Can. J. Math., Vol. XXIII, No. 3, 1971, pp. 503–506

QUASI-SPLITTING EXACT SEQUENCE

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1. Definitions. Let R be a ring with $1 \neq 0$ and α, β, γ R-endomorphisms of R-modules A, B, and C respectively. We shall denote by $\mathrm{M}(R)$ the category of R-modules, and by $\mathrm{End}(R)$ the category of R-endomorphisms. For objects α and β of $\mathrm{End}(R)$ a morphism $\lambda: \alpha \to \beta$ is an R-homomorphism such that $\lambda\alpha = \beta\lambda$. We shall denote by $\mathrm{Idm}(R)$ the full subcategory of $\mathrm{End}(R)$ whose objects are idempotents. $\mathrm{Idm}(R)$ is an abelian category. ker, coker and im are constructed in the naive way and hence

$$0 \to A \xrightarrow{\kappa} B \xrightarrow{\sigma} C \to 0$$

is exact in $\mathrm{M}(R)$ if and only if

$$0 \to \alpha \xrightarrow{\kappa} \beta \xrightarrow{\sigma} \gamma \to 0$$

is exact in $\mathrm{Idm}(R)$, where the domains of α, β, and γ are A, B, and C respectively. One sees that $\mathrm{End}(R)$ as well as $\mathrm{Idm}(R)$ is abelian. We observe that in $\mathrm{Idm}(R)$, the functors $\alpha \mapsto \ker \alpha$, $\alpha \mapsto \operatorname{coker} \alpha$ are naturally equivalent and are, as a consequence of the snake diagram, exact.

Definition 1. Call a long exact sequence

$$S: 0 \to A \xrightarrow{\kappa} B_{n-1} \to \ldots \to B_0 \xrightarrow{\sigma} C \to 0$$

quasi-splitting for α *and* γ in $\mathrm{End}(R)$ if there exist R-homomorphisms θ and τ such that $\theta\kappa = \alpha$ and $\sigma\tau = \gamma$. Clearly, this depends only on the extension class of S.

Definition 2. Define

$$\mathrm{Qsp}^0(\gamma, \alpha) = \{\lambda \in \mathrm{Hom}(C, A) | \alpha\lambda = \lambda\gamma = 0\}.$$

For $n > 0$, $\mathrm{Qsp}^n(\gamma, \alpha)$ is the subset of those elements of $\mathrm{Ext}^n(C, A)$ represented by long exact sequences that quasi-split wrt α and γ.

One has $\mathrm{Qsp}^n(0_C, 0_A) = \mathrm{Ext}^n(C, A)$ where 0_A is the zero-endomorphism of A. For $n = 1$, let E_1 be a short exact sequence quasi-splitting for α and γ, and let E_2 be congruent to E_1; then clearly E_2 is quasi-splitting for α and γ.
Given $\eta: \alpha \to \alpha'$, where the domain of α' is A', the associated pushout diagram implies that ηE_1 is quasi-splitting for α' and γ.

Received October 2, 1970, and in revised form, January 5, 1971. This research was supported in part by NSF Grant GP6483.

2. Results.

PROPOSITION 1. *For* $n \geqq 0$,

$$\operatorname{Qsp}^n(0_C, \alpha) = \ker[\operatorname{Ext}^n(C, A) \xrightarrow{\alpha_*} \operatorname{Ext}^n(C, A)], \textit{ and}$$

$$\operatorname{Qsp}^n(\gamma, 0_A) = \ker[\operatorname{Ext}^n(C, A) \xrightarrow{\gamma^*} \operatorname{Ext}^n(C, A)].$$

Proof. Recall that $\operatorname{Ext}^n(C, A) \approx \operatorname{H}^n(\underline{C}, A)$ where $\underline{C} = (C, \partial)$ is a projective resolution of C. Let $S: 0 \to A \to B_{n-1} \to \ldots \to B_0 \to C \to 0$ correspond to $[u]$ through this isomorphism; we have a push-out square

$$\begin{array}{ccc} \partial C_n & \longrightarrow & C_{n-1} \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle w} \\ A & \xrightarrow{\nu} & B_{n-1} \end{array}$$

where v restricts u, whence if $\alpha_*[u] = 0$, there is a map $\tau: C_{n-1} \to A$ such that $\tau\partial = \alpha u$ and we construct $\nu': B_{n-1} \to A$ with $\nu'\nu = \alpha$, $\nu' w = \tau$. This proves the first equality. The second is proved similarly.

