deduce various facts to carry out a form of automated reasoning. Second, logic offers the only formal approach to reasoning that has a sound theoretical foundation. This is especially important in order to mechanize or automate the reasoning process in that inferences should be correct and logically sound [47].

Subsequently, Montagna and Ono introduced a Kripke semantics for **MTL**$\forall$, and proved a completeness theorem of **MTL**$\forall$ with respect to the above mentioned Kripke semantics [36]. As an important monadic fragment of fuzzy predicate logic **MTL**$\forall$, monadic fuzzy predicate logic **mMTL**$\forall$, in which only a single individual variable occurs, has attracted an increasing amount of experts and scholars' attention in recent years [28, 48–54]. Along with the same lines of Hájek's equivalence between monadic fuzzy predicate logic **mBL**$\forall$ and fuzzy modal logic **S5**(**BL**) [30], monadic fuzzy predicate logic **mMTL**$\forall$ is actually equivalent to fuzzy modal logic **S5**(**MTL**) because there is a natural correspondence between formulas of both logics, and between corresponding models and between the corresponding truth degrees. Thus in this paper we will work in the language of the fuzzy modal logic **S5**(**MTL**) instead of in the monadic fuzzy predicate language of **mMTL**$\forall$. The logic **S5**(**MTL**) is defined semantically over the language of **MTL** augmented with the unary connectives $\Diamond$ and $\Box$, known as Kripke semantics, readers please refer to Section 4 for details. Several researchers initially were only interested in the tautologies of **S5**(**MTL**) and its axiomatic extensions, but in [30], a global consequence relation $\models_{\mathbf{S5(MTL)}}$, which is finitary since **mMTL**$\forall$ is a fragment of a finitary logic **MTL**$\forall$, is considered.

Latterly, Zahiri and Borumand Saeid [57] have also introduced monadic monoidal t-norm based logic **MMTL**, which is a system of many valued logic capturing the tautologies of monadic MTL-algebras. The logic **MMTL** is actually a Hilbert-style syntactic calculus in the language of **S5**(**MTL**), which is a logic **MTL** also together with the following axioms:

($\Diamond$1) $\varphi \Rightarrow \Diamond\varphi$,
($\Box$2) $\Box(\nu \Rightarrow \varphi) \Rightarrow (\nu \Rightarrow \Box\varphi)$,
($\Diamond$2) $\Box(\varphi \Rightarrow \nu) \Rightarrow (\Diamond\varphi \Rightarrow \nu)$,
($\Box$3) $\Box(\nu \sqcup \varphi) \Rightarrow (\nu \sqcup \Box\varphi)$,
($\Diamond$3) $\Diamond(\varphi \& \varphi) \equiv \Diamond\varphi \& \Diamond\varphi$,

where φ is any formula and ν is any propositional combination of formulas beginning with $\Diamond$ and $\Box$. The inference rules are Modus Ponens **MP**: φ, $\varphi \Rightarrow \phi \vdash \phi$ and Necessitation Rule **Nec** : $\varphi/\Box\varphi$ and the consequence relation is denoted by $\vdash_{\mathbf{S5(MTL)}}$.

The second main goal of this paper is to investigate the consistency of the syntactic and the semantic for the logic **mMTL**$\forall$. This aim can be equivalently transformed into studying the corresponding consistency of **S5**(**MTL**) by the equivalence of the logics **mMTL**$\forall$ and **S5**(**MTL**). In order to achieve this goal, we have to prove

$$\models_{\mathbf{S5(MTL)}} \text{ if and only if } \vdash_{\mathbf{S5(MTL)}},$$

and thus prove the completeness theorem of the logic **S5**(**MTL**). In Sections 3 and 4, we aim to prove this important result by using algebraic methods, and obtain some partial results towards this goal in Section 5.

This paper is organized as follows: Section 2 summarizes some notions about t-norm based residuated fuzzy logics and their corresponding monadic fuzzy logics. Section 3 surveys the present state of knowledge on monadic algebras of t-norm based fuzzy residuated logic. Section 4 proves the completeness theorem for the logic **mMTL**$\forall$ and its axiomatic extensions. Section 5 proves the strong completeness theorem for the logic **mIMTL**$\forall$.

§2. Preliminaries. In this section, we summarize some results about t-norm based residuated fuzzy logics and their corresponding monadic fuzzy predicate logics. In particular, all of t-norm based residuated fuzzy logics are extensions of the so-called monoidal logic (**ML** for short) [23], which is the extension of Full Lambek calculus with exchange and weakening. In order to make the paper as self-contained as possible, we begin with **ML** and concentrate on its most notable extensions.

Definition 2.1 [23]. **ML** *is the logic given by the Hilbert-style calculus with* **MP** *as its only inference rule and the following axioms*:

(1) $(\varphi \Rightarrow \phi) \Rightarrow ((\phi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \psi))$,
(2) $(\varphi \& \phi) \Rightarrow \varphi$,
(3) $(\varphi \& \phi) \Rightarrow (\phi \& \varphi)$,
(4) $(\varphi \Rightarrow (\phi \Rightarrow \psi)) \Rightarrow (\varphi \& \phi \Rightarrow \psi)$,
(5) $(\varphi \& \phi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow (\phi \Rightarrow \psi))$,
(6) $\varphi \sqcap \phi \Rightarrow \varphi$,
(7) $\varphi \sqcap \phi \Rightarrow \phi \& \varphi$,
(8) $\bar{0} \Rightarrow \varphi$,
(9) $\varphi \Rightarrow \varphi \sqcup \phi$,
(10) $\phi \Rightarrow \varphi \sqcup \phi$,
(11) $(\varphi \Rightarrow \psi) \Rightarrow ((\phi \Rightarrow \psi) \Rightarrow (\varphi \sqcup \phi \Rightarrow \psi))$,
(12) $(\varphi \Rightarrow \phi) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \phi \sqcap \psi))$.

The usual connectives in **ML** can be derived from $\&, \Rightarrow, \sqcap$ are as follows:

$$\bar{1} := \varphi \Rightarrow \varphi, \sim\varphi := \varphi \Rightarrow \bar{0},$$

$$\varphi \equiv \phi := (\varphi \Rightarrow \phi)\&(\phi \Rightarrow \varphi).$$

Outstanding logics up in the hierarchy of extensions of **ML** are presented.
Monoidal t-norm based logic **MTL** is **ML** plus the axiom of pre-linearity:

$$(\textbf{PRE}) \quad (\varphi \Rightarrow \phi) \sqcup (\phi \Rightarrow \varphi).$$

Divisible monoidal logic $\mathbf{ML}_\ell$ is **ML** plus the axiom of divisibility:

$$(\textbf{DIV}) \quad \varphi \sqcap \phi \Rightarrow \varphi\&(\varphi \Rightarrow \phi).$$

Involutive monoidal logic **IML** is **ML** plus the axiom of involution:

$$(\textbf{INV}) \quad \sim\sim\varphi \Rightarrow \varphi.$$

Intuitionistic logic **IL** is **ML** plus the axiom of idempotence:

$$(\textbf{IDE}) \quad \varphi \Rightarrow \varphi\&\varphi.$$

Basic fuzzy logic **BL** is **MTL** plus the axiom (**DIV**).

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Gödel logic **G** is **MTL** plus the axiom (**IDE**).[1]
Łukasiewicz logic **Ł** is **MTL** plus the following axiom:[2]

$$(\mathbf{MV}) \quad (\varphi \Rightarrow \phi) \Rightarrow \phi \Rightarrow (\phi \Rightarrow \varphi) \Rightarrow \varphi.$$

Involutive monoidal t-norm based logic **IMTL** is **MTL** plus the axiom (**INV**).
Nilpotent minimum logic **NM** is **IMTL** plus the axiom (**WNM**):[3]

$$(\mathbf{WNM}) \quad \sim(\varphi \& \phi) \sqcup ((\varphi \sqcap \phi) \Rightarrow (\varphi \& \phi)).$$

Classical logic **CL** is **MTL** plus the axiom:

$$(\mathbf{EM}) \quad \varphi \sqcup \sim \varphi.$$

As the author pointed out in [37], **ML** is an implicative logic in the sense of Blok and Pigozzi [1], and so it is an algebraic logic whose equivalent algebraic semantics Alg***ML** is the variety of residuated lattices that we usually denote by $\mathbb{RL}$.[4]

Definition 2.2 [55]. *An algebraic structure* $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *of type* $(2, 2, 2, 2, 0, 0)$ *is called a* residuated lattice *if it satisfies the following conditions*:

(1) $(L, \wedge, \vee, 0, 1)$ *is a bounded lattice.*
(2) $(L, \odot, 1)$ *is a commutative monoid.*
(3) $x \odot y \leq z$ *if and only if* $x \leq y \rightarrow z$,

for any $x, y, z \in L$.

In what follows, by L we denote the universe of a residuated lattice $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$. In any residuated lattice L, we define

$$\neg x = x \rightarrow 0, x \oplus y = \neg(\neg x \odot \neg y),$$

$$\neg\neg x = \neg(\neg x), x^0 = 1 \text{ and } x^n = x^{n-1} \odot x \text{ for } n \geq 1.$$

As a result of the general theory of algebraizability, every axiomatic extension of **ML** is algebraizable and its equivalent algebraic semantics is a subvariety of $\mathbb{RL}$. Namely, if **C** is an axiomatic extension of **ML**, then the equivalent algebraic semantics Alg***C** is the subvariety of $\mathbb{RL}$ determined by the equations of the form $\varphi \approx 1$, where φ ranges over the axiom schemata of **C** [35]. As a consequence of the general theory of algebraizability and Figure 1, Figure 2 is naturally obtained.

