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ON THE CLASS NUMBER OF A UNIT LATTICE OVER A RING OF REAL QUADRATIC INTEGERS

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§1. Introduction

Let K be a totally real algebraic number field. In a positive definite quadratic space over K a lattice E_n is called a unit lattice of rank n if E_n has an orthonormal basis $\{e_1, \cdots, e_n\}$. The class number one problem is to find n and K for which the class number of E_n is one. Dzewas ([1]), Nebelung ([3]), Pfeuffer ([6], [7]) and Peters ([5]) have settled this problem. The present state of this problem is: If $n \geqq 3$, then the class number of E_n is one if and only if "$K = \boldsymbol{Q}$, $n \leqq 8$", "$K = \boldsymbol{Q}(\sqrt{2})$, $n \leqq 4$", "$K = \boldsymbol{Q}(\sqrt{5})$, $n \leqq 4$", "$K = \boldsymbol{Q}(\sqrt{17})$, $n = 3$", "$K = K^{(49)}$, $n = 3$" or "$K = K^{(148)}$, $n = 3$", where $\boldsymbol{Q}$ is the rational number field and $K^{(49)}$ (resp. $K^{(148)}$) is the unique totally real cubic number field with discriminant 49 (resp. 148). The class number two problem has been studied by Pohst ([10]), who gets a nearly complete result for $n \geqq 4$: If $n \geqq 4$, then the class number of E_n is two only if "$K = K^{(49)}$, $n = 4$" or "$K = \boldsymbol{Q}(\sqrt{5})$, $n = 5, 6, 7$", and the class number of E_n is two in the first two cases. Pfeuffer ([8]) has shown that the class number of E_n is three for $K = \boldsymbol{Q}(\sqrt{5})$ and $n = 6$. In the special case that K is a real quadratic field, it remains to consider the class number of E_3 over K ($\neq \boldsymbol{Q}(\sqrt{2})$, $\boldsymbol{Q}(\sqrt{5})$, $\boldsymbol{Q}(\sqrt{17})$).

All former proofs of the "only if" assertions and nearly all proofs of the class number one (or two) for special fields K and special n use the Siegel Mass Formula. On the other hand we have another method by which Kneser ([2]) has found the class number of E_n for $\boldsymbol{Q}$. Using this method Salamon ([11]) has found the first result for $\boldsymbol{Q}(\sqrt{3})$. In this paper we shall prove the following theorem by using the Kneser method.

THEOREM. *In the case of real quadratic fields, the class number of* E_n *(with* $n \geqq 3$*) is two if and only if*

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$$\begin{array}{ll} \boldsymbol{Q}(\sqrt{2})\,, & n=5\,, \\ \boldsymbol{Q}(\sqrt{3})\,, & n=3\,, \\ \boldsymbol{Q}(\sqrt{5})\,, & n=5\,, \\ \boldsymbol{Q}(\sqrt{13})\,, & n=3\,, \\ \boldsymbol{Q}(\sqrt{33})\,, & n=3\,, \\ \boldsymbol{Q}(\sqrt{41})\,, & n=3\,. \end{array}$$

The class number of E_n is a monotone increasing function of n for a fixed K ([4], 105: 1). In Section 2 we discuss some properties of adjacent lattices. In Section 3 we find some special adjacent lattices to E_n and prove that the class number of E_n is more than two unless K is one of the exceptional eight fields (cf. Proposition 8). In Section 4 we treat the above exceptional cases and determine the class number by using the Kneser method. The notation used in this paper will generally be those of [4].

§2. Adjacent lattices

Let p be an odd prime number. Put

$$A_p^n = \left\{(a_1, \cdots, a_n) \in \boldsymbol{Z}^n;\ \sum_{i=1}^{n} a_i^2 \equiv 0 \bmod p,\ (a_1, \cdots, a_n) \not\equiv (0, \cdots, 0) \bmod p\right\},$$

where $\boldsymbol{Z}$ is the ring of rational integers. We define an equivalence relation $\sim$ on A_p^n: $(a_1, \cdots, a_n) \sim (b_1, \cdots, b_n)$ if and only if there is a permutation $\{1', 2', \cdots, n'\}$ of $\{1, 2, \cdots, n\}$ and an integer c prime to p such that $b_i^2 \equiv c a_{i'}^2 \bmod p$ for all i. In each equivalence class we can choose a representative $(a_1, \cdots, a_n)$ satisfying

$$0 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_n$$

and

$$\sum_{i=1}^{n} a_i^2 \leqq \sum_{i=1}^{n} b_i^2$$

for all $(b_1, \cdots, b_n)$ in the class. By R_p^n we denote the set of the above representatives. Let $(a_1, \cdots, a_n)$ and $(b_1, \cdots, b_n)$ be in the same class and $(a_1, \cdots, a_n) \in R_p^n$. We define the norm and the type of $(b_1, \cdots, b_n)$ (or the class):

$$N(b_1, \cdots, b_n) = \frac{1}{p} \sum_{i=1}^{n} a_i^2\,,$$

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$$T(b_1, \cdots, b_n) = \min\left\{\sum_{i=1}^{n} c_i^2;\ \sum_{i=1}^{n} c_i b_i \equiv 0 \bmod p,\ (c_1, \cdots, c_n) \neq (0, \cdots, 0)\right\}.$$

It is easy to prove the following

PROPOSITION 1. *The number of the equivalence classes of the specified type T in A_p^3 is as follows*:

	$T=1$	$T=2$	$T=3$	$T\geqq 4$
$p=3$	0	1	0	0
$p\equiv 1 \bmod 24$	1	1	1	$(p-25)/24$
$p\equiv 5 \bmod 24$	1	0	0	$(p-5)/24$
$p\equiv 7 \bmod 24$	0	0	1	$(p-7)/24$
$p\equiv 11 \bmod 24$	0	1	0	$(p-11)/24$
$p\equiv 13 \bmod 24$	1	0	1	$(p-13)/24$
$p\equiv 17 \bmod 24$	1	1	0	$(p-17)/24$
$p\equiv 19 \bmod 24$	0	1	1	$(p-19)/24$
$p\equiv 23 \bmod 24$	0	0	0	$(p+1)/24$

Moreover if the type is one or two, then the norm is one.

Let $K = \boldsymbol{Q}(\sqrt{D})$ be a real quadratic field over $\boldsymbol{Q}$ with a square-free rational integer D and $\mathfrak{o}$ be the ring of integers in K. By $\operatorname{gen} L$ we denote the genus containing a lattice L in a quadratic space V over K. A lattice L is said to be even if $Q(L) \subset 2\mathfrak{o}$. For vectors $x_1, \cdots, x_m$ in V, $[x_1, \cdots, x_m]$ denotes the lattice generated by $\{x_1, \cdots, x_m\}$ over $\mathfrak{o}$.

Let $\mathfrak{a}$ be a non-zero ideal of $\mathfrak{o}$ and L be a unimodular lattice in V. For $x \in \mathfrak{a}^{-1}L$ such that $Q(x) \in \mathfrak{o}$, we put

$$L(x) = \mathfrak{o}x + \{z \in L;\ B(x, z) \in \mathfrak{o}\},$$

which is called an $\mathfrak{a}$-adjacent lattice to L (Cf. [2]). The following Lemmas 1–4 are valid.

LEMMA 1. *Let L be a unimodular lattice and $L(x)$ be an $\mathfrak{a}$-adjacent lattice to L. Then $L(x)$ is unimodular. If $\mathfrak{a}$ is prime to $2\mathfrak{o}$ or $L(x)_\mathfrak{p} \simeq L_\mathfrak{p}$ for any dyadic spot $\mathfrak{p}$, then an $\mathfrak{a}$-adjacent lattice to L belongs to* $\operatorname{gen} L$.

