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LOCALIZATION IN NON-COMMUTATIVE NOETHERIAN RINGS

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1.1 Introduction and summary. To construct a well behaved localization of a noetherian ring R at a semiprime ideal S, it seems necessary to assume that the set $\mathscr{C}(S)$ of modulo S regular elements satisfies the Ore condition; and it is convenient to require the Artin Rees property for the Jacobson radical of the quotient ring R_S in addition: one calls such S *classical*. To determine the classical semiprime ideals is no easy matter; it happens frequently that a prime ideal fails to be classical itself, but is minimal over a suitable classical semiprime ideal.

The present paper studies the structure of classical semiprime ideals: they are built in a unique way from *clans* (minimal families of prime ideals with classical intersection), and each prime ideal belongs to at most one clan. We are thus led to regard the quotient rings R_S at the clans S as the natural localizations of a noetherian ring R. We determine these clans for rings which are finite as module over their centre, with an application to group rings, and for HNP-rings, and provide some preliminary results for enveloping algebras of solvable Lie algebras.

1.2. Terminology. By a ring, we mean a not necessarily commutative ring with identity; and unless stated otherwise, a module is a unitary right-module. Terms like noetherian, ideal, etc. mean left- *and* right-noetherian, -ideal, etc. unless specified by one of the prefixes left- or right-. A *regular* element is a non-zerodivisor.

$E(M)$ is the injective hull of the module M; $J(R)$ is the Jacobson radical of the ring R. A ring R is *semilocal* if $R/J(R)$ is semisimple artinian.

A (hereditary) *torsion theory* on the category mod R of R-modules may be described by its torsion class $\mathscr{T}$, torsion-free class $\mathscr{F}$, torsion radical ρ, Gabriel filter $\mathscr{D}$ of right-ideals, or equivalence class (qua mutual cogeneration) of injective modules; cf. [**27**]. A monomorphism (or submodule) is called *dense/closed* if its cokernel is torsion/torsionfree. The *closure* $\mathrm{cl}\, N = \mathrm{cl}_M N$ of a submodule N of M is the smallest closed submodule of M containing N. With any multiplicative subset Σ of R is associated, the torsion theory determined by $\mathscr{T}_\Sigma = \{X \in \text{mod}\, R\text{: for each } x \in X \text{ there is } s \in \Sigma \text{ with } xs = 0\}$ or $\mathscr{D}_\Sigma = \{D \text{ right-ideal of } R\text{: } r^{-1}D \cap \Sigma \neq \emptyset \text{ for all } r \in R\}$.

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1.3 Semiprime ideals. A proper ideal I of a ring R is (*semi*) *prime* if $aRb \subset I$ implies $a \in I$ or $b \in I$ (if $aRa \subset I$ implies $a \in R$); a semiprime ideal is the intersection of prime ideals. An ideal is said to have the right-AR-property, if for every right-ideal A there exists a number n such that $A \cap I^n \subset AI$.

In a right-noetherian ring R, a semiprime ideal has a unique representation as a finite irredundant intersection of prime ideals, and this establishes a one-to-one correspondence between semiprime ideals $S = \bigcap_{i=1}^n P_i$ and non-empty finite sets $\{P_1, \ldots, P_n\}$ of mutually incomparable prime ideals. Terminology like localizable, clan, etc. will be used to refer simultaneously to a semiprime ideal and its associated set of prime ideals.

For a semiprime ideal S of a right-noetherian ring R, the torsion theories determined by the injective module $E(R/S)$ and by the multiplicative set $\mathscr{C}(S) = \{c \in R: cx \in S \text{ implies } x \in S\}$ coincide; this is called the S-torsion theory $\mathscr{T}_S$ with quotient ring R_S (respectively ${}_SR$ if left-modules are considered). If $S = \bigcap_{i=1}^n P_i$, then $E(R/S) \cong \oplus_{i=1}^n E(R/P_i)$, $\mathscr{C}(S) = \bigcap_{i=1}^n \mathscr{C}(P_i)$ and for given elements $c_i \in \mathscr{C}(P_i)$ there exist $r_i \in R$ with $\sum_{i=1}^n c_i r_i \in \mathscr{C}(S)$ ([**15**; **12**; **17**]).