In particular:

- The equivalent algebraic semantics for $\mathbf{ML}_\ell$ is the variety $\mathbb{RL}_\ell$ of residuated ℓ-monoids, those residuated lattices satisfying (DIV) $x \wedge y = x \odot (x \rightarrow y)$.
- The equivalent algebraic semantics for **IML** is the variety $\mathbb{IRL}$ of involutive residuated lattices, those residuated lattices satisfying (INV) $\neg\neg x = x$.
- The equivalent algebraic semantics for **MTL** is the variety $\mathbb{MTL}$ of MTL-algebras, those residuated lattices satisfying (PRL) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

[1] Gödel logic **G** is also obtained by adding to **BL** the axiom (**IDE**).
[2] Łukasiewicz logic Ł is **BL** plus the axiom (**INV**).
[3] Weak nilpotent minimum logic **WNM** is **MTL** plus the axiom (**WNM**).
[4] Variety is a basic notion on Universal Algebra, which is a class of algebras that is closed under subalgebras, homomorphic images, and direct products, readers please refer to Definition 9.3 and its related basic notions in [3] for more details.

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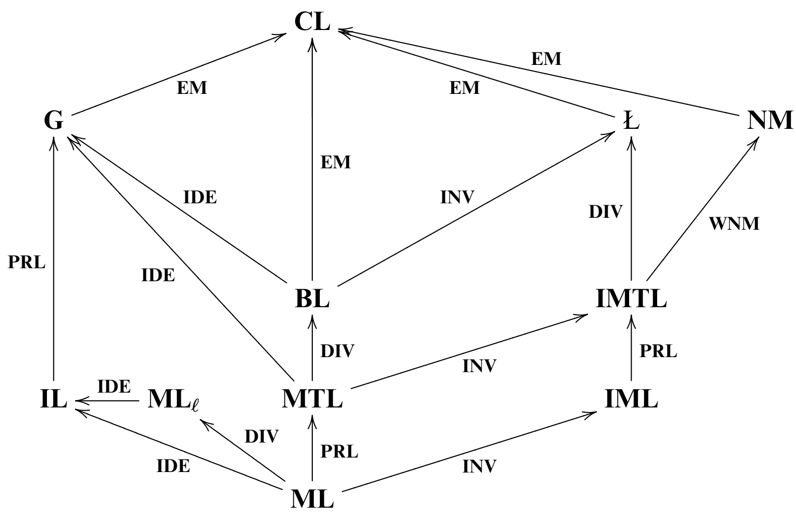

Figure 1. Relationships between t-norm based residuated fuzzy logics.

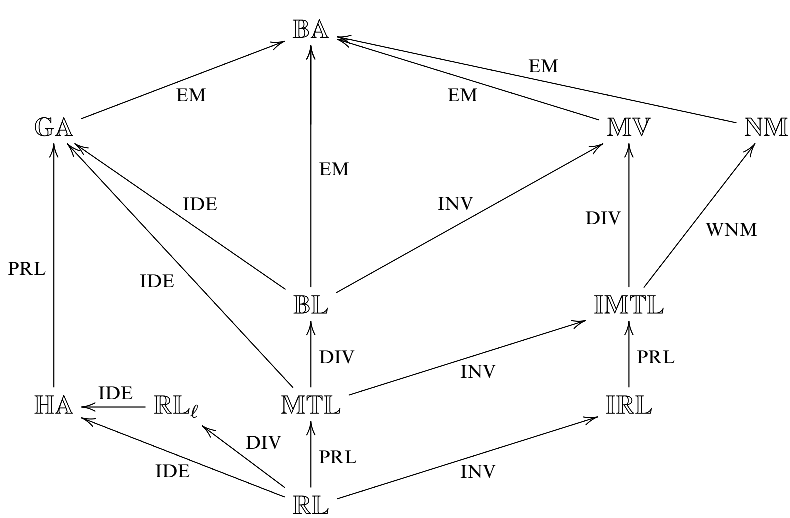

Figure 2. Relationships between algebras of t-norm based residuated fuzzy logic.

- The equivalent algebraic semantics for **IL** is the variety $\mathbb{HA}$ of Heyting algebras, those residuated lattices satisfying (IDE) $x \odot x = x$.
- The equivalent algebraic semantics for **IMTL** is the variety $\mathbb{IMTL}$ of IMTL-algebras, those MTL-algebras satisfying the algebraic equation (INV).
- The equivalent algebraic semantics for **BL** is the variety $\mathbb{BL}$ of BL-algebras, those MTL-algebras satisfying the algebraic equation (PRL).
- The equivalent algebraic semantics for **Ł** is the variety $\mathbb{MV}$ of MV-algebras, those BL-algebras satisfying the algebraic equation (INV).

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- The equivalent algebraic semantics for **G** is the variety $\mathbb{GA}$ of Gödel algebras, those MTL-algebras satisfying the algebraic equation (IDE).
- The equivalent algebraic semantics for **NM** is the variety $\mathbb{NM}$ of NM-algebras, those IMTL-algebras satisfying the algebraic equation (WNM) $\neg(x \odot y) \vee ((x \wedge y) \rightarrow (x \odot y)) = 1$.
- The equivalent algebraic semantics for **CL** is the variety $\mathbb{BA}$ of Boolean algebras, those MTL-algebras satisfying the algebraic equation (EM) $x \vee \neg x = 1$.

Here we review some notions about monadic fuzzy predicate logic **mMTL**∀ [30].

Definition 2.3 [16]. **MTL**∀ *is a fuzzy predicate logic and has the following axioms*:

(P) *the axioms resulting from the axioms of* **MTL** *by the substitution of the propositional variables by the* **Γ**-*formulas*:[5]

(∀1) $(\forall x)\varphi(x) \Rightarrow \varphi(t)$, *where* t *is substitutable for* x *in* φ,
(∃1) $\varphi(t) \Rightarrow (\exists x)\varphi(x)$, *where* t *is substitutable for* x *in* φ,
(∀2) $(\forall x)(\phi \Rightarrow \varphi) \Rightarrow (\phi \Rightarrow \forall(x)\varphi)$, *where* x *is not free in* ϕ,
(∃2) $(\forall x)(\varphi \Rightarrow \phi) \Rightarrow ((\exists x)\varphi \Rightarrow \phi)$, *where* x *is not free in* ϕ,
(∀3) $(\forall x)(\phi \sqcup \varphi) \Rightarrow \phi \sqcup \forall(x)\varphi)$, *where* x *is not free in* ϕ.

The deduction rules are those of **MTL** *and generalization*: *from* φ *infer* $(\forall x)\varphi$.

Definition 2.4 [28]. *Monadic fuzzy predicate logic* **mMTL**∀ *is the monadic fragment of fuzzy predicate logic* **MTL**∀, *consisting of the formulas with unary predicates and just one object variable.*

§3. Monadic algebras of t-norm based residuated fuzzy logic.

In this section, we survey the present state of knowledge on monadic algebras of t-norm based residuated fuzzy logic, and show that the relationships between these monadic algebras completely inherit those between corresponding algebras. Considering the algebras of t-norm based residuated fuzzy logic are all particular case of residuated lattices, we review monadic residuated lattices in advance.

Definition 3.1 [43]. *An algebraic structure* $(L, \wedge, \vee, \odot, \rightarrow, \forall, \exists, 0, 1)$ *is said to be a* monadic residuated lattice *if* $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *is a residuated lattice such that* ∀ *and* ∃ *satisfy the following identities*:

(∀1) $\forall x \rightarrow x = 1$,
(∃1) $x \rightarrow \exists x = 1$,
(∀2) $\forall(x \rightarrow \exists y) = \exists x \rightarrow \exists y$,
(∀3) $\forall(\exists x \rightarrow y) = \exists x \rightarrow \forall y$,
(∀4) $\forall(x \vee \exists y) = \forall x \vee \exists y$,
(∀5) $\forall\forall x = x$,
(∃2) $\exists\forall x = \forall x$,
(∃3) $\exists(\exists x \odot \exists y) = \exists x \odot \exists y$,
(∃4) $\exists(x \odot x) = \exists x \odot \exists x$,

for any $x, y \in L$.

[5] **Γ** is a predicate language (**P**, **F**, **A**), where **P** is a non-empty set of predicate symbols, **F** is a set of function symbols, and **A** is a function assigning to each predicate and function symbol a natural number called the arity of the symbol.

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A monadic residuated lattice $(L,\wedge,\vee,\odot,\rightarrow,\forall,\exists,0,1)$ will be denoted simply by $(L,\forall,\exists)$. The class of monadic residuated lattices will be denoted by $\mathbb{MRL}$. Clearly, in light of the above axiomatization, $\mathbb{MRL}$ forms a variety.

REMARK 3.2 [51]. $(\exists1)$, $(\forall5)$, *and* $(\exists3)$ *are redundant in Definition* 3.1.

Zahiri and Borumand Saeid introduced the variety $\mathbb{MMTL}$ of monadic MTL-algebras and gave some important representations of monadic MTL-algebras in [57].

DEFINITION 3.3 [57]. *An algebraic structure* $(L,\wedge,\vee,\odot,\rightarrow,\forall,\exists,0,1)$ *is said to be a* monadic MTL-algebra *if* $(L,\wedge,\vee,\odot,\rightarrow,0,1)$ *is an MTL-algebra such that* $\forall$ *and* $\exists$ *satisfy the following identities*: $(\forall1)$, $(\exists4)$, *and*

$$(\forall6)\ \forall(x\rightarrow\forall y)=\exists x\rightarrow\forall y,$$
$$(\forall7)\ \forall(\forall x\rightarrow y)=\forall x\rightarrow\forall y,$$
$$(\forall8)\ \forall(\exists x\vee y)=\exists x\vee\forall y.$$

The variety of monadic MTL-algebras is denoted by $\mathbb{MMTL}$.

The notion of monadic residuated lattices is a natural generalization of that with respect to monadic MTL-algebras.