COROLLARY 1. *For* $n \geqq 0$,

$$\operatorname{Qsp}^n(\gamma, \alpha) = \ker[\operatorname{Ext}^n(C, A) \xrightarrow{\gamma^*} \operatorname{Ext}^n(C, A)] \cap \ker[\operatorname{Ext}^n(C, A) \xrightarrow{\alpha_*} \operatorname{Ext}^n(C, A)]$$

COROLLARY 2. *For* $n \geqq 0$, $\operatorname{Qsp}^n(\text{-}, \text{-})$ *is an additive bifunctor on* $\operatorname{End}(R)$ *to the category of abelian groups. It is contravariant in the first variable and covariant in the second variable.*

In general, Qsp is not half-exact. The following is an example due to Whaples.

Let (a), (b), (b') and (c) be cyclic groups generated a, b, b' and c of orders 2^6, 2^4, 2^2 and 2 respectively. Let $B = (a) \oplus (b')$, $X = (a) \oplus (b) \oplus (c)$, and let A and D be the subgroups of $(a) \oplus (b)$ and $(b) \oplus (c)$ generated by $(2a, -b)$ and $(2b, -c)$ respectively. The following typical 9-diagram in which i, k send the announced generators to $(0, 2b, -c)$, $(2a, -b')$, respectively and μ sends $a \mapsto a$, $b \mapsto b'$, $c \mapsto 2c'$ stipulates the extension E' and the map σ.

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & A & = & A & & \\ & & & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} & & \\ E': 0 & \longrightarrow & D & \xrightarrow{i} & X & \xrightarrow{\mu} & B & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \sigma} & & \\ E'': 0 & \longrightarrow & D & \longrightarrow & Y & \longrightarrow & C & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

https://doi.org/10.4153/CJM-1971-053-9 Published online by Cambridge University Press

By a suitable choice of j, the bottom row E'' coincides with any given extension satisfying $\sigma^*E'' = E'$. (This follows by the 9-lemma). As the reader may easily verify, the extension E' quasi-splits for the map $\delta: D \to D$ obtained on multiplying by 2. On the other hand, E'' does not. Indeed, $j : A \to X$ is of the form $k + f$, where f is a homomorphism $A \to D$ followed by inclusion. Consequently, if $\beta' : D \oplus A \to X$ is the map induced by inclusion on each summand, a quasi-splitting of E'' would imply the existence of a homomorphism $\theta: X \to D$ such that the image of $\theta\beta'$ is contained in δD. This is clearly impossible. Finally k^*E' splits. We therefore conclude that Qsp′ is not half-exact.

PROPOSITION 2. *Let δ be an object and*

$$E: 0 \to \alpha \xrightarrow{\kappa} \beta \xrightarrow{\sigma} \gamma \to 0$$

be a short exact sequence in Idm(R). *Then the following two sequences, starting with* 0 *and* $n = 0$, *are exact.*

$$(1) \qquad \ldots\ldots \to \mathrm{Qsp}^n(\gamma, \delta) \xrightarrow{\sigma^*} \mathrm{Qsp}^n(\beta, \delta) \xrightarrow{\kappa^*} \mathrm{Qsp}^n(\alpha, \delta) \xrightarrow{E^*} \mathrm{Qsp}^{n+1}(\gamma, \delta) \to \ldots\ldots$$

$$(2) \qquad \ldots\ldots \to \mathrm{Qsp}^n(\delta, \alpha) \xrightarrow{\kappa_*} \mathrm{Qsp}^n(\delta, \beta) \xrightarrow{\sigma_*} \mathrm{Qsp}^n(\delta, \gamma) \xrightarrow{E_*} \mathrm{Qsp}^{n+1}(\delta, \alpha) \to \ldots\ldots$$

where E^* and E_* are natural.