A prime ideal P of a right-noetherian ring R is either dense or closed, for any torsion theory $\mathscr{T}$. (If $R/P \notin \mathscr{F}$, then there is a uniform R/P-right-ideal in $\mathscr{T}$; then all uniform R/P-right-ideals lie in $\mathscr{T}$ since they are mutually subisomorphic; then there is an essential R/P-right-ideal in $\mathscr{T}$, which contains a regular element; hence $R/P \in \mathscr{T}$. This observation was communicated to us by R. Richards.) It follows for $S = \bigcap_{i=1}^n P_i$ that a prime ideal Q is S-closed if and only if $Q \subset \bigcup_{i=1}^n P_i$, and that $S \to \mathscr{T}_S$ is a meet-semilattice embedding from the set of semiprime ideals S (ordered inverse to the set-inclusion of $\bigcup_{i=1}^n P_i$) into the lattice of all torsion theories.

Definition. A semiprime ideal S of a right-noetherian ring R is *right-localizable* if $\mathscr{C}(S)$ is a right-Ore set. It is *right-classical* if it is right-localizable and if in addition the Jacobson radical $J(R_S)$ of the quotient ring has the right-AR-property. A non-empty finite set $\{P_1, \ldots, P_n\}$ of mutually incomparable prime ideals will be called a *clan* if its associated semiprime ideal is classical but the associated semiprime ideals of all proper subsets are not.

If $S = \bigcap_{i=1}^n P_i$ is right-localizable, then R_S is right-noetherian and semilocal with $J(R_S) = SR_S$. IR_S is an ideal of R_S for every ideal I of R, and the spectrum of R_S consists precisely of the ideals PR_S for the prime ideals P of R contained in $\bigcup_{i=1}^n P_i$ ([**3**, 2.10]). If S is localizable, then ${}_SR = R_S$.

Criteria for S to be right-localizable or right-classical, are to be found in [**13**] and [**17**]; in particular S is right-localizable if and only if $\mathscr{C}(S)$ operates regularly on $E(R/S)$, and it is right-classical if and only if in addition each element of $E(R/S)$ is annihilated by some power of S. Over FBN- and HNP-rings, localizable semiprime ideals are automatically classical, and there seems to be no example known of a semilocal noetherian ring whose Jacobson radical doesn't have the AR-property.

2.1 Artinian rings.

LEMMA 1. *The following are equivalent for a semiprime ideal S of a right-artinian ring R:*

(1) *S is right-classical.*

(2) *S is right-localizable.*

(3) *S has the right-AR-property.*

(4) *$eR(1 - e) = 0$, where $e = e^2 \in R$ and $S = Re + J(R) = eR + J(R)$.*

Proof. (1) implies (2) trivially. If (2) is given, then R_S is right-artinian, hence $0 = J(R_S)^n = S^n R_S$ for some n. If A is any right-ideal of R and if $a \in A \cap S^n$, then $ac = 0$ for some $c \in \mathscr{C}(S)$ since $\bar{a} \in S^n R_S = 0$. Since $\bar{c}$ is regular hence invertible in the semisimple artinian ring R/S, $cr = 1 - s$ for suitable $r \in R$ and $s \in S$, hence $0 = acr = a(1 - s)$ or $a = as \in AS$, demonstrating (3).

Given (3), note first that in any right-artinian ring R there exists an idempotent e, unique and central modulo $J(R)$, with $S = eR + J(R)$. Using the right-AR-property on the ideal $A = fR + J(R)$ where $f = 1 - e$, one gets n with $A \cap S^n \subset AS$. Now $e \in S^n$ and $Rf \subset A$ yield $eRf \subset A \cap S^n \subset AS = (fR + J)(Re + J) \subset fRe + fJ + Je + J^2$, which produces by multiplication by e and f the inclusion $eRf \subset eJ^2 f \subset J(eRe)eRf + eRfJ(fRf)$, hence $eRf = 0$ by the nilpotence of $J(eRe)$ and $J(fRf)$.

That finally (4) implies (1), is checked by computation: one has the matrix representation $R = \begin{bmatrix} eRe & 0 \\ fRe & fRf \end{bmatrix}$ with $S = \begin{bmatrix} eRe & 0 \\ fRe & fJf \end{bmatrix}$ hence $\mathscr{C}(S) = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} : c \text{ is invertible in } fRf \right\}$. The right-Ore-condition is easily verified explicitly; and the quotient ring is the right-artinian ring $R_S = fRf$, whose Jacobson radical, being nilpotent, has obviously the AR-property.