THEOREM 3.4. *Let L be an MTL-algebra and $\forall: L\rightarrow L$ and $\exists: L\rightarrow L$ be two unary operations on L. Then the sets*

$$G=\{(\forall1),(\forall2),(\forall3),(\forall4),(\exists2),(\exists4)\},$$

$$W=\{(\forall1),(\forall6),(\forall7),(\forall8),(\exists4)\}$$

are equivalent.

Proof. G⇒ W:
$(\forall6)$ By $(\exists2)$ and $(\forall3)$, we have

$$\forall(\forall x\rightarrow y)=\forall(\exists\forall x\rightarrow y)=\exists\forall x\rightarrow\forall y=\forall x\rightarrow\forall y.$$

$(\forall7)$ By $(\exists2)$ and $(\forall2)$, we have

$$\forall(x\rightarrow\forall y)=\forall(x\rightarrow\exists\forall y)=\exists x\rightarrow\exists\forall y=\exists x\rightarrow\exists y.$$

$(\forall8)$ By $(\forall4)$ and $(\exists2)$, we have

$$\forall(x\vee\forall y)=\forall(x\vee\exists\forall y)=\forall x\vee\exists\forall y=\forall x\vee\forall y,$$

and by the arbitrariness of x,y, we also get

$$\forall(\forall x\vee y)=\forall(y\vee\forall x)=\forall y\vee\forall x=\forall x\vee\forall y.$$

Also, by $(\forall4)$ and $(\exists2)$, we have

$$\forall(\exists x\vee y)=\forall(\forall\exists x\vee y)=\forall\exists x\vee\forall y=\exists x\vee\forall y.$$

W⇒ G:
The proofs of $(\forall2)$, $(\forall3)$, $(\forall4)$, and $(\exists2)$ can be directly obtained from Lemma 3.1 (5), (15), (18), and (6), respectively, in [57]. □

Theorem 3.4 actually gives us an idea of introducing the notion of monadic MTL-algebra in a different way, which implies the following result immediately.

THEOREM 3.5. *The subvariety of* $\mathbb{MRL}$ *determined by the equation*

$$\text{(PRL)} \quad (x \to y) \vee (y \to x) = 1$$

is term-equivalent to the variety $\mathbb{MMTL}$.

Several academics introduced three kinds of monadic BL-algebras and tried to provide a right algebraic semantics for Hájek's monadic fuzzy logic **mBL**∀ [4, 13, 25]. Here we will review Draŭgulici's, Grigolia's and Castaño's axioms of monadic BL-algebras and discuss the relationship among them.

DEFINITION 3.6 ((Draŭgulici's axioms) [13]). *A* monadic BL-algebra *is a triple* $(L, \forall, \exists)$, *where* L *is a BL-algebra,* $\forall : L \to L$ *and* $\exists : L \to L$ *are two unary operations on* L *satisfying the identities*: $(\forall 1)$, $(\exists 1)$, $(\forall 4)$, $(\forall 6)$, $(\forall 7)$ *and*

$$(\forall 9)\ \forall(\forall x \odot \forall y) = \forall x \odot \forall y,$$

$$(\forall 10)\ \forall 1 = 1.$$

REMARK 3.7. (1) $(\forall 9)$, $(\exists 1)$ *and* $(\forall 10)$ *are redundant in Draŭgulici's axioms.*
(2) *The other one was introduced by Grigolia* [25] *as a triple* $(L, \forall, \exists)$ *that satisfies the identities*: $(\forall 1)$, $(\exists 1)$, $(\forall 2)$, $(\forall 3)$ *and* $(\forall 4)$. *It is also easily verified that*

$$\text{Grigolia's axioms} = \text{Drăgulici's axioms} + (\text{E2}).$$

It is worth noticing in [26] that the formula

$$(\exists x)(\alpha(x) \& \beta(x))) \equiv (\exists x)\alpha(x) \& (\exists x)\beta(x)$$

is a theorem in **BL**∀, and also belongs to **mBL**∀. However Example 3.8 shows that the last formula is independent of the axioms of Definition 3.6 and Remark 3.7(2).

EXAMPLE 3.8. *Let* $L = [0, 1]$. *Define binary operations* $\wedge, \vee, \odot, \to$ *by*

$$x \wedge y = \min\{x, y\}, \quad x \vee y = \max\{x, y\}, \quad \neg x = 1 - x,$$

$$x \odot y = \begin{cases} 0, & x \leq \neg y, \\ x \wedge y, & x > \neg y, \end{cases} \qquad x \to y = \begin{cases} 1, & x \leq y, \\ \neg x \vee y, & x > y. \end{cases}$$

Then $(L, \wedge, \vee, \to, \odot, 0, 1)$ *is a BL-algebra, and is called the standard MV-algebra. If we define two unary operations* $\forall : L \to L$ *and* $\exists : L \to L$ *by*

$$\forall x = \begin{cases} 1, & x = 1, \\ 0, & x \neq 1, \end{cases} \qquad \exists x = \begin{cases} 0, & x = 0, \\ 1, & x \neq 0, \end{cases}$$

then $\forall$ *and* $\exists$ *satisfy the axioms of Definition* 3.6 *and Remark* 3.7(2). *However, the unary operation* $\exists$ *does not satisfy the above formula, since*

$$\exists(\frac{1}{2} \odot \frac{1}{2}) = \exists 0 = 0 \neq 1 = \exists\frac{1}{2} \odot \exists\frac{1}{2}.$$

In order to solve the above drawback, Castaño et al. revised the variety $\mathbb{MBL}$ of monadic BL-algebras and proved that are the equivalent algebraic semantics of the monadic fuzzy logic **mBL**∀ [4].

Definition 3.9 [4]. *A* monadic BL-algebra *is a triple* $(L, \forall, \exists)$, *where* L *is a BL-algebra,* $\forall : L \to L$ *and* $\exists : L \to L$ *are two unary operations on* L *that satisfy* $(\forall 1), (\forall 6), (\forall 7), (\forall 8)$ *and* $(\exists 4)$.

Theorem 3.10. *The subvariety of* $\mathbb{MMTL}$ *determined by the equation*

$$\text{(DIV)} \;\; x \wedge y = x \odot (x \to y)$$

is term-equivalent to the variety $\mathbb{MBL}$.

Proof. Definitions 3.3 and 3.9 show that monadic BL-algebra and monadic MTL-algebra have the same axioms with respect to two quantifiers $\forall$ and $\exists$. □

Subsequently, in order to review the variety of monadic algebras of involutive monoidal t-norm based logic and their subvarieties, we study the variety $\mathbb{MIRL}$ of monadic involutive residuated lattices in advance.

Definition 3.11. *A* monadic involutive residuated lattice *is a pair* $(L, \forall)$, *where* L *is an involutive residuated lattice and* $\forall : L \to L$ *is an unary operation on* L *satisfying the following conditions*: $(\forall 1), (\forall 7)$ *and*

$$(\forall 11) \;\; \forall(x \vee \forall y) = \forall x \vee \forall y,$$

$$(\forall 12) \;\; \forall(x \to \neg x) = \forall x \to \neg \forall x.$$

Theorem 3.12. *The subvariety of* $\mathbb{MRL}$ *determined by the equation*

$$\text{(INV)} \;\; \neg\neg x = x$$

is term-equivalent to the variety $\mathbb{MIRL}$, *where* $\exists = \neg\forall\neg$.

Proof. Let $(L, \forall, \exists)$ be a monadic residuated lattice that satisfies $\neg\neg x = x$. Then L becomes an involutive residuated lattice. Here we will show that $(L, \forall)$ is a monadic involutive residuated lattice.

$(\forall 7)$: The proof is similar to that of G$\Rightarrow$ W of Theorem 3.4.
$(\forall 11)$: By $(\exists 2)$ and $(\forall 4)$, we have

$$\forall(\forall x \vee y) = \forall(\exists\forall x \vee y) = \exists\forall x \vee \forall y = \forall x \vee \forall y.$$

$(\forall 12)$: By $(\exists 4)$ and $\neg\exists\neg x = \forall x$, we have

$$\forall(\neg x \to x) = \neg\exists\neg(\neg x \to x) = \neg\exists(\neg x \odot \neg x) = \neg(\exists\neg x \odot \exists\neg x) = \neg\forall x \to \forall x.$$

Conversely, let $(L, \forall)$ be a monadic involutive residuated lattice. Here we show that $(L, \forall, \exists)$ is a monadic residuated lattice, where

$$\exists x := \neg\forall\neg x$$

for any $x \in L$. Actually, we only need to show that $(L, \forall, \exists)$ satisfies the axioms of W by Theorem 3.4, which is indeed equivalent to the axioms of monadic residuated lattice defined in Definition 3.1.
$(\forall 6)$: By Propositions 3.9(5), we have

$$\forall(x \to \forall y) = \forall(\neg\neg x \to \forall y) = \neg\forall\neg x \to \forall y = \exists x \to \forall y.$$

$(\forall 7)$: By Propositions 3.9(5), we have

$$\forall(\forall x \to y) = \forall(\exists\forall x \to y) = \exists\forall x \to \forall y = \forall x \to \forall y.$$

(∀8): By (M11) and Proposition 3.7(6), we have

$$\forall(\exists x \vee y) = \forall(\forall\exists x \vee y) = \forall\exists x \vee \forall y = \exists x \vee \forall y.$$

(∃4) Applying (∀12) and $\neg\forall\neg x = \exists x$, we have

$$\begin{aligned}\forall(\neg x \to x) &= \neg\forall x \to \forall x\\ &\Leftrightarrow \neg\exists\neg(\neg x \to x) = \exists\neg x \to \neg\exists\neg x\\ &\Leftrightarrow \exists(\neg x \odot \neg x) = \neg(\exists\neg x \to \neg\exists\neg x)\\ &\Leftrightarrow \exists(\neg x \odot \neg x) = (\exists\neg x \odot \exists\neg x)\\ &\Leftrightarrow \exists(x \odot x) = \exists x \odot \exists x.\end{aligned}$$

Therefore, $(L, \forall, \exists)$ is a monadic residuated lattice.