LEMMA 2. *Let L be a unimodular lattice in V, and $L(x)$ and $L(x')$ two $\mathfrak{a}$-adjacent lattices to L. If $B(x, x') \in \mathfrak{o}$ and $x - \gamma x' \in L$ for some $\gamma \in \mathfrak{o}$ prime to $\mathfrak{a}$, then $L(x) = L(x')$.*

LEMMA 3. *Let L be a unimodular lattice in V and $L(x)$ and $L(x')$ be two $\mathfrak{a}$-adjacent lattices to L. If $x' = \sigma x$ for some σ in $O(L)$, then $L(x) \simeq L(x')$.*

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LEMMA 4. *Let L be a unimodular lattice in V and $L(x)$ be an $\mathfrak{a}$-adjacent lattice to L. If there is a vector w in L such that $2/Q(x-w)$ and $(Q(x)-Q(w))/Q(x-w)$ are in $\mathfrak{a}$, then $L(x) \simeq L$.*

LEMMA 5. *Let p be an odd prime number dividing D and $\mathfrak{p}$ a prime ideal dividing p. Then a $\mathfrak{p}$-adjacent lattice to E_n is isometric to some $E_n(x)$ with $x = (\sqrt{D}/p)\sum_{i=1}^n a_i e_i$ and $(a_1, \cdots, a_n) \in R_p^n \cup \{(0, \cdots, 0)\}$.*

Proof. Note that $p\mathfrak{o} = \mathfrak{p}^2$ and $\mathfrak{o}/\mathfrak{p} \simeq Z/pZ$. Take an element $z = \sum_{i=1}^n \alpha_i e_i \in \mathfrak{p}^{-1}E_n$ with $Q(z) \in \mathfrak{o}$. We can find $a_i \in Z$ such that

$$\sqrt{D}\,\alpha_i \equiv \frac{D}{p} a_i \bmod \mathfrak{p}$$

since $\sqrt{D}\,\alpha_i \in \mathfrak{o}$ and D/p is prime to $\mathfrak{p}$. Put $x = (\sqrt{D}/p)\sum_{i=1}^n a_i e_i$. Then $x \in \mathfrak{p}^{-1}E_n$ and $z - x \in E_n$. We have $\sum_{i=1}^n a_i^2 \equiv 0 \bmod p$ since $Q(z) \in \mathfrak{o}$. Hence $Q(x) \in \mathfrak{o}$ and $(a_1, \cdots, a_n) \in A_p^n$ if $x \notin E_n$. Since $-2B(x, z) = Q(z-x) - Q(x) - Q(z) \in \mathfrak{o}$ and $B(x, z) \in \mathfrak{p}^{-2}$, we have $B(x, z) \in \mathfrak{o}$. By Lemma 2 we have $E_n(z) = E_n(x)$. Considering the structure of $O(E_n)$, we may have $(a_1, \cdots, a_n) \in R_p^n \cup \{(0, \cdots, 0)\}$ by Lemmas 2 and 3.

§ 3. Special adjacent lattices to E_n

PROPOSITION 2. *Let $b_1, \cdots, b_n$ be positive rational integers satisfying $\sum_{i=1}^n b_i^2 = D$. Assume $n \geqq 3$. Consider the lattice $\bar{A} = E_n(z) = [z] \perp A$ with $z = (1/\sqrt{D})\sum_{i=1}^n b_i e_i$. Then*

1) $\bar{A} \in \operatorname{gen} E_n$,
2) *A is even if $n \equiv b_1 \equiv \cdots \equiv b_n \equiv 1 \bmod 2$,*
3) *$A \in \operatorname{gen} E_{n-1}$ unless $n \equiv b_1 \equiv \cdots \equiv b_n \equiv 1 \bmod 2$,*
4) *$A \simeq E_2$ if $n = 3$, $D \equiv 1 \bmod 4$ and $b_i = b_j$ for some $i < j$,*
5) *$1 \notin Q(A)$ unless $n = 3$, $D \equiv 1 \bmod 4$ and $b_i = b_j$ for some $i < j$.*

Proof. (i) Suppose that D is odd. By Lemma 1 we have $\bar{A} \in \operatorname{gen} E_n$ Let $\mathfrak{p}$ be a dyadic spot on K. We can assume that b_1 is odd. Put $v_i = b_1 e_i - b_i e_1$ for $i = 2, 3, \cdots, n$. Then $A_\mathfrak{p} = [v_2, \cdots, v_n]_\mathfrak{p}$ with $B(v_i, v_j) \in Z$ and $\det(B(v_i, v_j)) = b_1^{2(n-2)}D \equiv 1 \bmod 2$. The assertion (2) is clear. We shall show (3). Consider a lattice $M = [v_2, \cdots, v_n]$ over Z. Then $M_2 \simeq \langle 1\rangle \perp \cdots \perp \langle 1\rangle \perp \langle D\rangle$ or $M_2 \simeq \langle 1\rangle \perp \cdots \perp \langle 1\rangle \perp \langle D\rangle \perp \langle D\rangle \perp \langle D\rangle$ since M is not even and the Hasse symbol of M_2 takes the value $+1$, where M_2 is the $2Z$-completion of M. So $A_\mathfrak{p} \simeq E_{n-1\mathfrak{p}}$ since $\mathfrak{o}_\mathfrak{p} \supset Z_2$ and $\sqrt{D} \in K$. By Lemma 1 we have the assertion (3).

(ii) Suppose that D is even. We can assume that b_1 and b_2 are odd. Let $\mathfrak{p}$ be dyadic. Then $A_\mathfrak{p} = [b_1 z - \sqrt{D}e_1, v_3, \cdots, v_n]_\mathfrak{p}$ with $\det(B(v_i, v_j)) = b_1^{2(n-3)}(D - b_2^2) \equiv 1 \bmod 4$. Thus $A_\mathfrak{p} \simeq [v_3, \cdots, v_n]_\mathfrak{p} \perp \langle D - b_2^2 \rangle$. By a similar argument as in (i) we have $A_\mathfrak{p} \simeq E_{n-1\mathfrak{p}}$. By Lemma 1 we have $A \in \operatorname{gen} E_{n-1}$, and so $\bar{A} \in \operatorname{gen} E_n$.

(iii) Suppose that $n = 3$, $D \equiv 1 \bmod 4$ and $b_1 = b_2$. Thus $b_3 \equiv 1 \bmod 2$. Take f and g in $\boldsymbol{Z}$ such that $2b_2 f - b_3 g = 1$. Put

$$w_1 = -(b_3 f + b_2 g)z + f\sqrt{D}e_3 + \tfrac{1}{2}(1 + g\sqrt{D})e_1 + \tfrac{1}{2}(-1 + g\sqrt{D})e_2$$

and $w_2 = w_1 - e_1 + e_2$. Then $A = [w_1] \perp [w_2] \simeq E_2$.

(iv) We shall show the assertion (5). Any non-zero vector $u \in A$ can be written as $u = -az + \sum_{i=1}^n (c_i + d_i\sqrt{D})e_i$ with $a = \sum_{i=1}^n b_i d_i \in \boldsymbol{Z}$, $\sum_{i=1}^n b_i c_i = 0$, $|a| \leqq \frac{1}{2}D$, $c_i \in \frac{1}{2}\boldsymbol{Z}$, $d_i \in \frac{1}{2}\boldsymbol{Z}$ and $c_i - d_i \in \boldsymbol{Z}$ for all i. Thus

$$\begin{aligned} Q(u) &= \sum_{i=1}^n c_i^2 + D\sum_{i=1}^n d_i^2 - a^2 + 2\sqrt{D}\sum_{i=1}^n c_i d_i \\ &= \sum_{i=1}^n c_i^2 + \sum_{i<j}(b_i d_j - b_j d_i)^2 + 2\sqrt{D}\sum_{i=1}^n c_i d_i\,. \end{aligned}$$