COROLLARY 2. *The only localizable semiprime ideal of a ring-directly indecomposable artinian ring is the Jacobson radical.*

Remark. An indecomposable artinian ring may have many semiprime ideals which are one-sided localizable. For instance for the ring of $n \times n$-upper triangular matrices over a field, there are n prime ideals P_i (defined by putting a zero in the i-th diagonal position), and the right-localizable semiprime ideals are precisely the $P_1 \cap \ldots \cap P_i$ for $i = 1, \ldots, n$.

2.2 Completions. We collect a number of facts on the completion of a semilocal right-noetherian ring R with respect to the J-adic topology, $J = J(R)$. They seem to be generally known, but we couldn't locate a systematic source of reference (cf. however [**29**; **16**; **17**; and **13**]).

For simplicity assume $\bigcap_{n=1}^{\infty} J^n = 0$; then R may be regarded as a subring of its Hausdorff completion $\hat{R}$. $\hat{R}$ is semiperfect with $J(\hat{R}) = \hat{J}$. The ideals $(J^n)^\wedge$ determine the completion topology, and $\hat{R}/(J^n)^\wedge \cong R/J^n$; there results a one-to-one correspondence between the open right-ideals of R and $\hat{R}$. For an arbitrary right-ideal A of R, the completion $\hat{A}$ in the relative topology coincides with the topological closure of A in $\hat{R}$; and the completion in the J-adic topology (which is defined by the AJ^n) is $A\hat{R}$.

If J has the right-AR-property, then every right-ideal A of R is closed, and the relative and the J-adic topologies on A agree. Hence $(AB)^\wedge = \hat{A}\hat{B}$ for right-ideals A and B; in particular $(J^n)^\wedge = (\hat{J})^n$ hence the completion topology on $\hat{R}$ coincides with the $J(\hat{R})$-adic topology. $\hat{R}$ is the bicommutator of $E = E(R/J)$, since the latter is the E-adic completion of R, and since the E-adic topology coincides with the J-adic one due to the right-AR-property. Note also that the right-AR-property implies our assumption $\bigcap J^n = 0$.

Whether $\hat{R}$ is right-noetherian, and whether $\hat{J}$ possesses the right-AR property if J does, seem to be open questions.

2.3 Main theorems.

LEMMA 3. *Let R be a right-noetherian ring with the right-localizable semiprime ideal $S = \bigcap_{i=1}^{n} P_i$, and let $T = \bigcap_{i=1}^{t} P_i$, $t \leqq n$. Then $\mathscr{C}(T)$ is right-Ore in R, if and only if $\mathscr{C}(TR_S)$ is right-Ore in R_S.*

Proof. One shows first that $c \in \mathscr{C}(T)$ if and only if $cb^{-1} \in \mathscr{C}(TR_S)$ for one/all $b \in \mathscr{C}(S)$; then the lemma is easily verified.

THEOREM 4. *Let R be a noetherian ring with the classical semiprime ideal $S = \bigcap_{i=1}^{n} P_i$. Then there is a one-to-one correspondence between the central idempotents of $\hat{R}_S$ and the localizable subsets of $\{P_1, \ldots, P_n\}$. Such subsets are automatically classical.*

Proof. Note that ${}_SR = R_S$ is a semilocal noetherian ring with Jacobson radical $J = J(R_S) = SR_S$. Let $\{P_1, \ldots, P_t\}$ be a localizable subset, and put $T = \bigcap_{i=1}^{t} P_i$. Then TR_S is localizable by Lemma 3, hence so is $\overline{TR}_S$ in $\bar{R}_S = R_S/J^n$ for all n. Hence by Lemma 1, since the factor rings $\bar{R}_S$ are artinian, there is a unique central idempotent $\bar{e}_n \in \bar{R}_S$ with $\overline{TR}_S = \bar{e}_n\bar{R}_S + \bar{J}$; i.e. $TR_S = e_nR_S + J$. By uniqueness, the sequence $e_n \in R_S$ of (arbitrary) inverse images is Cauchy, hence $e = \lim e_n$ exists in $\hat{R}_S$, and is a central idempotent.