Inspired by Theorem 3.12, we have the corresponding result of monadic IMTL-algebras and monadic MTL-algebras. □

THEOREM 3.13. *The subvariety of* $\mathbb{MMTL}$ *determined by the equation*

$$\text{(INV)} \quad \neg\neg x = x$$

is term-equivalent to the variety $\mathbb{MIMTL}$, *where* $\exists = \neg\forall\neg$.

Inspired by the above observation, we introduced the notion of monadic NM-algebras and proved that is the equivalent algebraic semantics of **mNM**∀ in [49].

DEFINITION 3.14 [49]. *A* monadic NM-algebra *is a pair* $(L, \forall)$, *where* L *is an NM-algebra and* $\forall : L \to L$ *is an unary operation on* L *satisfying the following conditions*: (∀1), (∀7), (∀11) *and* (∀12).

THEOREM 3.15. *The subvariety of* $\mathbb{MIMTL}$ *determined by the equation*

$$\text{(WNM)} \quad (x \odot y \to 0) \vee (x \wedge y \to x \odot y) = 1$$

is term-equivalent to the variety $\mathbb{MNM}$.

Proof. The proof is trivial and hence we omit them. □

The variety $\mathbb{MMV}$ of monadic MV-algebras were introduced by Rutledge [45] as an algebra $(L, \oplus, \odot, ^*, \forall, 0, 1)$ satisfies the following identities: (∀1), (∀9)

$$\text{(∀13)} \quad \forall(x \wedge y) = \forall x \wedge \forall y,$$

$$\text{(∀14)} \quad \forall\neg\forall x = \neg\forall x,$$

$$\text{(∀15)} \quad \forall(x \odot x) = \forall x \odot \forall x,$$

$$\text{(∀16)} \quad \forall(x \oplus x) = \forall x \oplus \forall x.$$

REMARK 3.16. *J. Rach ů nek and F. Švrček were first attempt to define unary operators with some properties of quantifiers for residuated* ℓ*-monoids using only the universal quantifier as the initial one, the resulting class of algebras called monadic residuated* ℓ*-monoids* [44]. *The variety of monadic residuated* ℓ*-monoids is denoted by* $\mathbb{MRL}\ell$, *which is analogously as for the variety* $\mathbb{MMV}$ *of monadic MV-algebras. But it seems to be more appropriate to introduce such monadic algebras similarly as the monadic residuated lattices, MTL-algebras and BL-algebras. The reason is that MV-algebras satisfy De Morgan and double negation laws, and thus in the definition of the corresponding monadic* *algebras, it is possible to use only one of the existential and universal quantifiers as primitive, the other being definable as the dual of the one defined, readers can refer to Proposition* 4 *in* [42] *for details. Namely, if* $\exists$ *is an existential quantifier and* $\forall$ *is a universal one on an* MV*-algebra* L*, then* $\forall_\exists : L \to L$ *and* $\exists_\forall : L \to L$ *such that for any* $x \in L$,

$$\forall_\exists x = \neg(\exists\neg x) \ \textit{ and } \ \exists_\forall x = \neg(\forall\neg x),$$

is a universal and an existential quantifier on L*, respectively, and moreover*

$$\exists_{(\forall_\exists)} = \exists \ \textit{ and } \ \forall_{(\exists_\forall)} = \forall.$$

However, definitions of monadic BL-algebras, residuated lattices and residuated ℓ*-monoids require the introduction of both kinds of quantifiers simultaneously, because these quantifiers are not mutually interdefinable. As a consequence of Theorems* 3.4 *and* 3.10*, monadic BL-algebra,* MTL*-algebra, residuated* ℓ*-monoid and residuated lattice have the same axioms, that is, a monadic residuated* ℓ*-monoid is indeed a triple* $(L, \forall, \exists)$ *satisfying the identities*: $(\forall 1), (\forall 6), (\forall 7), (\forall 8)$ *and* $(\exists 4)$.

Figallo Orellano also gave a characterization of monadic MV-algebras in [19].

THEOREM 3.17 [19]. *Let* L *be an* MV*-algebra and* $\forall$ *a unary operation on* L*. Then* $(L, \forall)$ *is a monadic* MV*-algebra if and only if it satisfies* $(\forall 1), (\forall 7), (\forall 12)$.

THEOREM 3.18. *The subvariety of* $\mathbb{MMTL}$ *determined by the equation*

$$(\mathrm{MV}) \ \ (x \to y) \to y = y \to (y \to x)$$

is term-equivalent to the variety $\mathbb{MMV}$.

Proof. Let $(L, \forall, \exists)$ be a monadic MTL-algebra satisfying the condition (MV). Then L becomes an MV-algebra, and by Definition 3.14 and Theorem 3.17, $(L, \forall)$ is a monadic MV-algebra.

Conversely, let $(L, \forall)$ be a monadic MV-algebra. Here we show that $(L, \forall, \exists)$ is a monadic MTL-algebra, where

$$\exists x := \neg\forall\neg x$$

for any $x \in L$. It is noted first that $(\forall 11)$ hold. Indeed, for any $x, y \in L$, we have

$$\forall(x \vee \forall y) = \forall((x \to \forall y) \to \forall y) \leq \forall(x \to \forall y) \to \forall y \leq (\forall x \to \forall y) \to \forall y = \forall x \vee \forall y,$$

which implies $\forall(x \vee \forall y) \leq \forall x \vee \forall y$. The reverse inequality holds trivially.

The rest of proof can be deduced directly form Theorem 3.13. □

As a consequence of Theorem 3.18, the following results are obtained.

COROLLARY 3.19. *The subvariety of* $\mathbb{MIMTL}$ *determined by the equation*

$$(\mathrm{DIV}) \ \ x \wedge y = x \odot (x \to y)$$

is term-equivalent to the variety $\mathbb{MMV}$.

COROLLARY 3.20. *The subvariety of* $\mathbb{MBL}$ *determined by the equation*

$$(\mathrm{INV}) \ \ \neg\neg x = x$$

is term-equivalent to the variety $\mathbb{MMV}$.

Monadic Boolean algebras were introduced by Halmos [32], as algebraic models for classical predicate logics for languages having one placed predicates and a single quantifier. More precisely, an algebra $(L, \wedge, \vee, \neg, 0, 1, \exists)$ is said to be *a monadic Boolean algebra* if $(L, \wedge, \vee, \neg, 0, 1)$ is a Boolean algebra and in addition $\exists$ satisfies the identities: $(\exists 1)$,

$$(\exists 5)\ \ \exists 0 = 0,$$

$$(\exists 6)\ \ \exists(\exists x \wedge y) = \exists x \wedge \exists y.$$

The variety of monadic Boolean algebras is denoted by $\mathbb{MBA}$.

THEOREM 3.21. *The subvariety of* $\mathbb{MRL}$ *determined by adding the equation*

$$(\mathrm{EM})\ \ \neg x \vee x = 1$$

is term-equivalent to the variety $\mathbb{MBA}$.

Proof. Let $(L, \forall, \exists)$ be a monadic residuated lattice satisfying the condition (EM). Then L becomes a Boolean algebra.

$(\exists 5)$ The proof is similar to that of Proposition 3.1(9) in [43].
$(\exists 6)$ Applying $(\forall 11)$, for any $x, y \in L$, we have

$$\begin{aligned}\forall(x \vee \forall y) &= \forall x \vee \forall y \\ &\Leftrightarrow \neg\exists\neg(x \vee \neg\exists\neg y) = \neg\exists\neg x \vee \neg\exists\neg y \\ &\Leftrightarrow \neg\exists(\neg x \wedge \exists\neg y) = \neg(\exists\neg x \wedge \exists\neg y) \\ &\Leftrightarrow \exists(\neg x \wedge \exists\neg y) = \exists\neg x \wedge \exists\neg y \\ &\Leftrightarrow \exists(x \wedge \exists y) = \exists x \wedge \exists y,\end{aligned}$$

and by the arbitrariness of x, y, we also get $\exists(\exists x \wedge y) = \exists x \wedge \exists y$. □

REMARK 3.22. *Figures 2 and 3 show that the relationships between monadic algebras of* t*-norm based residuated fuzzy logic completely inherit those between corresponding*

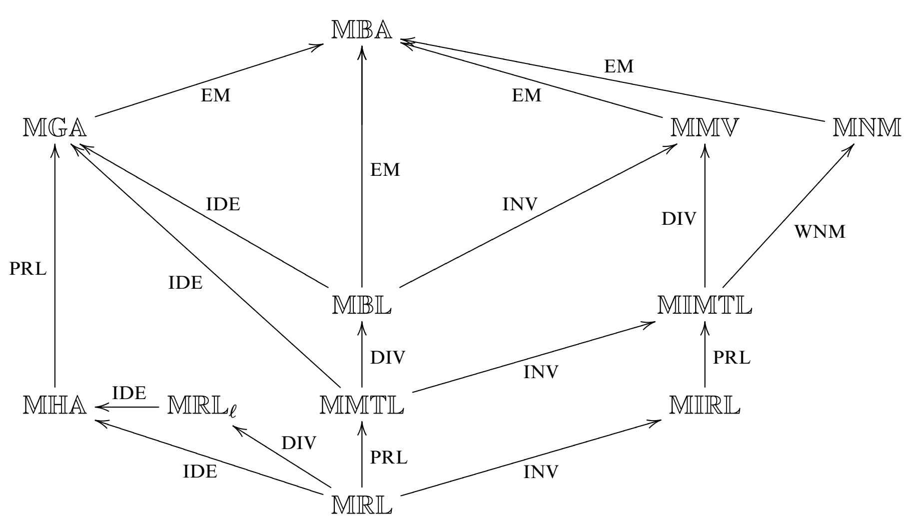

Figure 3. Relationships between monadic algebras of *t*-norm based residuated fuzzy logic.

algebras of t*-norm based residuated fuzzy logic, which solve the drawback in the paragraph 2 in the Introduction.*

§4. Completeness for monadic fuzzy predicate logic mMTL∀. In this section, the main aim of us to show that the variety of monadic MTL-algebras is the equivalent algebraic semantics of monadic fuzzy predicate logic **mMTL**∀, and give an algebraic proof of completeness for this logic.