If the number of the pairs (i, j) such that $b_i d_j - b_j d_i \neq 0$ and $i < j$ is less than $n - 1$, then $b_i d_j - b_j d_i = 0$ for all i and j. Hence $d_1/b_1 = \cdots = d_n/b_n = c$ for some $c \in \boldsymbol{Q}$. Since the g.c.d. of b_i's is one, we have $c \in \frac{1}{2}\boldsymbol{Z}$ or $c \in \boldsymbol{Z}$ according as $D \equiv 1 \bmod 4$ or not. Thus $a = \sum_{i=1}^n b_i d_i = c\sum_{i=1}^n b_i^2 = cD$. This implies $c = 0$ and $a = d_1 = \cdots = d_n = 0$. Hence $c_i \in \boldsymbol{Z}$ for all i and so $\sum_{i=1}^n c_i^2 \geqq 2$. This shows $Q(u) \neq 1$. Suppose that the number of the pairs (i, j) such that $b_i d_j - b_j d_i \neq 0$ and $i < j$ is not less than $n - 1$. If all d_i's are in $\boldsymbol{Z}$, then $\sum_{i<j}(b_i d_j - b_j d_i)^2 \geqq n - 1 \geqq 2$, so $Q(u) \neq 1$. Thus we may assume that $D \equiv 1 \bmod 4$ and $d_{i'} \notin \boldsymbol{Z}$ for some i'. Thus $c_{i'} \notin \boldsymbol{Z}$. Then $\sum_{i=1}^n c_i^2 \geqq \frac{1}{2}$ since $b_1 b_2 \cdots b_n \neq 0$. Hence

$$\sum_{i=1}^n c_i^2 + \sum_{i<j}(b_i d_j - b_j d_i)^2 \geqq \tfrac{1}{2} + \tfrac{1}{4}(n-1) = \tfrac{1}{4}(n+1) \geqq 1$$

and the equality holds only when $n = 3$ and $\sum_{i=1}^n c_i^2 = \frac{1}{2}$. This case occurs only when $n = 3$ and $b_i = b_j$ for some $i < j$ since $\sum_{i=1}^n b_i c_i = 0$. But this is excluded.

PROPOSITION 3. *Let $D \not\equiv 1 \bmod 4$. Let p be an odd prime dividing D. Consider the lattice $B = E_3(y)$ with $y = (\sqrt{D}/p)\sum_{i=1}^3 a_i e_i$ and $(a_1, a_2, a_3) \in R_p^3$. Then*

(1) $B \in \text{gen}\, E_3$,

(2) $B \simeq E_1 \perp B'$ *and* $1 \notin Q(B')$ *if* $D = p = \sum_{i=1}^3 a_i^2$ *or if* $T(a_1, a_2, a_3) = 1$,

(3) $1 \notin Q(B)$ *if* $T(a_1, a_2, a_3) \geqq 2$ *and unless* $D = p = \sum_{i=1}^3 a_i^2$.

Proof. By Lemma 1 we have $B \in \text{gen}\, E_3$. Suppose that $T(a_1, a_2, a_3) \geqq 2$ and $Q(u) = 1$ for some $u \in B$. We can write $u = ay + \sum_{i=1}^3 (c_i + d_i\sqrt{D})e_i$ where $a \in Z$, $c_i \in Z$, $d_i \in Z$, $\sum_{i=1}^3 a_i c_i \equiv 0 \bmod p$ and $|a| < \frac{1}{2}p$. Then

$$1 = Q(u) = \sum_{i=1}^3 c_i^2 + \frac{D}{p}\frac{1}{p}\sum_{i=1}^3 (aa_i + pd_i)^2 + \frac{2\sqrt{D}}{p}\sum_{i=1}^3 c_i(aa_i + pd_i)\,.$$

Hence we have $\sum_{i=1}^3 c_i^2 = 0$ and $D = p = \sum_{i=1}^3 (aa_i + pd_i)^2$ since $T(a_1, a_2, a_3) \geqq 2$. Thus the assertion (3) holds. Now let $D = p = \sum_{i=1}^3 a_i^2$. Then $B = [y] \perp B'$ and $Q(B') \not\ni 1$ by (5) of Proposition 2. If $T(a_1, a_2, a_3) = 1$, then $a_1 = 0$, $D \neq p$ and $B = [e_1] \perp B'$. Similarly we have $1 \notin Q(B')$.

PROPOSITION 4. *Let* $D \equiv 1 \bmod 4$ *and* p *be a prime dividing* D. *Consider the lattice* $B = E_n(y)$ *with* $y = (\sqrt{D}/p)\sum_{i=1}^n a_i e_i$ *and* $(a_1, \cdots, a_n) \in R_p^n$. *Assume that* $n \geqq 3$. *Put* $T = T(a_1, \cdots, a_n)$ *and* $N = N(a_1, \cdots, a_n)$. *Then*

(1) $B \in \text{gen}\, E_n$,

(2) $B \simeq E_1 \perp B'$ *with* $1 \notin Q(B')$ *if* $n = 3$, $D \neq p$ *and* $T = 1$,

(3) $1 \notin Q(B)$ *and* $2 \in Q(B)$ *if* $D \neq p$ *and* $T = 2$,

(4) $1 \notin Q(B)$ *and* $2 \in Q(B)$ *if* $D \neq p$ *and* $T \geqq 3$,

(5) $B \simeq E_3$ *if* $n = 3$, $D = p$ *and* $T \leqq 2$,

(6) $B \simeq E_1 \perp B'$ *with* $1 \notin Q(B')$ *if* $D = p$, $N = 1$ *and* $T \geqq 3$,

(7) $1 \notin Q(B)$ *if* $D = p$, $N = 2$ *and* $T \geqq 3$,

(8) $1 \notin Q(B)$ *if* $D = p$, $N \geqq 3$ *and* $T \geqq 2$,

(9) $2 \notin Q(B)$ *if* $n = 3$, $D = p$, $N \geqq 3$ *and* $T \geqq 3$,

(10) $2 \in Q(B)$ *if* $D = p$ *with* $N = 2$ *or if* $T = 2$.

Proof. By Lemma 1 we have (1). (10) holds trivially. Take a non-zero vector u in B and write

$$u = ay + \sum_{i=1}^n (c_i + d_i\sqrt{D})e_i$$

with $a \in Z$, $|a| < \frac{1}{2}p$, $c_i \in \frac{1}{2}Z$, $d_i \in \frac{1}{2}Z$, $c_i - d_i \in Z$ and $2\sum_{i=1}^n a_i c_i \equiv 0 \bmod p$. Then

$$Q(u) = X + Y + \frac{2\sqrt{D}}{p}\sum_{i=1}^n c_i(aa_i + pd_i)\,,$$

where $X = \sum_{i=1}^n c_i^2$ and $Y = (D/p)(1/p)\sum_{i=1}^n (aa_i + pd_i)^2$. If $Y = 0$, then

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$a = d_i = 0$ for all i, so $c_i \in Z$ for all i. Thus $X \geqq T$. If $X = 0$ and $Y \neq 0$, then $c_i = 0$ for all i and $Y \geqq DN/p$. If $X \neq 0$ and $Y \neq 0$, then $X + Y \geqq (T/4) + (D/p)(N/4)$. Thus (3), (7), (8) and the half of (4) hold. Now suppose that $D \neq p$ and $T \geqq 3$ or that $D = p$, $T \geqq 3$, $N \geqq 3$ and $n = 3$. Thus $X \geqq \frac{3}{4}$ and $Y \geqq \frac{3}{4}$. If $X = \frac{3}{4}$ with $Y = \frac{5}{4}$ or $X = \frac{5}{4}$ with $Y = \frac{3}{4}$, then we have $2 \equiv 4X - 4Y \equiv \sum_{i=1}^{n}(2c_i)^2 - \sum_{i=1}^{n}(2d_i)^2 \equiv 0 \bmod 4$, which is a contradiction. If $X = Y = 1$, then $D = p$ and $\sum_{i=1}^{n} c_i^2 = 1$. Thus $D = p$ and $n \geqq 4$, which is a contradiction. Hence (9) and the rest of (4) hold. If $n = 3$, $D \neq p$ and $T = 1$, then $a_1 = 0$ and $B = [e_1] \perp B'$ with $B' = [e_2, e_3](y)$. Hence we have $1 \notin Q(B')$ by a direct calculation. So the assertion (2) holds. Assume that $n = 3$, $D = p$ and $T \leqq 2$. Then $N = 1$. If $T = 1$, then $B = [e_1] \perp [y] \perp [y'] \simeq E_3$ with $y' = (1/\sqrt{p})(a_3e_2 - a_2e_3)$. If $T = 2$, then $B \simeq E_3$ by Proposition 2, (4). Thus (5) holds. Finally (6) follows from Proposition 2, (5).