Conversely for a given central idempotent $e \in R_S$, define $T = \epsilon^{-1}(e\hat{R}_S + \hat{J})$ for the natural map $\epsilon : R \to R_S \to \hat{R}_S$; T is a semiprime ideal of R; moreover $TR_S = R_S \cap (e\hat{R}_S + \hat{J})$. Clearly $\bar{e}$ is a central idempotent of the artinian ring $\hat{R}_S/(J^n)^\wedge \cong R_S/J^n$, hence $(e\hat{R}_S + \hat{J})/(J^n)^\wedge \cong TR_S/J^n$ is localizable by Lemma 1, hence $\mathscr{C}(\overline{TR}_S) = \overline{\mathscr{C}}(TR_S)$ is an Ore-set in R_S/J^n, for every n. Thus for given $c \in \mathscr{C}(TR_S)$ and $r \in R_S$, there exist $c_n \in \mathscr{C}(TR_S)$, $r_n \in R_S$ and $h_n \in J^n$ with $cr_n - rc_n = h_n$. For the right-ideal $A = \sum_{n=1}^{\infty} h_nR_S$ of R_S, there is by the right-AR-property of J, a number N with $h_N \in A \cap J^N \subset AJ = \sum_{n=1}^{\infty} h_n J$. Therefore $h_N = \sum_{n=1}^{m} h_n j_n$ with $j_n \in J$, hence $r(c_N - \sum_{n=1}^{m} c_n j_n) = cr_N - h_N - \sum_{n=1}^{m} (cr_n - h_n)j_n = c(r_N - \sum_{n=1}^{m} r_n j_n)$, where

$$c_N - \sum_{n=1}^{m} c_n j_n \in \mathscr{C}(TR_S)$$

since $j_n \in J \subset TR_S$. Therefore TR_S, and consequently T by Lemma 3, are localizable.

That T corresponds to a subset of $\{P_1, \ldots, P_n\}$, and that both constructions are inverse, is readily checked. If $T = \bigcap_{i=1}^{t} P_i$ is localizable, then $P_k R_T = R_T$ for $t < k \leqq n$, since $P_k \not\subset \bigcup_{i=1}^{t} P_i$; hence $SR_T = TR_T$. Then for $e \in E(R/T) \subset E(R/S)$ there is $eS^N = 0$, hence $0 = eS^N R_T = eT^N R_T$ hence $0 = eT^N$, using that $E(R/T)$ is an R_T-module. This proves that T is classical (cf. Section 1.3).

THEOREM 5. *A prime ideal of a noetherian ring belongs to at most one clan.*

Proof. Let $S = \bigcap_{i=1}^{n} P_i$ and $T = \bigcap_{j=1}^{m} Q_j$ be two clans of the noetherian ring R, and assume $P_1 \subset Q_1$. Let $P_1, \ldots, P_s$ be precisely the P_i's contained in $\bigcup_{j=1}^{m} Q_j$, and put $A = \bigcap_{i=1}^{s} P_i$. Since $P_i \not\subset Q_j$ for $i > s$ and arbitrary j, $P_i R_T = R_T$ hence $SR_T = \bigcap_{i=1}^{n} P_i R_T = \bigcap_{i=1}^{s} P_i R_T = AR_T$.

Suppose that A is not right-localizable; then there exist

$$e \in E(R/A) \subset E(R/S) \quad \text{and} \quad c \in \mathscr{C}(A) \quad \text{with} \quad ec = 0.$$

Since A is T-closed, $E(R/A)$ is an R_T-module; therefore $0 = eS^n$ for suitable n implies $0 = eS^n R_T = eA^n R_T$ hence $eA^n = 0$ (cf. Section 1.3).

$c \in \mathscr{C}(A)$ implies $\bar{c} \in \mathscr{C}(AR_S)$; hence $\bar{c}$ is regular in R_S/AR_S, which is a factor of $R_S/SR_S = R_S/J(R_S)$ hence semisimple artinian. Consider $\bar{R}_S = R_S/A^n R_S$; one has $J(\bar{R}_S) = \overline{AR}_S$ since $\bar{R}_S/\overline{AR}_S \cong R_S/AR_S$ is semisimple artinian, and since $\overline{AR}_S{}^n = 0$. Thus $\bar{R}_S$ is artinian; and $\bar{c}$ regular in $R_S/AR_S \cong \bar{R}_S/J(\bar{R}_S)$ implies that $\bar{c}$ is invertible in $\bar{R}_S/J(\bar{R}_S)$ hence in $\bar{R}_S$: $\overline{cq} = \bar{1}$ where $\bar{q} \in \bar{R}_S$, or $cq - 1 = u \in A^n R_S$. Then $q = bt^{-1}$ and $u = at^{-1}$ with $b \in R$, $a \in A^n$ and $t \in \mathscr{C}(S)$, hence $cb - t = a$. Therefore $0 = ec$ yields $0 = ecb = e(t + a) = et$ since $ea \in eA^n = 0$, contradicting the localizability of S.