Along with the same line of Hájek's excellent work in [30], similarly, we can prove that formulas of **S5**(**MTL**) are in the obvious one–one isomorphic correspondence with formulas of monadic fuzzy predicate calculus **mMTL**∀ with unary predicates and just one object variable x, the atomic formula $P_i(x)$ corresponding to propositional variable p_i and modalities $\Box$ and $\Diamond$ correspond to the quantifiers $(\forall x)$ and $(\exists x)$. Kripke semantics correspond in the obvious way to semantics for **mMTL**∀, the correspondence maps tautologies of the fuzzy modal logic to tautologies of the monadic fuzzy predicate logic and the same for standard tautologies. Using the above equivalence, we continue the algebraic tradition of naming monadic the algebraic semantics of monadic fragments of several logics (Boolean, intuitionistic, Łukasiewicz, etc.), and opt to call the algebras corresponding to the fuzzy modal logic **S5**(**MTL**) monadic MTL-algebras. However, in this section we will work in the language of the fuzzy modal logic **S5**(**MTL**) instead of in the monadic fuzzy language of **mMTL**∀ and prove that the variety of monadic MTL-algebras is the equivalent algebraic semantics of the logic **S5**(**MTL**) in advance.

Here we first study m-relatively complete subalgebras with respect to monadic MTL-algebras. In particular, we characterize those subalgebras of a given MTL-algebra that may be the range of the quantifiers ∀ and ∃. As a consequence of this characterization, we build the most important examples of monadic MTL-algebras, which will be called functional monadic MTL-algebras.

Given an MTL-algebra L, we say that a subalgebra $S \leq L$ is *m-relatively complete* if the following conditions hold:

(S1) For every $a \in L$, the subset $\{s \in S \mid s \leq a\}$ has a greatest element and $\{s \in S \mid s \geq a\}$ has a least element.

(S2) For every $a \in L$ and $s_1, s_2 \in S$ such that $s_1 \leq s_2 \vee a$, there exists $s_3 \in S$ such that $s_1 \leq s_2 \vee s_3$ and $s_3 \leq a$.

(S3) For every $a \in L$ and $s_1 \in S$ such that $a \odot a \leq s_1$, there exists $s_2 \in S$ such that $a \leq s_2$ and $s_2 \odot s_2 \leq s_1$.

THEOREM 4.1. *Consider an MTL-algebra L and an m-relatively complete subalgebra S. If we define on L the operation*

$$\forall a = \max\{s \in S \mid s \leq a\}, \quad \exists a = \min\{s \in S \mid s \geq a\}$$

then $(L, \forall, \exists)$ *is a monadic MTL-algebra such that* $\forall L = \exists L = S$. *Conversely, if* $(L, \forall, \exists)$ *is a monadic MTL-algebra, then* $\forall L = \exists L$ *is an m-relatively complete subalgebra of L.*

Proof. Clearly condition (S1) guarantees the existence of $\forall x$ and $\exists x$ for any $x \in L$. Then it remains to show that $(L, \forall, \exists)$ satisfies axioms (∀1), (∀6)–(∀8) and (∃4).

(∀1) From the definition of $\forall x$, it is clear that $\forall x \leq x$. Thus $\forall x \to x = 1$.

(∀6) From $x \leq \forall x$, we get $\exists x \to \forall y \leq x \to \forall y$. Then $\exists x \to \forall y \in \{s \in S | s \leq x \to \forall y\}$. Let us see that $\exists x \to \forall y = \max\{s \in S | s \leq x \to \forall y\}$. Indeed, if $s \in S$, and $s \leq$

$x \to \forall y$, then $x \le s \to \forall y$. Then, by definition of $\exists x$, we have $\exists x \le s \to \forall y$. Thus $s \le \exists x \to \forall y$, which implies that $\forall(x \to \forall y) = \exists x \to \forall y$.

($\forall 7$) From $\forall y \le y$, we get $\forall x \to \forall y \le \forall x \to y$, In addition, if $s \in S$ and $s \le \forall x \to y$, then $s \odot \forall x \le y$. Thus $s \odot \forall x \le \forall y$ and $s \le \forall x \to \forall y$. Hence we have shown that $\forall(\forall x \to y) = \forall x \to \forall y$.

($\forall 8$) Since $\forall y \le y, \exists x \vee \forall y \le \exists x \vee y$. Now, if $s \in S$ and $s \le \exists x \vee y$, by condition (S2), there must be $s' \in S$ such that $s \le \exists x \vee s'$ and $s' \le y$. Then $s' \le \forall y$ and $s \le \exists x \vee \forall y$. Thus, we have shown that $\forall(\exists x \vee y) = \exists x \vee \forall y$.

($\exists 4$) Clearly, $x \odot x \le \exists x \odot \exists x$. In addition, if $s \in S$ and $x \odot x \le s$, by condition (S3), there is $s' \in S$ such that $s' \odot s' \le s$ and $x \le s'$. Then $\exists x \le s'$ and $\exists x \odot \exists x \le s' \odot s' \le s$. Thus $\exists(x \odot x) = \exists x \odot \exists x$.

Conversely, let $(L, \forall, \exists)$ be a monadic MTL-algebra. Then we have that $\forall L$ is a subalgebra of L. Now, let us show that conditions (S1)–(S3) hold.

(S1) Using the basic properties of monadic MTL-algebra, we have that if $s \le x$ for some $s \in \forall L$, then $s = \forall s \le \forall x \le x$. Thus $\forall x = \max\{s \in \forall L | s \le x\}$. Analogously $\exists x = \min\{s \in \forall L | s \ge x\}$.

(S2) Assume $s_1 \le s_2 \vee x$ for some $s_1, s_2 \in \forall L$, $x \in L$. Then, using the basic properties of monadic MTL-algebras, we have $s_1 \le s_2 \vee \forall x$ and $\forall x \le x$.

(S3) If $x \odot x \le s$ for some $s \in \forall L$ and $x \in L$, then $\exists x \odot \exists x \le s$ and $x \le \exists x$, $\exists x \in \forall L$. This shows that $\forall L$ is an m-relatively complete subalgebra of L. □

The following is the most important example of monadic MTL-algebras built according to the previous theorem.

EXAMPLE 4.2. *Considering a linearly ordered MTL-algebra L and a nonempty set X, we restrict our attention to those elements $f \in L^X$ such that*

$$\inf\{f(x) \mid x \in X\}, \quad \sup\{f(x) | x \in X\}$$

both exist in L. We denote by M the subset of L^X of these safe elements. For every $f \in M$, we define

$$\forall_f(x) = \inf\{f(y) \mid y \in X\}, \ \exists_f(x) = \sup\{f(y) \mid y \in X\}, \textit{ for } x \in L$$

and $\forall_f$ and $\exists_f$ are both constant maps.

REMARK 4.3. *Let G be a subalgebra of L^X contained in M such that for every* f, *$\forall_f, \exists_f \in G$. We will prove that G has a natural structure of monadic MTL-algebra. Let S be the subset of constant maps of L^X. Then we prove that $G \cap S$ is an m-relatively complete subalgebra of G. Indeed, since G and S are subalgebra of L^X, it is clear that $G \cap S$ is a subalgebra of G.*

(S1) *If $f \in G$, then $\forall_f \in G$, so $\max\{s \in G \cap S | s \le f\} = \forall_f \in G$. Analogously, $\min\{s \in G \cap S | s \ge f\} = \exists_f \in G$.*

(S2) *Since $G \cap S$ is totally ordered, we may check condition (s_2). If $1 = s \vee f$ for some $f \in G$ and $s \in G \cap S$. Putting $s(x) = s_0 \in L, x \in X$. Then $s_0 \vee f(x) = 1$ for every $x \in X$. As L is linearly ordered, either $s_0 = 1$ or $f(x) = 1$ for every $x \in X$.*

(S3) *If $f \odot f \le c$ for some $f \in G$ and $c \in G \cap S$, then $f(x) \odot f(x) \le c_0$ for any $x \in X$. Moreover, $f(x) \odot y \le c_0$ for any $x, y \in X$, since*

$$f(x) \odot f(y) \le (f(x) \vee f(y))^2 = f(x)^2 \vee f(y)^2 \le c_0.$$

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Hence $f(x) \leq f(y) \rightarrow c_0$ *for a fixed* $y \in X$ *and any* $x \in X$. *Thus* $\exists_f(x) \leq f(y) \rightarrow c_0$, *where* $\exists_f = \sup\{f(y)|y \in X\}$. *Now,* $f(y) \leq \exists_f(x) \rightarrow c_0$ *for any* $y \in Y$. *Then* $\exists_f(x) \leq \exists_f(x) \rightarrow c_0$ *and* $\exists_f(x) \odot \exists_f(x) \leq c_0$, *which concludes the proof that* $G \cap S$ *is an m-relatively complete subalgebra of* G.

Then it follows from Theorem 4.1 that $(G, \forall_f, \exists_f)$ *is a monadic MTL-algebra. Monadic MTL-algebras of this form are called* functional monadic MTL-algebras. *Also, if* L *is* $|X|$*-complete, then* $S = L^X$ *and* $(L^X, \forall_f, \exists_f)$ *is a functional monadic MTL-algebra.*

Here we show that the variety of monadic MTL-algebras is the equivalent algebraic semantics of monadic fuzzy predicate logic **mMTL**∀, which will be used to prove the main result of this section.