PROPOSITION 5. *Let* $D \equiv 3 \bmod 4$. *Consider the lattice* $C = E_3(x) = [e_3] \perp C'$ *with* $x = \frac{1}{2}(e_1 + \sqrt{D}\, e_2)$. *Then*

(1) $C \in \operatorname{gen} E_3$,

(2) $1 \notin Q(C')$ *if* $D > 3$,

(3) C' *is even if and only if* $D \equiv 7 \bmod 8$.

Proof. We have

$$C' = [x, 2e_2] \simeq \begin{pmatrix} \frac{1}{4}(D+1) & \sqrt{D} \\ \sqrt{D} & 4 \end{pmatrix}.$$

Let $\mathfrak{p}$ be dyadic. If $D \equiv 3 \bmod 8$, then C' is not even and $C'_{\mathfrak{p}} \simeq \langle \frac{1}{4}(D+1) \rangle \perp \langle \frac{1}{4}(D+1) \rangle \simeq E_{2\mathfrak{p}}$ since $3 \in K_{\mathfrak{p}}^2$. If $D \equiv 7 \bmod 8$, then C' is even and $C'_{\mathfrak{p}} \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so $C_{\mathfrak{p}} \simeq \langle 1 \rangle \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \simeq E_{3\mathfrak{p}}$ since $-1 \in K_{\mathfrak{p}}^2$. Thus (1) and (3) are proved by Lemma 1. It is easy to show (2) directly.

PROPOSITION 6. *Let* $D \equiv 5 \bmod 12$. *Consider the lattice* $G = E_3(x) = [e_3] \perp G'$ *with* $x = \frac{1}{3}(e_1 + D\sqrt{D}\, e_2)$. *Then*

(1) $G \in \operatorname{gen} E_3$ and $G' \in \operatorname{gen} E_2$,

(2) $1 \notin Q(G')$ and $2 \notin Q(G')$ if $D \geqq 29$.

Proof. (1) follows from Lemma 1. We have $G' = [x, 3e_2]$. It is easy to show (2) by a direct calculation.

PROPOSITION 7. *Let* D *be a prime* $p \equiv 1 \bmod 12$. *Then the number of the classes in* A_p^3 *whose type is six is one or zero according as*

$p \equiv 1 \bmod 24$ or $p \equiv 13 \bmod 24$. *Let* $(a_1, a_2, a_3) \in R_p^3$ *with* $T = T(a_1, a_2, a_3) \geqq 3$ *and* $N(a_1, a_2, a_3) = 2$. *Put* $x = (1/\sqrt{p})(a_1e_1 + a_2e_2 + a_3e_3)$. *If there are two vectors* u_1 *and* u_2 *in* $B = E_3(x)$ *such that* $Q(u_1) = Q(u_2) = 2B(u_1, u_2) = 2$, *then* $T = 3$ *or* $T = 6$.

Proof. Let $(b_1, b_2, b_3) \in A_p^3$ whose type is six. Thus we may assume that $b_3 = 2b_1 + b_2$. Hence $0 \equiv \sum_{i=1}^3 b_i^2 \equiv 2(b_1 + b_2)^2 + 3b_1^2 \bmod p$. So $(-6/p) = 1$, i.e., $p \equiv 1 \bmod 24$. If $p \equiv 1 \bmod 24$, then there is an integer c such that $c^2 \equiv -6 \bmod p$. Hence $\pm c(b_1 + b_2) \equiv 3b_1 \bmod p$. Thus $(b_1, b_2, b_3) \sim (c, 3 - c, 3 + c)$, i.e., there is one and only one class whose type is six. We shall show $T = 3$ or $T = 6$. Suppose that $T \neq 3$ and $T \neq 6$. Thus $T = 5$ or $T \geqq 7$. Take a vector u in B with $Q(u) = 2$ and write

$$u = ax + \sum_{i=1}^{3} (c_i + d_i\sqrt{p})e_i$$

with $a \in Z$, $|a| < \frac{1}{2}p$, $c_i \in \frac{1}{2}Z$, $d_i \in \frac{1}{2}Z$, $c_i - d_i \in Z$, $2\sum_{i=1}^3 a_ic_i \equiv 0 \bmod p$. Then $Q(u) = X + Y + (2/\sqrt{p})\sum_{i=1}^3 c_i(aa_i + pd_i)$, where $X = \sum_{i=1}^3 c_i^2$ and $Y = (1/p)\sum_{i=1}^3 (aa_i + pd_i)^2$. Hence we have one of the following:

(i) $X = 0$ and $Y = 2$,

(ii) $X = \frac{5}{4}$ and $Y = \frac{3}{4}$,

(iii) $X = \frac{3}{2}$ and $Y = \frac{1}{2}$.

In the case (ii) we have $1 \equiv \sum_{i=1}^3 (2c_i)^2 \equiv \sum_{i=1}^3 (2d_i)^2 \equiv \sum_{i=1}^3 (2aa_i + 2pd_i)^2 = 3p \equiv 3 \bmod 4$. This is a contradiction. In the case (iii) we have $(a_1, a_2, a_3) \sim (c, 3 - c, 3 + c)$ for an integer c with $c^2 + 6 \equiv 0 \bmod p$ by the argument used above since $X = \frac{6}{4}$. Since T must be five we have $c \equiv \pm 1, \pm 2, \pm 3, \pm 6$ or $\pm 9 \bmod p$, which is a contradiction to the fact that p divides $c^2 + 6$. In the case (i) we have $c_i = 0$ and $d_i \in Z$ for all i. Hence we can write $u_1 = ax + \sqrt{p}\sum_{i=1}^3 d_ie_i = (1/\sqrt{p})\sum_{i=1}^3 f_ie_i$ and $u_2 = a'x + \sqrt{p}\sum_{i=1}^3 d_i'e_i = (1/\sqrt{p})\sum_{i=1}^3 f_i'e_i$ with $a, a', d_i, d_i', f_i, f_i' \in Z$. Thus $f_i \equiv aa_i \bmod p$ and $f_i' \equiv a'a_i \bmod p$. Hence $f_if_j' - f_jf_i' \equiv 0 \bmod p$. Since $3p^2 = (2p)^2 - p^2 = \sum_{i=1}^3 f_i^2 \sum_{i=1}^3 f_i'^2 - (\sum_{i=1}^3 f_if_i')^2 = \sum_{i<j} (f_if_j' - f_jf_i')^2$, we have $f_if_j' - f_jf_i' = h_{ij}p = \pm p$ whenever $i \neq j$. Since $0 = f_1(f_2f_3' - f_3f_2') + f_2(f_3f_1' - f_1f_3') + f_3(f_1f_2' - f_2f_1')$, we have $0 = f_1h_{23} + f_2h_{31} + f_3h_{12}$, i.e., $a_1h_{23} + a_2h_{31} + a_3h_{12} \equiv 0 \bmod p$. This implies that $T \leqq 3$. This is a contradiction.

LEMMA 6. *Let* D *be a square-free positive integer. In order that* $D = b_1^2 + b_2^2 + b_3^2 + b_4^2$ *for some positive integers* b_1, b_2, b_3 *and* b_4, *it is necessary and sufficient that* $D \neq 1, 2, 3, 5, 6, 11, 14, 17, 29, 41$.