Proof. This follows from the corresponding exact sequence for Ext and the exactness of ker.

Define an object $\rho \in \mathrm{Idm}(R)$ to be I-projective if $\mathrm{Qsp}^0(\rho, \text{-})$ is exact. An element of $\mathrm{Qsp}^0(\rho, \alpha)$ is determined by a map coker $\to$ ker α. Because of the equivalence of ker and coker it follows that ρ is I-projective if and only if ker ρ is projective in M(R). One may verify that if ρ is I-projective, then ρ is projective in Idm(R). The converse is not true: let $\rho = 0 \oplus 1$ where the domain of ρ is $\mathbf{Z} \oplus \mathbf{Z}_2$, and consider the epimorphism $\mathbf{Z}_4 \to \mathbf{Z}_2$ and $\mathbf{Z} \oplus \mathbf{Z}_2$, where $\mathbf{Z}_4$, $\mathbf{Z}_2$ are subjected to the identical automorphisms.

PROPOSITION 3. End(R) *and* Idm(R) *have enough projectives.*

Proof. Let $g: S \to S$ be a map of sets and $F(g): F(S) \to F(S)$ the induced map of the associated free R-module. Each set map $\eta: g \to B$ determines a unique R-linear extension $\bar{\eta}: F(g) \to B$. Thus, if $\bar{\eta}$ is given and $k: \alpha \to \beta$ is surjective in End$R$, η lifts to $\xi: g \to \alpha$ with $\xi \circ k = \eta$ and consequently $F(\xi) \circ k = F(\eta)$. It follows that $F(g)$ is projective. For g in End$R$, the identical map induces the surjective $F(g) \to g$. Consequently End$R$ has enough projectives. Finally, if g is in Idm$R$ then so is $F(g)$. Consequently, Idm$R$ also has enough projectives.

COROLLARY 3. Idm(R) *has enough I-projectives.*

In End(R), it follows that the satellites of Qsp^0 are not Qsp's, by Whaples's counterexample.

THEOREM. *Let there be given a family of contravariant functors* $\mathrm{Qs}^n(\text{-})$, $n \geqq 0$, *from the category* $\mathrm{Idm}(R)$ *into the category of abelian groups. For each n and each exact sequence*

$$E: 0 \rightarrow \alpha \xrightarrow{\kappa} \beta \xrightarrow{\sigma} \gamma \rightarrow 0$$

in $\mathrm{Idm}(R)$, *let there be given a homomorphism* E^n: $\mathrm{Qs}^n(\alpha) \rightarrow \mathrm{Qs}^{n+1}(\gamma)$ *which is natural. Suppose that for a fixed object* δ, *and a short exact sequence E given as above, in* $\mathrm{Idm}(R)$

$$\mathrm{Qs}^0(\alpha) = \mathrm{Qsp}^0(\alpha, \delta) \textit{ for all } \alpha \textit{ in } \mathrm{Idm}(R)$$
$$\mathrm{Qs}^n(\rho) = 0 \textit{ for } n > 0 \textit{ and all I-projectives.}$$

and the following sequence is exact

$$0 \rightarrow \mathrm{Qs}^0(\gamma) \rightarrow \ldots\ldots \rightarrow \mathrm{Qs}^n(\gamma) \xrightarrow{\sigma^*} \mathrm{Qs}^n(\beta) \xrightarrow{\kappa^*} \mathrm{Qs}^n(\alpha) \xrightarrow{E^n} \mathrm{Qs}^{n+1}(\gamma) \rightarrow \ldots\ldots$$

Then there is a natural equivalence ψ^n: $\mathrm{Qs}^n(\text{-}) \rightarrow \mathrm{Qsp}^n(\text{-}, \delta)$ *for all n and E, and* $\psi^{n+1}E^n = E^*\psi^n$.

Proof. The argument in [**2**, p. 99] generalizes immediately, since Qsp^n, $n > 0$, are zero for the class of I-projectives.

In particular, for α, β, γ and δ being zero-endomorphisms, these are just part of the well-known results for Ext.

Acknowledgements. The author wishes to thank M. Adelman, T. C. Kuo, and G. Whaples for their suggestions and comments.

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