DEFINITION 4.4. *A* Kripke model *for* **S5**(**MTL**) *is a triple* $K = (X, e, L)$ *where* X *is a nonempty set of worlds,* L *is a linearly ordered MTL-algebra and* $e : Prop \times X \rightarrow L$ *is an evaluation map. The evaluation map extends to any formula*:

(e_1) $e(0, x) = 0$, $e(1, x) = 1$,
(e_2) $e(\varphi \sqcap \psi, x) = e(\varphi, x) \wedge e(\psi, x)$,
(e_3) $e(\varphi \sqcup \psi, x) = e(\varphi, x) \vee e(\psi, x)$,
(e_4) $e(\varphi \& \psi, x) = e(\varphi, x) \odot e(\psi, x)$,
(e_5) $e(\varphi \Rightarrow \psi, x) = e(\varphi, x) \rightarrow e(\psi, x)$,
(e_6) $e(\Box\varphi, x) = \inf\{e(\varphi, y) : y \in X\}$,
(e_7) $e(\Diamond\varphi, x) = \sup\{e(\varphi, y) : y \in X\}$.

We can also define truth degree $\|\varphi\|_{K,\omega}$ *of a formula* φ *in* K *at the world* ω, *which is due done recursively on the structure of* φ. *For propositional variables* $p \in Prop$, *we have that* $\|p\|_{K,\omega} = e(\omega, p)$. *The definition of the truth value is then extended for the logical connectives of monoidal t-norm based logic in the usual way, and for the modal connectives by*

$$\|\Box\varphi\| = \inf_{w \in W} \|\varphi\|_{K,w},$$

$$\|\Diamond\alpha\| = \sup_{w \in W} \|\varphi\|_{K,w},$$

the infima and suprema above may not exist in general; hence, we restrict our attention only to safe structures, that is, structure K *for which* $\|\varphi\|_{K,\omega}$ *is defined for every formula* φ *at every world* ω. *However, we are only interested in the tautologies of this logic, but here considered albeit implicitly, a global consequence relation* $\models_{\mathbf{S5(MTL)}}$. *Given a set of formulas* Γ *we say that a safe structure* $K = (X, e, L)$ *is a model of* Γ *if for every* $\varphi \in \Gamma$ *and every* $\omega \in W$ *we have that* $\|\varphi\|_{K,\omega} = 1$. *Thus, given a set of formulas* Γ *and a formula* φ *we write* $\Gamma \models_{\mathbf{S5(MTL)}} \varphi$ *if and only if for every safe model* $K = (X, e, L)$ *of* Γ *we have that* $\|\varphi\|_{K,\omega} = 1$ *for every* $\omega \in W$.

We already noted that **S5**(**MTL**) *is actually equivalent to* **mMTL**∀ *because there is a natural correspondence between formulas of both logics, between corresponding models and between the corresponding truth degrees. Thus, since the latter is finitary (being a fragment of a finitary logic), the consequence relation* **S5**(**MTL**) *is also finitary.*

THEOREM 4.5. *The fuzzy modal logic* **S5**(**MTL**) *is strongly complete with respect to its general semantics, that is, the following statements are equivalent for every set of formulas* $\Gamma \cup \{\varphi\}$:

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(1) $\Gamma \vdash \varphi$,
(2) $K \models \varphi$ *for every safe model* K *of* Γ.

Proof. The proof is similar to that of Theorem 2 in [30]. □

Considering a safe Kripke model $K = (X, e, L)$, we can turn the map $e : Prop \times X \to L$ into a map $\overline{e} : Prop \to L^X$ given by the relation $\overline{e}(p)(x) = e(p, x)$. Since K is safe, $\overline{e}$ extends to formulas in the following way:

(e_1) $\overline{e}(x) = 0, \overline{e}(1) = 1$,
(e_2) $\overline{e}(\varphi \sqcap \psi) = \overline{e}(\varphi) \wedge \overline{e}(\psi)$,
(e_3) $\overline{e}(\varphi \sqcup \psi) = \overline{e}(\varphi) \vee \overline{e}(\psi)$,
(e_4) $\overline{e}(\varphi \& \psi) = \overline{e}(\varphi) \odot \overline{e}(\psi)$,
(e_5) $\overline{e}(\varphi \Rightarrow \psi) = \overline{e}(\varphi) \to \overline{e}(\psi)$,
(e_6) $\overline{e}(\Box\varphi) = \forall_f \overline{e}(\varphi)$,
(e_7) $\overline{e}(\Diamond\varphi) = \exists_f \overline{e}(\varphi)$.

Thus it is clear that $\{\overline{e}(\varphi) : \varphi \text{ formula}\} \subseteq L^X$ is the universe of a functional monadic MTL-algebra in Example 4.2.

Then we have the following result similar to Theorem 4.5.

THEOREM 4.6. *The following statements are equivalent for every set of formulas* $\Gamma \cup \{\varphi\}$:

(1) $\Gamma \vdash \varphi$,
(2) $\overline{e}(\varphi) = 1$ *for every* $\overline{e} : Prop \to G$, *where* $(G, \forall_f, \exists_f)$ *is any functional monadic MTL-algebra and* $\overline{e}(\gamma) = 1$ *for every* $\gamma \in \Gamma$.

THEOREM 4.7. *The variety* $\mathbb{MMTL}$ *is the equivalent algebraic semantics for the logic* **S5**(**MTL**).

Proof. It is enough to show that the next two conditions hold for set for formulas $\Gamma \cup \{\varphi, \phi\}$:

(1) $\Gamma \vdash \varphi$ if and only if $\{\psi \equiv 1 | \psi \in \Gamma\} \vdash_{\mathbb{MMTL}} \varphi \equiv 1$,
(2) $\varphi \equiv \phi \models_{\mathbb{MMTL}} (\varphi \Rightarrow \phi) \sqcap (\phi \Rightarrow \varphi) \equiv 1$.

Condition (2) is trivially verified. We only show the condition (1). For the forward implication, if $\Gamma \vdash \varphi$, there exists a proof of φ from Γ and the axioms of **S5**(**MTL**) by successive application of the reference rules **MP** and **Nec**. Thus, it is enough to show that the equation $\varphi \equiv 1$ is valid in $\mathbb{MMTL}$ for every axiom φ of **S5**(**MTL**) and that the inference rules preserve validity. The former statement follows from the basic properties of monadic MTL-algebras. The preservation of **MP** and **Nec** are also easily verified. For the converse implication, simply observe that, since $(G, \forall_f, \exists_f)$ is a monadic MTL-algebra, condition (2) of Theorem 4.6 holds. □

Thus, from the general theory of Algebraic Logic, we get the next result.

COROLLARY 4.8. *There is a one–one correspondence between axiomatic extensions of* **S5**(**MTL**) *and subvarieties of* $\mathbb{MMTL}$.

Zahiri and Borumand Saeid give a Hilbert-style syntactic calculus in the language of **MMTL**, which is indeed equivalent to **S5**(**MTL**) whose consequence we denote by $\vdash_{\mathbf{S5(MTL)}}$. The axioms of this calculus are the instantiations of the axioms schemata **MTL** for formulas in the language of **S5**(**MTL**), plus the following axioms:

($\lozenge$1) $\varphi \Rightarrow \lozenge\varphi$,
($\Box$2) $\Box(\nu \Rightarrow \varphi) \Rightarrow (\nu \Rightarrow \Box\varphi)$,
($\lozenge$2) $\Box(\varphi \Rightarrow \nu) \Rightarrow (\lozenge\varphi \Rightarrow \nu)$,
($\Box$3) $\Box(\nu \sqcup \varphi) \Rightarrow (\nu \sqcup \Box\varphi)$,
($\lozenge$3) $\lozenge(\varphi\&\varphi) \equiv \lozenge\varphi\&\lozenge\varphi$,

where φ is any formula and ν is any propositional combination of formulas beginning with $\Box$ or $\lozenge$. The inference rules are **MP**: $\varphi, \varphi \Rightarrow \phi \vdash \phi$ and **Nec** : $\varphi/\Box\varphi$.

Here we achieve the main aim of this section, namely, giving an algebraic proof of the completeness theorem for monadic fuzzy predicate logic **mMTL**$\forall$.

THEOREM 4.9. *The following statements are equivalent for every set of formulas* $\Gamma \cup \{\varphi\}$:

(1) **S5**(**MTL**) *is complete, i.e., the following two statements are equivalent*:
  (i) $\Gamma \vdash_{\mathbf{S5(MTL)}} \varphi$,
  (ii) $\Gamma \models_{\mathbf{S5(MTL)}} \varphi$.
(2) *The variety* $\mathbb{MMTL}$ *is generated by the functional monadic MTL-algebras.*

Proof. We prove first that (2) implies (1). The soundness and implication is straightforward and does not depend on the hypothesis. Assume $\Gamma \models_{\mathbf{S5(MTL)}} \varphi$ and since $\varphi \models_{\mathbf{S5(MTL)}}$ is finitary, we may also assume that Γ is finite. By way of contradiction, suppose $\Gamma \nvdash_{\mathbf{S5(MTL)}} \varphi$. Then $\Gamma \nvDash_{\mathbf{S5(MTL)}} \varphi$ (this is the consequence relation associated with the variety of $\mathbb{MMTL}$), which is equivalent to saying that $\mathbb{MMTL}$ does not satisfy the quasi-equation

$$\gamma_1 \approx \& \cdots \&\gamma_n \approx 1 \Rightarrow \varphi \approx 1,$$

where $\{\gamma_1 \dots \gamma_n\} = \Gamma$. By hypothesis, there is a functional algebra $(M, \forall_f, \exists_f) \in \mathbb{MMTL}$ with $M \leq L^X$ and L is an MTL-algebra, and a valuation h into $(M, \forall_f, \exists_f)$ such that $h(\Gamma) \subseteq \{1\}$ but $h(\varphi) \neq 1$. Consider the structure $K = (X, e, L)$ with $e(x, p) = h(p)(x)$ for $x \in X$ and $p \in Prop$. For every $\gamma \in \Gamma$ and $x \in X$ we have that $\|\gamma\|_K, x = h(\gamma)(x) = 1$, but $\|\gamma\|_K, x = h(\gamma)(x) \neq 1$ for some $x \in X$, which is a contradiction.