PROPOSITION 8. *Let $n \geqq 3$. Then the class number of E_n is more than two unless D is one of the following*: 2, 3, 5, 13, 17, 29, 33, 41.

Proof. It is enough to find two lattices L and M in gen E_3 such that $L \neq E_3$, $M \neq E_3$ and $L \neq M$.

(i) Let $D \equiv 2 \bmod 4$. For L we take the lattice A in Proposition 2 if $D = 10$. If $D \neq 10$, then there is an odd prime $q\,(\neq 5)$ dividing D. By Proposition 1 there is an element $(a_1, a_2, a_2) \in R_q^3$ whose type is more than one, for which we consider the lattice B in Proposition 3. Then put $L = B$ if $D \neq 10$. Next take an odd prime p dividing D. If $p \equiv 1 \bmod 4$, then there is an element $(a_1, a_2, a_3) \in R_p^3$ whose type is one, for which we consider the lattice B in Proposition 3. If $p \equiv 3 \bmod 4$, then we can consider the lattice $\bar{A}$ in Proposition 2. Then put $M = B$ or $M = \bar{A}$ according as $p \equiv 1 \bmod 4$ or $p \equiv 3 \bmod 4$. Note that $1 \notin Q(L)$ and $M \simeq E_1 \perp M'$ with $1 \notin Q(M')$.

(ii) Let $D \equiv 3 \bmod 8$. For L we take a lattice $\bar{A} \simeq E_1 \perp A$ with an even lattice A in Proposition 2. For M we take the lattice $C \simeq E_1 \perp C'$ with an odd lattice C' and $1 \notin Q(C')$ in Proposition 5.

(iii) Let $D \equiv 7 \bmod 8$. For L we take a lattice A with $1 \notin Q(A)$ in Proposition 2 and for M we take the lattice $C \simeq E_1 \perp C'$ with $1 \notin Q(C')$ in Proposition 5.

(iv) Let $D \equiv 1 \bmod 4$ and not a prime. If no prime divisor of D is congruent to $7 \bmod 8$, then by Proposition 1 we have two elements $(a_1, a_2, a_3) \in R_p^3$ and $(a_1', a_2', a_3') \in R_q^3$ for some prime divisors p and q of D (possibly $p = q$) such that $T(a_1, a_2, a_3) = 1$ and $T(a_1', a_2', a_3') \geqq 2$ or such that $T(a_1, a_2, a_3) = 2$ and $T(a_1', a_2', a_3') \geqq 3$. For L and M we take the lattice B for (a_1, a_2, a_3) and the lattice B for (a_1', a_2', a_3') in Proposition 4. If D has a prime divisor $p \equiv 7 \bmod 8$, then there is an element $(a_1, a_2, a_3) \in R_p^3$ whose type is more than two, for which we can consider the lattice B with $Q(B) \not\ni 1$ in Proposition 4. Put $L = B$. There are positive integers b_1, b_2 and b_3 such that $b_1^2 + b_2^2 + b_3^2 = D$ since $D \equiv 1 \bmod 4$ and $p \equiv 3 \bmod 4$. We have $b_i \neq b_j$ whenever $i \neq j$ since $(-2/p) = -1$. Hence we can consider a lattice $\bar{A} = E_1 \perp A$ with $1 \notin Q(A)$. Put $M = \bar{A}$.

(v) Let D be a prime $p \equiv 1 \bmod 12$. Since $p = 3a^2 + b^2$ for some positive integers a and b, we can consider the lattice A for (a, a, a, b) in Proposition 2. Put $L = A$. Then $1 \notin Q(L)$ and there are two vectors u_1 and u_2 in L such that $Q(u_1) = Q(u_2) = 2B(u_1, u_2) = 2$. First suppose $p \equiv 1 \bmod 24$. Then there are at least two elements (a_1, a_2, a_3)

and (a'_1, a'_2, a'_3) in R_p^3 whose types are more than three by Proposition 1. Hence we can assume that $T(a_1, a_2, a_3) \neq 6$ by Proposition 7. We put $M = B$ for (a_1, a_2, a_3) in Proposition 4. Hence $M \not\simeq E_3$. And $M \not\simeq L$ if $N(a_1, a_2, a_3) \neq 2$. If $M \simeq L$ and $N(a_1, a_2, a_3) = 2$, then (noting the existence of the pair $\{u_1, u_2\}$) we have $T(a_1, a_2, a_3) = 3$ or 6 by Proposition 7. This is a contradiction. Secondly suppose that $p \equiv 13 \bmod 24$. There is an element $(a_1, a_2, a_3) \in R_p^3$ whose type is more than three by Proposition 1. For M we take the lattice B for (a_1, a_2, a_3) in Proposition 4. If $N(a_1, a_2, a_3) = 1$, then $B = E_1 \perp B'$ with $1 \notin Q(B')$. If $N(a_1, a_2, a_3) \geqq 3$, then $1 \notin Q(B)$ and $2 \notin Q(B)$. If $N(a_1, a_2, a_3) = 2$, then $1 \notin Q(B)$ and $B \not\simeq L$ by Proposition 7.

(vi) Let D be a prime $p \equiv 5 \bmod 12$. For L we take the lattice A with $1 \notin Q(A)$ in Proposition 2. For M we take the lattice $G = E_1 \perp G'$ with $1 \notin Q(G')$ in Proposition 6.

§ 4. Special values of D

For the explicit value of the class number of E_n we use the Kneser Method. Following [4] we state the method. By J we denote the group of ideles of the field K. For a finite spot $\mathfrak{p}$ on K we put

$$J^{\mathfrak{p}} = \{\boldsymbol{i} = (\boldsymbol{i}_{\mathfrak{q}}) \in J;\ \boldsymbol{i}_{\mathfrak{q}} \text{ is a unit in } \mathfrak{O}_{\mathfrak{q}} \text{ for all finite spot } \mathfrak{q} \neq \mathfrak{p}\}\,.$$

Put $V = KE_n$ and $P = \theta(O^+(V))$, where θ is the spinor norm and $O^+(V)$ is the proper orthogonal group of V. Consider P as the image of P under the natural isomorphism $K^* \to J$. Recall Theorem 104: 9 in [4]:

LEMMA 7. *Let $n \geqq 3$, $V_{\mathfrak{p}}$ be isotropic and $J = PJ^{\mathfrak{p}}$. Then for any $L \in \operatorname{gen} E_n$ there is a lattice M isometric to L such that $M_{\mathfrak{q}} = E_{n\mathfrak{q}}$ for all finite spot $\mathfrak{q} \neq \mathfrak{p}$.*

By Proposition 101: 8 in [4] we have

LEMMA 8. *Let $n \geqq 3$ and the ideal class number of K be one. Assume that the norm of the fundamental unit in K is -1 or that the norm of a generator of $\mathfrak{p}$ is negative. Then $J = PJ^{\mathfrak{p}}$.*

LEMMA 9. *Let $n \geqq 3$, $\mathfrak{p}$ be a spot dividing D and $M \in \operatorname{gen} E_n$ with $M_{\mathfrak{q}} = E_{n\mathfrak{q}}$ for all finite spot $\mathfrak{q} \neq \mathfrak{p}$. Assume that n is odd and $D = 2$ if $\mathfrak{p}$ is dyadic. Then there is a chain of lattices*

$$E_n = L_0, L_1, \cdots, L_t = M$$

in $\operatorname{gen} E_n$ with L_{i+1} $\mathfrak{p}$-adjacent to L_i.