Consequently A is localizable, hence $A = S$ by Theorem 4, hence $\bigcup_{i=1}^{n} P_i \subset \bigcup_{j=1}^{m} Q_j$; i.e. $\mathscr{T}_S \supset \mathscr{T}_T$. In particular if $P = P_1 = Q_1$ belongs to both clans, then symmetry implies $\mathscr{T}_S = \mathscr{T}_T$, completing the proof.

If every prime ideal of a noetherian ring belongs to a clan, we say that the ring has *enough clans.*

By Theorem 4, every classical set of prime ideals of a noetherian ring is uniquely partitioned into clans (since the identity element of the semiperfect ring $\hat{R}_S$ has a unique decomposition into central and centrally indecomposable orthogonal idempotents), and the corresponding torsion theory if the meet $\mathscr{T}_S = \wedge \mathscr{T}_{S_k}$. Moreover by the proof of Theorem 5, the prime ideals of any two clans are either all incomparable, or the two torsion theories are comparable.

Therefore the meet of two *classical* torsion theories (i.e. torsion theories corresponding to classical semiprime ideals) $\mathscr{T}_1$ and $\mathscr{T}_2$ is again classical: write both as meets of torsion theories $\mathscr{T}_{S_k}$ corresponding to clans and delete the non-minimal ones among these; the union $\{P_1, \ldots, P_n\}$ of the remaining clans is incomparable, hence for $S = \bigcap_{i=1}^n P_i$ one has $\mathscr{T}_1 \wedge \mathscr{T}_2 = \mathscr{T}_S$, $\mathscr{C}(S) = \bigcap \mathscr{C}(P_i) = \bigcap \mathscr{C}(S_k)$ and $E(R/S)$ maps injectively into $\oplus E(R/S_k)$. If $e \in E(R/S)$ and $c \in \mathscr{C}(S)$ with $ec = 0$ are given, and if e maps to $(e_1, \ldots, e_m)$, then $e_k c = 0$ hence $e_k = 0$ since $c \in \mathscr{C}(S) \subset \mathscr{C}(S_k)$ operates regularly on $E(R/S_k)$; hence $e = 0$ and $\mathscr{C}(S)$ is an Ore set. Moreover for any $e \in E(R/S)$ one has $e_k S_k^N = 0$ for suitable N and all k since all S_k are classical, hence $eS^N = 0$ and S is classical as desired.

From these and the observations in Section 1.3 follows readily a lattice-theoretical formulation of our results.

COROLLARY 6. *For any noetherian ring, the classical torsion theories form a sub-meet-semilattice of the lattice of all torsion theories. In this meet-semilattice, the meet-irreducible elements are the torsion theories corresponding to clans, and each element is uniquely a finite irredundant meet of meet-irreducibles.*

3.1 Rings which are finite over their centre. Let R be a noetherian ring which is finitely generated as module over a subring A of its centre Γ. By [**6**], A and Γ are (commutative) noetherian rings. Such a ring R can be satisfactorily localized at the prime ideals of A or Γ; and it is reassuring that our non-commutative approach leads to these very same localizations.

For any prime ideal Q of A, there exists at least one and at most finitely many (automatically mutually incomparable) prime ideals P of R lying over Q, i.e. with $P \cap A = Q$. They arise as the inverse images of the maximal ideals of R_Q/QR_Q, which is a finite-dimensional algebra over the field A_Q/QA_Q. If S is the intersection of these prime ideals, then S_Q is the Jacobson radical of R_Q, and the latter is semilocal (cf. [**8**; **24**; **26**]).

For any $c \in \mathscr{C}(S)$, $\bar{c}$ is regular in R/S hence in $(R/S)_Q \cong R_Q/S_Q = R_Q/J(R_Q)$, hence invertible in this semisimple artinian ring, hence invertible in R_Q. Consequently $\mathscr{C}(S)$ is an Ore set and $R_Q = R_S$, in fact the two torsion theories coincide. From the commutative Artin Rees Lemma ([**29**, p. 255]) follows that S is even classical.