Conversely, we now prove that (1) implies (2). Assume $\mathbb{MMTL}$ is not generated by its functional members. Thus, there is an identity

$$\gamma_1 \approx 1\& \cdots \&\gamma_n \approx 1 \Rightarrow \varphi \approx 1$$

that is on every functional monadic MTL-algebra, but is not true on the variety $\mathbb{MTL}$. From the properties of algebraizable logics, we get that $\{\gamma_1 \dots \gamma_n\} \nvdash_{\mathbf{S5(MTL)}} \varphi$. However, we claim that $\{\gamma_1 \dots \gamma_n\} \models_{\mathbf{S5(MTL)}} \varphi$. Indeed, assume $K = (X, e, L)$ is a model of $\{\gamma_1 \dots \gamma_n\}$, where L is a linearly ordered MTL-algebra. Then $\|\gamma_i\|_K, x = h(\gamma_i)(x) = 1$ for any $x \in X$ and $1 \leq i \leq n$. Let Fm be the set of propositional formulas in the language of **S5**(**MTL**). For each $\psi \in Fm$, we define $f_\psi : X \to L$ such that $f_\psi(x) = \|\psi\|_K, x$ for every $x \in X$. Consider $B = \{f_\psi | \psi \in Fm\}$. Then $B \subseteq L^X$ and, in addition, B is a subuniverse of the MTL-algebra L^X. Moreover, it is straightforward to check that $\forall_f \psi = f\Box\psi, \exists_f \psi = f\lozenge\psi$. Then $(L, \forall_f, \exists_f)$ is a functional monadic MTL-algebra. Consider now the interpretation $e' : Fm \to L$ given by $e'(\psi) = f_\psi$ for any $\psi \in Fm$. Then $e'(\gamma_i) = 1$ for $1 \leq i \leq n$. By hypothesis, $e'(\varphi) = 1$, that is, $f_\varphi = 1$, so $\|\varphi\|_K, x = 1$ for any $x \in X$. This completes the proof that $\{\gamma_1 \dots \gamma_n\} \vdash_{\mathbf{S5(MTL)}} \varphi$. $\Box$

§5. Strong completeness for monadic fuzzy predicate logic mIMTL$\forall$. In this section, we obtain the completeness theorem of some axiomatic extensions for monadic

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fuzzy predicate logic **mMTL**$\forall$, and then give a major application, proving the strong completeness theorem for monadic fuzzy predicate logic **mIMTL**$\forall$.

Along with the same line of Section 4, we naturally extend some basic notions, including Kripke model and its related concepts, from the fuzzy modal logic $\mathbf{S5}(\mathbf{MTL})$ to a more general one, $\mathbf{S5}(\mathbf{C})$ which is indeed an S5-modal expansion of an axiomatic extension $\mathbf{C}$ of the logic **MTL**, and denote a global consequence relation by $\models_{\mathbf{S5}(\mathbf{C})}$.

Here we also introduce a Hilbert-style syntactic calculus in the language of $\mathbf{S5}(\mathbf{C})$ whose consequence relation is denoted by $\vdash_{\mathbf{S5}(\mathbf{C})}$. The axiom of this calculus are the instantiations of the axiom schemata of $\mathbf{C}$ for formulas in the language of $\mathbf{S5}(\mathbf{C})$, plusing the axioms: ($\Diamond$1), ($\Box$2), ($\Diamond$2), ($\Box$3) and ($\Diamond$3). Moreover, Theorem 4.7 shows that the variety $\mathbb{MMTL}$ is the equivalent algebraic semantics of the logic $\mathbf{S5}(\mathbf{MTL})$. As a consequence of the general theory of algebraizability all axiomatic extensions of $\mathbf{S5}(\mathbf{MTL})$ are algebraizable logics and their equivalent algebraic semantics are precisely the subvarieties of $\mathbb{MMTL}$, see Corollary 4.8. In particular, given any axiomatic extension $\mathbf{C}$ of **MTL**, the logic $\mathbf{S5}(\mathbf{C})$ is an axiomatic extension of $\mathbf{S5}(\mathbf{MTL})$ with an equivalent algebraic semantics $\mathrm{Alg}^*\mathbf{S5}(\mathbf{C})$, denote also by $\mathbb{MMTL}_{\mathbb{C}}$. It is also worth noticing that $\mathbf{S5}(\mathbf{C})$ is actually equivalent to the monadic fragment in one variable (without constants) of the first-order logic $\mathbf{C}\forall$ because there is a natural correspondence between formulas of both logics, between corresponding models and between the corresponding truth degrees.

Given an axiomatic extension $\mathbf{C}$ of **MTL**, we say that a monadic MTL-algebra is $\mathbb{C}$-functional if it is built in the previously defined way (Example 4.2 and Remark 4.3) from a linearly ordered MTL-algebra that belongs to $\mathrm{Alg}{*}\mathbb{C}$.

Along with the same line of Theorem 4.9, the following corollary naturally hold.

COROLLARY 5.1. *Let* $\mathbf{C}$ *be an axiomatic extension of* **MTL**. *Then the following statements are equivalent*:

(1) *For any formula* φ *and any set of formulas* Γ, *we have*

$$\Gamma \vdash_{\mathbf{S5}(\mathbf{C})} \varphi \text{ if and only if } \Gamma \models_{\mathbf{S5}(\mathbf{C})} \varphi.$$

(2) *The variety* $\mathbb{MMTL}_{\mathbb{C}}$ *is generated, as a quasivariety, by its* $\mathbb{C}$*-functional algebras*

Theorem 3.13 shows that a monadic IMTL-algebra is a monadic MTL-algebra that satisfies the identity $\neg\neg x = x$. In other words, a monadic IMTL-algebra is a monadic MTL-algebra whose MTL-reduct is an IMTL-algebra. This monadic algebra was first introduced in [53] as an equivalent algebraic semantic for monadic fuzzy predicate logic based on involutive monoidal t-norm logic **mIMTL**$\forall$.

Here we prove a representation theorem of monadic IMTL-algebras, and use it to derive strong completeness of the calculus with respect to the chain-based models. In order to make this section as self-contained as possible, we recall that some results about model theory in advance, which is indeed a straightforward application of model theoretic techniques [6].

Let A be a structure in the first-order language $\mathcal{L}$, and let $\{c_a : a \in A\}$ be a set of new constant symbols and put $\mathcal{L}_{\mathcal{A}} = \mathcal{L} \cup \{c_a : a \in A\}$. Given an $\mathcal{L}$-structure B and a family $\{b_a : a \in A\}$ of elements of B, we denote by $(B, \{b_a\}_{a\in A})$ the expansion of B to the language $\mathcal{L}_{\mathcal{A}}$ that results by defining b_a as the interpretation of the constant c_a for any $a \in A$. The diagram of A, written diag(A), is the set of all atomic sentences and negations of atomic sentences in the language $\mathcal{L}_{\mathcal{A}}$ which are true in $(A, \{b_a\}_{a\in A})$.

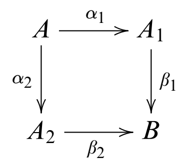

Figure 4. An amalgamation diagram of $\mathbb{K}$.

Proposition 5.2 [6]. *Let A, B be two $\mathcal{L}$-structures and let $f : A \to B$ be a function. Then the following statements are equivalent*:

(1) f *is an embedding of A into B*,
(2) $(B, \{f(a)\}_{a \in A})$ *is a model of diag* (A).

Let $\mathbb{K}$ be a class of structures in a first-order language. A V-formation in $\mathbb{K}$ is a 5-tuple $(A, A_1, A_2, \alpha_1, \alpha_2)$ consisting of three structures A, A_1, A_2 in $\mathbb{K}$ and two embeddings $\alpha_1 : A \to A_1$ and $\alpha_2 : A \to A_2$. An amalgam in $\mathbb{K}$ of the V-formation is a triple (B, β_1, β_2) consisting a structure B in $\mathbb{K}$ and two embeddings $\beta_1 : A_1 \to B$ and $\beta_2 : A_2 \to B$ satisfying $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$.

The class $\mathbb{K}$ has the amalgamation property provided every V-formation in $\mathbb{K}$ has an amalgam in $\mathbb{K}$. Here (Figure 4) we define a generalization of the notion of V-formation and amalgam. Given a set I, an I-formation in $\mathbb{K}$ is a triple $(A, (A_i)_{i \in I}, \{a_i\}_{i \in I})$ where A and A_i are a structure in $\mathbb{K}$ and each $\alpha_i : A \to A_i$ is an embedding. An amalgam in $\mathbb{K}$ of the I-formation is a pair $(B, \{\beta_i\}_{i \in I})$, where B is a structure in $\mathbb{K}$ and $\beta_i : A_i \to B$ are embeddings such that $\beta_i \circ \alpha_i = \beta_j \circ \alpha_j$ for any $i, j \in I$. Here we have that $\mathbb{K}$ has the amalgamation property over I if any I-formation in $\mathbb{K}$ has an amalgam in $\mathbb{K}$, readers refer to [6, 22, 40] for more details about results and examples of amalgamation.

In particular, Pierce studied amalgamations of lattice ordered groups in [40].

Example 5.3 [40]. *The class of totally ordered Abelian ℓ-groups has the amalgamation property.*

Theorem 5.4 [6]. *Let $\mathbb{K}$ be a class of structures in a first-order language with the amalgamation property. Then $\mathbb{K}$ has the amalgamation property over any finite set I. If $\mathbb{K}$ is an elementary class, then $\mathbb{K}$ has the amalgamation property over any set I.*

Theorem 5.4 can be applied to the class of totally ordered IMTL-algebras.