Proof. Following the proof of 106: 4 in [4], we can prove this assertion. It is enough to find a chain of lattices $E_{n\mathfrak{p}} = L_0^{(\mathfrak{p})}, L_1^{(\mathfrak{p})}, \cdots, L_t^{(\mathfrak{p})} = M_\mathfrak{p}$ in $V_\mathfrak{p}$ with $L_{i+1}^{(\mathfrak{p})}$ $\mathfrak{p}$-adjacent to $L_i^{(\mathfrak{p})}$ and $L_i^{(\mathfrak{p})} \simeq E_{n\mathfrak{p}}$. Put $L_0 = E_n$. Then $M_\mathfrak{p} = \sigma L_{0\mathfrak{p}}$ for some $\sigma \in O(V_\mathfrak{p})$. By expressing σ as a product of symmetries on $V_\mathfrak{p}$ we see that it is enough if we assume that σ is a symmetry. Then $\sigma = \tau_u$ with u a maximal anisotropic vector in $L_{0\mathfrak{p}}$. Then there is either a 1- or 2-dimensional unimodular sublattice K of $L_{0\mathfrak{p}}$ which contains u. If the rank of K is one, then $L_{0\mathfrak{p}} = \tau_u L_{0\mathfrak{p}} = M_\mathfrak{p}$, so $M_\mathfrak{p}$ is $\mathfrak{p}$-adjacent to $L_{0\mathfrak{p}}$. If the rank of K is two, then we take the splitting $L_{0\mathfrak{p}} = K \perp K'$. Then $K' = \tau_u K' \subset M_\mathfrak{p}$ and so we have a splitting $M_\mathfrak{p} = K'$ $K = \mathfrak{p}^r x + \mathfrak{o}_\mathfrak{p} y$ and $K'' = \mathfrak{o}_\mathfrak{p} x + \mathfrak{p}^r y$ with a non-negative ir :e we may put $L_0^{(\mathfrak{p})} = L_{0\mathfrak{p}}$, $L_1^{(\mathfrak{p})} = (\mathfrak{p}^{r-1}x + \mathfrak{p}y) \perp K' = L_0^{(\mathfrak{p})}(\pi^{r-1}x), \cdots, L_r^{(\mathfrak{p})} = M_\mathfrak{p} = K'' \perp K' = (\mathfrak{o}_\mathfrak{p} x + \mathfrak{p}^r y) \perp K' = L_{r-1}^{(\mathfrak{p})}(x)$, where $\mathfrak{p} = \pi\mathfrak{o}_\mathfrak{p}$. We must show that $L_i^{(\mathfrak{p})} \simeq E_{n\mathfrak{p}}$. It is trivial when $i = 0$ or $i = r$. Assume that $1 \leqq i \leqq r-1$. If $\mathfrak{p}$ is non-dyadic, then $\mathfrak{p}^{r-i}x + \mathfrak{p}^i y \simeq \langle 1 \rangle \perp \langle -1 \rangle \simeq K$, so $L_i^{(\mathfrak{p})} \simeq K \perp K' \simeq E_{n\mathfrak{p}}$. If $\mathfrak{p}$ is dyadic, then n is odd, hence $K' = [z] \perp K'''$ with $Q(z) = \varepsilon$ a unit in $\mathfrak{o}_\mathfrak{p}$. It is enough to show that $(\mathfrak{p}^r x + \mathfrak{o}_\mathfrak{p} y) \perp \mathfrak{o}_\mathfrak{p} z \simeq (\mathfrak{p}^{r-i}x + \mathfrak{p}^i y) \perp \mathfrak{o}_\mathfrak{p} z$ for $1 \leqq i \leqq r-1$. We can assume that $\mathfrak{p}^r B(x, y) = \mathfrak{o}_\mathfrak{p}$. Since $y \in K \subset L_{0\mathfrak{p}} \simeq E_{n\mathfrak{p}}$ and $\mathfrak{p} = \sqrt{2}\,\mathfrak{o}_\mathfrak{p}$, we have $Q(y) \equiv 0$ or $1 \bmod 2$. Similarly $Q(x) \equiv 0$ or $1 \bmod 2$ and $\varepsilon \equiv 1 \bmod 2$. If $Q(y) \equiv 0 \bmod 2$, then $(\mathfrak{p}^r x + \mathfrak{o}_\mathfrak{p} y) \perp \mathfrak{o}_\mathfrak{p} z \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \langle \varepsilon \rangle \simeq (\mathfrak{p}^{r-i}x + \mathfrak{p}^i y) \perp \mathfrak{o}_\mathfrak{p} z$. If $Q(y) \equiv 1 \bmod 2$ and $\pi^{2r}Q(x) \equiv 0 \bmod 8$, then $(\mathfrak{p}^r x + \mathfrak{o}_\mathfrak{p} y) \perp \mathfrak{o}_\mathfrak{p} z \simeq \langle Q(y) \rangle \perp \langle -Q(y) \rangle \perp \langle \varepsilon \rangle \simeq \langle \varepsilon \rangle \perp \langle -\varepsilon \rangle \perp \langle \varepsilon \rangle \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \langle \varepsilon \rangle \simeq (\mathfrak{p}^{r-i}x + \mathfrak{p}^i y) \perp \mathfrak{o}_\mathfrak{p} z$. If $Q(x) \equiv Q(y) \equiv 1 \bmod 2$, $r = 2$ and $i = 1$, then $(\mathfrak{p}^2 x + \mathfrak{o}_\mathfrak{p} y) \perp \mathfrak{o}_\mathfrak{p} z \simeq \langle Q(y) \rangle \perp \langle 3Q(y) \rangle \perp \langle \varepsilon \rangle \simeq \langle \varepsilon \rangle \perp \langle 3\varepsilon \rangle \perp \langle \varepsilon \rangle \simeq \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp \langle \varepsilon \rangle \simeq (\mathfrak{p}x + \mathfrak{p}y) \perp \mathfrak{o}_\mathfrak{p} z$.

PROPOSITION 9. *Let $D = 2$. Then the class number of E_n is one if $n \leqq 4$, two if $n = 5$ and more than two if $n \geqq 6$.*

Proof. There are three lattices E_6, $E_6((1/\sqrt{2})(e_1 + \cdots + e_4))$ and $E_6((1/\sqrt{2})(e_1 + \cdots + e_6))$ in gen E_6, any two of which are not isometric. Let $n = 5$. Take $\mathfrak{p} = (\sqrt{2})$ and a $\mathfrak{p}$-adjacent lattice $E_5(x)$ in gen E_5. Write $x = (1/\sqrt{2}) \sum_{i=1}^5 \alpha_i e_i$. Note that $O(E_5)$ contains all permutations of $\{e_1, \cdots, e_5\}$. And note that $Q(x) \equiv 0$ or $1 \bmod 2$ since $E_5(x) \in$ gen E_5. By Lemmas 2 and 3 we have only to consider the following three cases: (i) $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$. Then $E_5(x) = E_5$. (ii) $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = \alpha_4 = \alpha_5 = 0$. Then $E_5(x) \simeq E_5$. (iii) $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$ and $\alpha_5 = 0$. Then