THEOREM 7. *Let R be a noetherian ring which is finitely generated as module over its centre Γ. Then the clans are the sets of prime ideals of R lying over the various prime ideals Π of Γ. In particular, R has enough clans.*

Proof. The preceding consideration has shown that the sets in question are classical; it remains to see that they are minimal such. By Theorem 4, this amounts to proving that $\hat{R}_S = \hat{R}_\Pi$ contains no nontrivial central idempotent. Since $\hat{R}_\Pi \cong R_\Pi \otimes_{\Gamma_\Pi} \hat{\Gamma}_\Pi$, and since $\Gamma_\Pi \to \hat{\Gamma}_\Pi$ is flat, the lemma on p. 432 of [**8**] yields $\text{centre}(\hat{R}_\Pi) \cong \text{centre}(R_\Pi \otimes_{\Gamma_\Pi} \hat{\Gamma}_\Pi) \cong \text{centre}(R_\Pi) \otimes_{\Gamma_\Pi} \hat{\Gamma}_\Pi \cong \hat{\Gamma}_\Pi$, since $\operatorname{centre}(R_\Pi) = \Gamma_\Pi$ follows readily. As $\hat{\Gamma}_\Pi$ is a local ring, the claim follows.

Remark. Examples for the present situation are separable algebras and classical maximal orders. In both cases, there is exactly one prime ideal P of R over each prime Π of Γ (cf. [**1**] and [**28**]), hence all clans are *trivial* (i.e. one-element sets). Nontrivial clans arise plentifully from the next example, which we discuss in some detail.

3.2 Group rings. Consider the group ring $R = AG$ of a finite group G over a commutative noetherian ring A. The centre of AG is $\Gamma = \{\sum a_g g: a_g = a_h \text{ if } g, h \text{ are conjugate}\}$. The (probably well known) next lemma describes the relevant features of the spectrum of AG.

LEMMA 8. *Let Q be a prime ideal of A, and let K be the quotient field of A/Q. Then the prime ideals Π of Γ over Q correspond to the blocks of KG, and the prime ideals P of AG over any such Π correspond to the maximal ideals of the block.*

Proof. According to the consideration at the beginning of Section 3.1, the prime ideals of R respectively Γ over Q correspond to the maximal ideals of R_Q/QR_Q respectively $\Gamma_Q/Q\Gamma_Q$. Now

$$R_Q/QR_Q = A_QG/QA_QG \cong (A_Q/QA_Q)G = KG,$$

hence the prime ideals of $R = AG$ over Q correspond to the maximal ideals of KG. The restriction of the map $R_Q \to R_Q/QR_Q \to KG$ to Γ_Q induces an isomorphism $\Gamma_Q/Q\Gamma_Q \cong \operatorname{centre}(KG)$, using the explicit description of the centre of a group ring. Thus prime ideals of Γ correspond to maximal ideals of $\operatorname{centre}(KG)$, i.e. to blocks of KG.

Remark. If a block of KG is simple, it produces a trivial clan. Hence by Maschke's Theorem, nontrivial clans can arise only if the characteristic of K divides the order $|G|$ of G, i.e. if $|G| \in Q$. In particular if A is a Dedekind domain of characteristic zero (hence $A/|G|A$ is artinian), there are only finitely many such prime ideals Q, and consequently at most finitely many nontrivial clans of AG.

A group is called *q-nilpotent* for a prime number q, if G contains a normal subgroup N whose order is not divisible by q, such that G/N is a q-group [**10**]. Formally, we call every group also *0-nilpotent.* A group is nilpotent in the usual sense, if and only if it is q-nilpotent for all q. For any prime ideal Q of A, define the Q-augmentation ideal of AG by $\Delta_Q = \{\sum a_g g: \sum a_g \in Q\}$; it is a prime ideal of AG.

PROPOSITION 9. *Let Q be a prime ideal of the commutative noetherian ring A, and let q be the characteristic of the quotient field K of A/Q. Then the following statements on the group ring AG of a finite group G are equivalent:*

(1) *all prime ideals of AG over Q are classical,*

(2) *the Q-augmentation ideal is classical,*

(3) *G is q-nilpotent.*

Proof. By Maschke's Theorem, the proposition is trivial unless q divides $|G|$. (1) implies (2) trivially. If Δ_Q is classical, then the principal block of KG has only one irreducible representation by Lemma 8, which must be the trivial one, and consequently G is q-nilpotent by [**22**]. If G is q-nilpotent, then every block of KG has a unique simple module by [**23**, Corollary 3.6], hence every clan of AG is trivial by Lemma 8 and Theorem 7.