Corollary 5.5. *The class of totally ordered IMTL-algebras has amalgamation property.*

Proof. This result is a straightforward consequence of Example 5.3 by using Mundici's well-known correspondence between totally ordered Abelian ℓ-groups and totally ordered IMTL-algebras in [17]. □

As a consequence of Theorem 5.4, we get the following stronger result.

Theorem 5.6. *The class of totally ordered IMTL-algebras has amalgamation property over any set*.

With the aid of the general amalgamation property we prove here that finitely subdirectly irreducible monadic IMTL-algebras are isomorphic to $\mathcal{L}$-functional algebras. As a consequence we derive the strong completeness theorem of monadic involutive monoidal t-norm based predicate logic.

Proposition 5.7. *Let $(L, \forall, \exists)$ be a finitely subdirectly irreducible monadic IMTL-algebra. Then for any $x \in L$, there is a prime filter F of the IMTL-algebra L such that $x/F = \exists x/F$ and $F \cap \exists L = \{1\}$.*

Proof. Denote

$$\mathcal{F} = \{F \in F[L] \mid F \cap \exists L = \{1\}, \exists x \to x \in F\}.$$

Here we now prove that there is a prime filter P in $\mathcal{F}$.

Obviously $\mathcal{F}$ is non-empty since $\langle \exists x \to x \rangle \in \mathcal{F}$. Indeed, if $y \in \langle \exists x \to x \rangle \cap \exists L$, then $y \geq (\exists x \to x)^{2^n}$ for some positive integer. So

$$y = \exists y \geq \exists(\exists x \to x)^{2^n} = (\exists(\exists x \to x))^{2^n} = (\exists x \to \exists x)^{2^n} = 1^{2^n} = 1.$$

The union of a chain of filters in $\mathcal{F}$ is also a filter in $\mathcal{F}$, and hence by Zorn's Lemma, there is a maximal filter M in $\mathcal{F}$. Here we show that M is indeed a prime filter. If M is not prime, then there are $x, y \in L$ such that $x \to y, y \to x \notin M$. Thus M is properly contained in

$$\langle M \cup \{x \to y\} \rangle \text{ and } \langle M \cup \{y \to x\} \rangle.$$

By the maximality of M in $\mathcal{F}$, we have that

$$\langle M \cup \{x \to y\} \rangle \cap \exists L \neq \{1\} \text{ and } \langle M \cup \{y \to x\} \rangle \cap \exists L \neq \{1\}.$$

If

$$1 \neq z \in \langle M \cup \{x \to y\} \rangle \cap \exists L, 1 \neq w \in \langle M \cup \{y \to x\} \rangle \cap \exists L,$$

then

$$z \vee w \in \langle M \cup \{x \to y\} \rangle \cap \langle M \cup \{y \to x\} \rangle = \langle M \cup \{(x \to y) \vee (y \to x)\} = M.$$

Thus $z \vee w \in M \cap \exists L = \{1\}$, that is, $z \vee w = 1$. Also, since $\exists L$ is totally ordered, $z = 1$ or $w = 1$, which is a contradiction. This shows that M is a prime filter. □

Proposition 5.8. *Given a finite subdirectly irreducible*[6] *monadic IMTL-algebra $(L, \forall, \exists)$, there exists a subdirect embedding*

$$\alpha : L \to \prod_{i \in I} L_i,$$

where each L_i is a totally ordered IMTL-algebra and

$$\pi_i \circ \alpha|_{\exists L} : \exists L \to L_i$$

[6] An algebra A is subdirectly irreducible if for every subdirect embedding

$$\alpha : A \to \prod_{i \in I} A_i$$

there is an $i \in I$ such that

$$\pi_i \circ \alpha : A \to A_i$$

is an isomorphism. Readers please refer to Definition 8.3 in [3] for more details. Moreover, Proposition 5.8 shows that a monadic IMTL-algebra $(L, \forall, \exists)$ is finite subdirectly irreducible if and only if $\exists L$ is totally ordered.

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is[7] *an embedding, where* π_i *is the* i-*th projection map. Also, for any* $x \in L$, *there are* $i, j \in I$ *such that*

$$(\pi_i \circ \alpha)(\exists x) = (\pi_i \circ \alpha)(x), \ \ (\pi_j \circ \alpha)(\forall x) = (\pi_j \circ \alpha)(x).$$

Proof. It suffices to consider the subdirect representation of an IMTL-algebra L given by

$$L \to \prod_{P \in \mathcal{P}} L/P,$$

where

$$\mathcal{P} = \{P \in PF[L] \mid P \cap \exists L = \{1\}\}.$$

Along with the same line of Theorem 16 in [11] that $\bigcap \mathcal{P} = \{1\}$, so the representation is subdirect. The condition $P \cap \exists L = \{1\}$ guarantees that the projections on L/P are one–one correspondence on $\exists L$. Now, given $x \in L$, there is $P \in \mathcal{P}$ such that $\exists x/P = x/P$. Also, there is $P_1 \in \mathcal{P}$ such that $\exists \neg x/P_1 = \neg x/P_1$, so $\forall x/P_1 = \neg\exists\neg x/P_1 = \neg\neg x/P_1 = x/P_1$. □

THEOREM 5.9. *Every finitely subdirectly irreducible monadic IMTL-algebra* $(L, \forall, \exists)$ *is isomorphic to an* $\mathcal{L}$*-functional monadic IMTL-algebra, that is, there exist a totally ordered IMTL-algebra B, an index set I and an embedding* $\alpha : L \to B^I$ *such that*

$$\alpha(\exists x)(i) = \sup\{\alpha(x)(j) : j \in I\}, \alpha(\forall x)(i) = \inf\{\alpha(x)(j) : j \in I\}$$

for any $x \in L$ *and* $i \in I$.

Proof. Proposition 5.8 produces a family $\{L_i | i \in I\}$ of totally ordered IMTL-algebras and embedding

$$\alpha : L \to \prod_{i \in I} L_i,$$

such that

$$\pi_i \circ \alpha|_{\exists L} : \exists L \to L_i$$

is an embedding for any $i \in I$. Consider the I-formation $(\exists L, \{L_i | i \in I\}, \pi_i \circ \alpha|\exists L | i \in I\}$ in the elementary class $\mathbb{K}$ of totally ordered IMTL-algebras. By Theorem 5.6, the I-information has an amalgam $(B, \{\beta_i | i \in I\})$ in $\mathbb{K}$. Let

$$\beta := \prod \beta_i : \prod_{i \in I} L_i \to B^I$$

and note that

$$\beta \circ \alpha : A \to B^I$$

is an embedding. Here we prove that

$$(\beta \circ \alpha)(\exists x)(i) = \sup\{(\beta \circ \alpha)(x)(j) : j \in I\},$$

$$(\beta \circ \alpha)(\forall x)(i) = \inf\{(\beta \circ \alpha)(x)(j) : j \in I\},$$

for any $x \in L$ and $i \in I$. □

[7] Which is equivalent to $\pi_i \circ \alpha|_{\forall L} : \forall L \to L_i$ since $\forall L = \exists L$ for any monadic IMTL-algebra $(L, \forall, \exists)$.

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Indeed, because of the amalgamation of the I-formation, we have that

$$\begin{aligned}(\beta \circ \alpha)(\exists x)(i) &= \beta(\alpha(\exists x))(i)\\ &= \beta_i(\alpha(\exists x))(i)\\ &= \beta_i(\pi_i(\alpha(\exists x)))\\ &= \beta_j(\pi_j(\alpha(\exists x)))\\ &= \beta_j(\alpha(\exists x))(j)\\ &= \beta(\alpha(\exists x))(j)\\ &= (\beta \circ \alpha)(\exists x)(j),\end{aligned}$$

for any $i, j \in I$, and also know that there is $i_0 \in I$ such that

$$(\pi_{i_0} \circ \alpha)(\exists x) = (\pi_{i_0} \circ \alpha)(x).$$

Then

$$\begin{aligned}(\beta \circ \alpha)(x)(i_0) &= (\beta \circ \alpha)(\exists x)(j)\\ &\geq (\beta \circ \alpha)(\exists x)(j)\\ &\geq (\beta \circ \alpha)(x)(j),\end{aligned}$$

for any $j \in I$, which shows that

$$(\beta \circ \alpha)(\exists x)(i_0) = \sup\{(\beta \circ \alpha)(x)(j) \mid j \in I\}.$$

The proof of $\forall x$ is similar to that of $\exists x$.

COROLLARY 5.10. *The variety of monadic IMTL-algebras is generated, as a quasivariety, by its $\mathcal{L}$-functional members.*

The last result of this section is a strong completeness theorem with respect to chain-based models.

THEOREM 5.11. *For any formula φ and any set of formulas Γ, we have*

$$\Gamma \vdash_{S5(\mathcal{L})} \varphi \text{ if and only if } \Gamma \models_{S5(\mathcal{L})} \varphi.$$

§6. Concluding remarks and future work. The motivation of this paper is to give an algebraic proof of completeness for monadic fuzzy predicate logic **mMTL**∀ and some of its axiomatic extensions. In order to achieve our aim, we first survey the present state of knowledge on monadic algebras of t-norm based residuated fuzzy logic and show that the relationships for monadic algebras of *t*-norm based fuzzy residuated logic completely inherit that for corresponding algebras of *t*-norm based fuzzy residuated logic. The results of Section 4 proved that the variety of monadic MTL-algebras is the equivalent algebraic semantics of the logic **mMTL**∀ and gave an algebraic proof of completeness for this logic. Along with the same line, we further obtained the completeness theorem of some axiomatic extensions of the logic **mMTL**∀, and proved the strong completeness theorem of monadic fuzzy predicate logic **mIMTL**∀ via functional representation of finitely subdirectly irreducible algebras in Section 5.

Hopefully this first step will allow us to prove a strong result, namely the strong standard completeness for monadic fuzzy predicate logic **mIMTL**∀ that is equivalent to fuzzy modal logic **S5**(**IMTL**), which is one of the topics in subsequent discussions.

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