$E_5(x) = E_4^0 \perp [e_5]$, where $E_4^0 = E_4(u)$ with $u = (1/\sqrt{2})(e_1 + \cdots e_4)$. Hence a $\mathfrak{p}$-adjacent lattice to E_5 in $\operatorname{gen} E_5$ is isometric to E_5 or $E_5' = E_4^0 \perp [e_5]$. Next take a $\mathfrak{p}$-adjacent lattice $E_5'(y)$ to E_5' in $\operatorname{gen} E_5$ and write $\sqrt{2}\, y = w + \alpha e_5$ with $\alpha \in \mathfrak{o}$ and $w \in E_4^0$. Since $Q(y) \in \mathfrak{o}$ and E_4^0 is even, we have $\alpha \in \mathfrak{p}$. By Lemma 2 we have $E_5'(y) = E_5'((1/\sqrt{2})w) = E_4^0((1/\sqrt{2})w) \perp [e_5]$. Hence we may write $\sqrt{2}\, y = au + \sum_{i=1}^4 \alpha_i e_i$ where $\alpha_i \in \mathfrak{o}$, $a \in \{0, 1\}$ and $\sum_{i=1}^4 \alpha_i \equiv 0 \bmod \sqrt{2}$. Note that $Q(y) \equiv 0$ or $1 \bmod 2$ since $E_5'(y) \in \operatorname{gen} E_5$. If $a = 0$, then we have the following four cases by Lemmas 2 and 3: (i) $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. Then $E_5'(y) = E_5'$. (ii) $\alpha_1 = \sqrt{2}$ and $\alpha_2 = \alpha_3 = \alpha_4 = 0$. Then $E_5'(y) = E_5$. (iii) $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = \alpha_4 = 0$. Then $E_5'(y) = E_4^0(y) \perp [e_5] = E_2((1/\sqrt{2})(e_1 + e_2)) \perp E_2((1/\sqrt{2})(e_3 + e_4)) \perp [e_5] \simeq E_5$. (iv) $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = \sqrt{2}$ and $\alpha_4 = 0$. Then $E_5'(y) \simeq E_5'$. Next consider the case of $a = 1$. Since $\tau_{e_1} \in O(E_5')$, we have $\sqrt{2}\, \tau_{e_1} y = u + (-\alpha_1 - \sqrt{2})e_1 + \alpha_2 e_2 + \cdots$. Hence we may assume that $\alpha_i \equiv 0 \bmod 2$ or $\alpha_i \equiv 1 \bmod 2$. Thus we have only to consider the following cases by Lemmas 2 and 3: (v) $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ or $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$. Then $E_5'(y) \simeq E_5$. (vi) $\alpha_1 = -1$, $\alpha_2 = 1$ and $\alpha_3 = \alpha_4 = 0$. Apply Lemma 4 to this case taking $w = u - \sqrt{2}\, e_1$. Then $E_5'(y) \simeq E_4^0 \perp [e_5] \simeq E_5'$. Thus a $\mathfrak{p}$-adjacent lattice to E_5' is isometric to E_5 or E_5'. Hence $\{E_5', E_5\}$ is a set of all representatives of classes in $\operatorname{gen} E_5$ by Lemmas 7, 8 and 9. By Theorem 105:1 in [4] this implies that the class number is one if $n \leqq 4$.

From [11] we have

Proposition 10. *Let $D = 3$. Then the class number of E_n is one if $n \leqq 2$, two if $n = 3$ and more than two if $n \geqq 4$.*

Proposition 11. *Let $D = 5$. Then the class number of E_n is one if $n \leqq 4$, two if $n = 5$ and more than two if $n \geqq 6$.*

Proof. Put $x = (1/\sqrt{5})(e_1 + \cdots + e_5)$, $y_1 = (1/\sqrt{5})(e_1 + e_2 + 2e_3 + 2e_4)$ and $x_1 = (1/\sqrt{5})(e_1 + e_2 + e_3 + 2e_4 + 2e_5 + 2e_6)$. Consider the lattice $E_5' = E_5(x) = [x] \perp E_4^0$. Then E_4^0 is even. If $n = 6$, then we have three lattices E_6, $E_6(x)$ and $E_6' = E_6(x_1)$ in $\operatorname{gen} E_6$. By Proposition 4, (8) we have $1 \notin Q(E_6')$. Thus the class number of E_6 is more than two. Let $\mathfrak{p} = (\sqrt{5})$ and $n = 5$. Take a $\mathfrak{p}$-adjacent lattice $E_5(y)$ to E_5. By Lemma 5 we may consider $y = (1/\sqrt{5}) \sum_{i=1}^5 a_i e_i$ with $(a_1, \cdots, a_5) \in R_5^5$. Hence $y \in \{0, x, (1/\sqrt{5})(e_1 + 2e_2), y_1\}$. Thus $E_5(0) = E_5$, $E_5(x) = E_5'$ and $E_5((1/\sqrt{5})(e_1 + 2e_2)) \simeq E_5$. By Lemma 4 $E_5(y_1) \simeq E_5$ taking $w = \frac{1}{2}(1 + \sqrt{5})(e_3 + e_4) \in E_5$. Take a $\mathfrak{p}$-adjacent lattice $E_5'(z)$ to E_5'. By Lemma 2 we may assume that $\sqrt{5}\, z = ax + \sum_{i=1}^5 \alpha_i e_i$

where $a = 0$ or 1, $\alpha_i = a_i + b_i\sqrt{5} \in \boldsymbol{Z}[\sqrt{5}]$ and $\sum_{i=1}^{5} a_i \equiv 0 \bmod 5$. If $a = 0$, then we have only to consider the following three cases by Lemmas 2 and 3:

(i) $a_1 = \cdots = a_5 = 0$. Then $z = \sum_{i=1}^{5} b_i e_i$, so $E_5'(z) = E_5'$ or $E_5'(z) = E_5$ according as $\sum_{i=1}^{5} b_i \equiv 0 \bmod 5$ or not.

(ii) $a_1 = \cdots = a_5 = 1$. Thus $z = x + \sum_{i=1}^{5} b_i e_i$ with $\sum_{i=1}^{5} b_i \equiv 0 \bmod 5$, so $z \in E_5'$. Hence $E_5'(z) = E_5'$.

(iii) $a_1 = 1$, $a_2 = -1$, $a_3 = 2$, $a_4 = -2$, $a_5 = b_1 = \cdots = b_4 = 0$ and $b_5 \in \{0, 1\}$. If $b_5 = 0$, then $z \in E_5'$, so $E_5'(z) = E_5'$. If $b_5 = 1$, then we have $E_5'(z) = [z_0, z_1, \cdots, z_4] \simeq E_5$, where $z_0 = z + \bar{\zeta}e_3 + \zeta e_4 - e_5$, $z_1 = z + x + \bar{\zeta}e_1 + \bar{\zeta}e_3 - e_5$, $z_2 = z + 2x + \bar{\zeta}e_1 - \sqrt{5}\,e_3 - \zeta e_5$, $z_3 = z - 2x + \zeta e_2 + \sqrt{5}\,e_4 - \bar{\zeta}e_5$ and $z_4 = z - x + \zeta e_2 + \zeta e_4 - e_5$, where $\zeta = \frac{1}{2}(1 + \sqrt{5})$.

If $a = 1$, then we have only to consider the following six cases by Lemma 2 (note that $O(E_5')$ contains all permutations of $\{e_1, \cdots, e_5\}$):

(iv) $\alpha_1 = \cdots = \alpha_4 = 0$ and $\alpha_5 = 2\sqrt{5}$. Thus $E_5'(z) = [2z - 5e_5, 2z - e_1 - 4e_5, 2z - e_2 - 4e_5, 2z - e_3 - 4e_5, 2z - e_4 - 4e_5] \simeq E_5$.

(v) $\alpha_1 = \alpha_2 = \alpha_3 = 0$, $\alpha_4 = 2$ and $\alpha_5 = 3 + 3\sqrt{5}$. Thus $E_5'(z) = [z + 2x - \zeta e_4 - (3 + \sqrt{5})e_5, z - 2x - \bar{\zeta}(e_1 + e_2 + e_3) - (3 + \zeta)e_5, z - e_1 - \zeta e_4 - (3 + \zeta)e_5, z - e_2 - \zeta e_4 - (3 + \zeta)e_5, z - e_3 - \zeta e_4 - (3 + \zeta)e_5] \simeq E_5$.

(vi) $\alpha_1 = \alpha_2 = \alpha_3 = 0$, $\alpha_4 = 1$ and $\alpha_5 = 4 + \sqrt{5}$. Thus we have $E_5'(z) = [2z + 2x - \zeta(e_1 + e_2 + e_3 + e_4) - (3 + 2\sqrt{5})e_5, 2z - 2x - \bar{\zeta}(e_1 + e_2 + e_3) - e_4 + (3\zeta - 4)e_5, 2z - (e_1 + e_2) - \zeta e_4 - (3\zeta + 1)e_5, 2z - (e_2 + e_3) - \zeta e_4 - (3\zeta + 1)e_5, 2z - (e_1 + e_3) - \zeta e_4 - (3\zeta + 1)e_5] \simeq E_5$.