COROLLARY 10. *All clans of a group ring AG are trivial, if and only if G is q-nilpotent for all prime numbers q which are not invertible in A. In particular all clans of $\mathbf{Z}G$ are trivial, if and only if G is nilpotent.*

Remark. These considerations generalize slightly results of Smith [**26**] obtained by a different method avoiding representation theory. A combination of his and our approach should produce ring-theoretical proofs of the representation-theoretical results used above.

3.3 Hereditary noetherian prime rings.

THEOREM 11. *A nonzero semiprime ideal of an HNP-ring is classical/localizable if and only if it is invertible.*

Remarks. Consequently for *HNP*-rings, the clans coincide with the cycles defined in [**7**]. The result in [**18**] that bounded *HNP*-rings have enough invertible ideals, establishes that such rings have enough clans. For prime ideals, our theorem is in [**5**].

Proof. That an invertible semiprime ideal is classical, was proved in [**14**] for *HNP*-rings, and holds true for arbitrary noetherian rings: the standard argument (cf. e.g. [**4**]) for invertible prime ideals, which shows that they have the *AR*-property, and that ordinary and symbolic powers coincide, goes through.

Conversely if A is localizable, then ${}_SR = R_S$ is a semilocal *HNP*-ring with $J(R_S) = SR_S \neq 0$. Then by [**21**, Satz 4.5] or [**7**, Theorem 4.13] $J(R_S)$ is invertible. By [**7**], $S = X \cap Y = XY = YX$ where X is an invertible and Y is an eventually idempotent semiprime ideal. Then $J(R_S)^{m+1} = S^{m+1}R_S = X^{m+1}Y^{m+1}R_S = X^{m+1}Y^mR_S = XJ(R_S)^m$ hence $J(R_S) = XR_S$, hence $S = X$ is invertible.

3.4 Enveloping algebras of Lie algebras. We consider finite-dimensional Lie algebras L over an algebraically closed field K of characteristic zero, and their enveloping algebras $U(L)$ which are noetherian domains. If L is nilpotent, then all prime ideals of $U(L)$ are classical [**20**], hence $U(L)$ has enough clans and all of them are trivial. We study first the two-dimensional noncommutative Lie algebra L_2, with basis x, y and $[x, y] = x$, and use then the fact that each non-nilpotent solvable Lie algebra L maps onto L_2 ([**3**, Lemma on p. 71]) to deduce that no such algebra has enough clans.

LEMMA 12. *The spectrum of* $U(L_2)$ *consists of the prime ideals* 0, $\langle x\rangle$, $\langle x, y - a\rangle$ *for all* $a \in K$. 0 *and* $\langle x\rangle$ *are classical, while no other semiprime ideal is even right- or left-localizable.*

Proof. The determination of the spectrum is routine. 0 is trivially classical, and $\langle x\rangle$ is so since x is a normal element ([**20**; **19**]). Any other semiprime ideal must be of the form $S = \bigcap_{i=1}^{n} \langle x, y - a_i\rangle$; then there exists $b \in K$, different from all the a_i but equal to a suitable $a_j - 1$, since the characteristic is zero. Then $y - b \in \mathscr{C}(S)$; and if $\mathscr{C}(S)$ is right-Ore, then there exist $c \in \mathscr{C}(S)$ and $r \in U(L_2)$ with $(y - b)\, r = xc \in \langle x\rangle$. As $y - b \notin S$ hence $y - b \notin \langle x\rangle$, and as prime ideals are here completely prime, $r \in \langle x\rangle$ hence $r = xr'$ hence $xc = (y - b)\, r = (y - b)\, xr' = x(y - 1 - b)r' = x(y - a_j)r'$. Cancellation of x yields $c = (y - a_j)r' \in \langle x, y - a_j\rangle$, a contradiction.

PROPOSITION 13. *If* L *is any non-nilpotent solvable Lie algebra, then* $U(L)$ *does not have enough clans.*

Proof. Select a surjective Lie homomorphism $L \rightarrow L_2$, and consider the induced ring homomorphism $U(L) \rightarrow U(L_2)$; let P_a be the inverse image of the maximal ideal $\langle x, y - a\rangle$ of $U(L_2)$ in $U(L)$. Suppose that one of them, say P_0, belongs to a clan S. Then $\mathscr{C}(S)$ is a right-Ore set of $U(L)$ hence its image $\Sigma = \overline{\mathscr{C}(S)}$ is a right-Ore set of $U(L_2)$. Only finitely many P_a belong to the clan S; let $a_1 = 0, \ldots, a_m$ be the corresponding $a \in K$. Then $P_{a_i} \cap \mathscr{C}(S) = \emptyset$ hence $\langle x, y - a_i\rangle \cap \Sigma = \emptyset$; but if $a \neq a_1, \ldots, a_m$ then since P_a is maximal, $P_a \cap \mathscr{C}(S) \neq \emptyset$ hence $\langle x, y - a\rangle \cap \Sigma \neq \emptyset$.

Select again some $b \in K$, different from all $a_1, \ldots, a_m$ but equal to a suitable $a_j - 1$. Then $\langle x, y - b\rangle \cap \Sigma \ni s$, which can be written as $s = x\phi(x, y) + (y - b)\psi(y)$. Then $sx = x\phi x + (y - b)\psi(y)x = x\phi x + x(y - 1 - b)\psi(y - 1) = xs^*$ with $s^* = \phi x + (y - 1 - b)\psi(y - 1) = \phi x + (y - a_j)\psi(y - 1) \in \langle x, y - a_j\rangle$. By the right-Ore condition on Σ there are $r \in U(L_2)$ and $s' \in \Sigma$ with $sr = xs' \in \langle x\rangle$, hence $r \in \langle x\rangle$ since $s \in \Sigma$ implies $s \notin \langle x, y\rangle$ hence $s \notin \langle x\rangle$. Thus $xs' = sr = sxr' = xs^*r'$ hence $s' = s^*r' \in \langle x, y - a_j\rangle \cap \Sigma = \emptyset$, a contradiction.

Remarks. (1) The argument actually shows that there is no right- or left-localizable semiprime ideal S of $U(L)$ over which any of the P_a is minimal.

(2) Enveloping algebras of solvable Lie algebras L differ from the rings considered in the preceding sections insofar as they are never fully bounded, unless L is commutative. This follows for nilpotent Lie algebras from the facts that primitive factor algebras are Weyl algebras and that 0 is the intersection of primitive ideals, and for non-nilpotent solvable Lie algebras L from the existence of a surjective map $U(L) \rightarrow U(L_2)$ and the fact that $R = U(L_2)$ is not bounded; indeed the right-ideal yR does not contain any nonzero ideal.

(3) From investigations of low-dimensional non-nilpotent solvable Lie algebras, several common features emerge which might hold true in general: all clans are trivial; if P belongs to a clan and if $P \supset Q$, then Q belongs to a

clan; primes of codimension one never belong to a clan; primes of height one always belong to a clan.

3.5 Two counterexamples. (1) The ring $R = \begin{bmatrix} \mathbf{Z}_p & \mathbf{Q} \\ 0 & \mathbf{Q} \end{bmatrix}$ is right-noetherian, with polynomial identity hence fully bounded, and with Krull dimension one. (Though it is not left-noetherian, our main results can be deduced from these properties.) There are three prime ideals $P_0 = \begin{bmatrix} 0 & \mathbf{Q} \\ 0 & \mathbf{Q} \end{bmatrix}$, $P_1 = \begin{bmatrix} p\mathbf{Z}_p & \mathbf{Q} \\ 0 & \mathbf{Q} \end{bmatrix}$ and $P_2 = \begin{bmatrix} \mathbf{Z}_p & \mathbf{Q} \\ 0 & 0 \end{bmatrix}$, only one clan $\{P_0, P_2\}$ and the additional localizable set $\{P_1, P_2\}$. Thus R doesn't have enough clans, while P_2 belongs to two different localizable sets of prime ideals, illustrating that the assumption of the AR-property is essential in Theorem 5.

(2) The split extension $R = A \times N$ of a commutative noetherian ring A by a bimodule N with $N^2 = 0$, is noetherian if N is finitely generated on both sides, and satisfies the polynomial identity $S_2{}^2$. The prime ideals of R correspond naturally to the prime ideals of A. If N is the bimodule A, with the natural module structure modified by an automorphism σ of A on one side, then a prime ideal P of R belongs to a clan if and only if the set $\{\sigma^n(\bar{P}) : n \in \mathbf{Z}\}$ is finite, and then the clan containing P is just the set of corresponding prime ideals of R. For instance for $A = K[x]$ where K is a field of characteristic zero, $\sigma(x) = x + 1$ and $\bar{P} = xA$, one has $\sigma^n(\bar{P}) = (x + n)A$ hence P doesn't belong to a clan. (Though these facts may be verified directly, they follow naturally from a detailed study of the links between prime ideals in FBN-rings which we intend to describe elsewhere.)

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