(vii) $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = \alpha_4 = -1$ and $\alpha_5 = 0$. Thus we have $E_5'(z) = [z + x + \bar{\zeta}e_1 - \zeta e_2, z + x + \bar{\zeta}e_2 - \zeta e_1, z - x + \zeta e_3 - \bar{\zeta}e_4, z - x + \zeta e_4 - \bar{\zeta}e_3, z] \simeq E_5$.

(viii) $\alpha_1 = \alpha_2 = 2$, $\alpha_3 = \alpha_4 = -2$ and $\alpha_5 = -5$. By Lemma 4 we have $E_5'(z) \simeq E_5'$ by taking $w = 3x - \sqrt{5}\,(e_3 + e_4 + e_5)$.

(ix) $\alpha_1 = 2$, $\alpha_2 = 1$, $\alpha_3 = -1$, $\alpha_4 = -2$ and $\alpha_5 = 2\sqrt{5}$. Then we have $E_5'(z) = [2z + x - \sqrt{5}\,e_1 - \zeta e_2 - \bar{\zeta}e_4 - 4e_5, 2z - 2x - \zeta e_1 + \sqrt{5}\,e_3 + \sqrt{5}\,e_4 + (\zeta - 5)e_5, 2z + 2x - \sqrt{5}\,e_1 - \sqrt{5}\,e_2 - \bar{\zeta}e_4 - (\zeta + 4)e_5, 2z - x - \zeta e_1 - \bar{\zeta}e_3 + \sqrt{5}\,e_4 - 4e_5, 2z - \sqrt{5}\,e_1 - \zeta e_2 - \bar{\zeta}e_3 + \sqrt{5}\,e_4 - 4e_5] \simeq E_5$.

Hence gen E_5 contains just two classes by Lemmas 7, 8 and 9. $\{E_5, E_5'\}$ is a set of all representatives of classes in gen E_5.

PROPOSITION 12. *Let $D = 13$. Then the class number of E_n is one if $n \leqq 2$, two if $n = 3$ and more than two if $n \geqq 4$.*

Proof. Let $n = 4$. Then there are three lattices E_4, $E_4(y_1)$ and $E_4(y_2)$ in

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gen E_4 with $\sqrt{13}y_1 = e_1 + 2e_2 + 3e_3 + 5e_4$ and $\sqrt{13}y_2 = e_1 + 3e_2 + 4e_3$. By Proposition 4 $Q(E_4(y_1)) \not\ni 1$, $Q(E_3(y_2)) \not\ni 1$ and $E_4(y_2) = E_3(y_2) \perp [e_4]$. Thus the class number of E_4 is more than 2. Let $n = 3$ and $\mathfrak{p} \in (\sqrt{13})$. Take a $\mathfrak{p}$-adjacent lattice $E_3' = E_3(x)$ to E_3. By Lemma 5 and Proposition 4(5) we may consider $\sqrt{13}x = e_1 + 3e_2 + 4e_3$. Thus

$$Q(E_3') \not\ni 1 \quad \text{and} \quad E_3' = [x, y, z] \simeq \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & \sqrt{13} \\ 1 & \sqrt{13} & 5 \end{pmatrix},$$

where $y = e_1 + e_2 - e_3$ and $z = 6x - \sqrt{13}(e_2 + 2e_3)$. Next consider a $\mathfrak{p}$-adjacent lattice $E_3'' = E_3'(u)$ to E_3' with $\sqrt{13}u = \alpha x + \beta y + \gamma z \in E_3' - \mathfrak{p}E_3'$. If $\beta \in \mathfrak{p}$, then we may assume that $\beta = 0$ and $\alpha = 1$ by Lemma 2. Thus we may assume that $\gamma = 2$ or $\gamma = -5$ since $Q(u) \in \mathfrak{o}$ by Lemma 4. If $\gamma = 2$, then $E_3'' = [u, y, \sqrt{13}z] = [u - y,\ 4u + 2y - \sqrt{13}z,\ 3u + 2y - \sqrt{13}z] \simeq E_3$. If $\gamma = -5$, then $E_3'' = [u, y, \sqrt{13}z] = [u + 2y,\ 3u + y + \sqrt{13}z,\ 4u + 2y + \sqrt{13}z] \simeq E_3$. Let $\beta \notin \mathfrak{p}$. Then by Lemma 2 we may assume that α and $\beta \in \boldsymbol{Z}$ such that $|\alpha| \leqq 6$ and $|\gamma| \leqq 6$ and that $\beta \equiv 2 \bmod \sqrt{13}$. Since $\tau_{x+\sqrt{13}y-3z}$, $\tau_{2x+\sqrt{13}y-3z}$, $\tau_x \in O(E_3')$, we may assume that $\sqrt{13}u = x(\pm 2 + 2\sqrt{13})y - 6z$. Hence $E_3'' = [u,\ \pm(5u - 9y) + x + (\pm 2\sqrt{13} - 2)z,\ 4u \pm 2x - 4y - (\pm 2 - \sqrt{13})z] \simeq E_3'$. By Lemmas 7, 8 and 9 we have the assertion.

PROPOSITION 13. *Let $D = 17$. Then the class number of E_n is one if $n \leqq 3$ and more than two if $n \geqq 4$.*

Proof. Let $n = 4$. Then there are three lattices E_4, $E_4(y_1)$ and A in gen E_4, where $\sqrt{17}y_1 = e_1 + 3e_2 + 4e_3 + 5e_4$, $\sqrt{17}y_2 = e_1 + 2(e_2 + e_3 + e_4 + e_5)$ and $E_5(y_2) = [y_2] \perp A$. By Proposition 4 $Q(E_4(y_1)) \not\ni 1, 2$. By Proposition 2 $Q(A) \not\ni 1$ and $Q(A) \ni 2$. Hence the class number of E_4 is more than two. Let $n = 3$. Then by Lemma 5 and Proposition 4 a $(\sqrt{17})$-adjacent lattice to E_3 is isometric to E_3.

PROPOSITION 14. *Let $D = 29$. Then the class number of E_n is more than two if $n \geqq 3$.*

Proof. There are three lattices E_3, $E_3(y)$ and $E_3(y')$ in gen E_3, where $\sqrt{29}y = 2e_1 + 3e_2 + 4e_3$ and $2y' = e_1 + \frac{1}{2}(1 + \sqrt{29})e_2 + \frac{1}{2}(1 - \sqrt{29})e_3$. Then $E_3(y) = [y] \perp M$ with $1 \notin Q(M)$. Clearly $Q(E_3(y')) \not\ni 1$ since

$$E_3(y') = [y', 2e_1, \sqrt{29}e_1 - e_2 + e_3] \simeq \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 2\sqrt{29} \\ 0 & 2\sqrt{29} & 31 \end{pmatrix}.$$

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PROPOSITION 15. *Let* $D = 33$. *Then the class number of* E_n *is one if* $n \leqq 2$, *two if* $n = 3$ *and more than two if* $n \geqq 4$.

Proof. There are three lattices $E_4, E_4(x_1)$ and $E_4(x_2)$ in gen E_4 with $x_1 = (\sqrt{33}/11)(e_1 + e_2 + 3e_3)$ and $x_2 = (\sqrt{33}/11)(e_1 + e_2 + 2e_3 + 4e_4)$. Then $E_4(x_1) = E_3(x_1) \perp [e_4]$ with $1 \notin Q(E_3(x_1))$ and $1 \notin Q(E_4(x_2))$ by Proposition 4. Thus the class number of E_4 is more than two. Put $\pi = 11 + 2\sqrt{33}$ and $\omega = 6 + \sqrt{33}$. Let $n = 3$ and $\mathfrak{p} = (\pi)$. Then a $\mathfrak{p}$-adjacent lattice to E_3 is isometric to E_3 or $E_3(x_1)$ by Proposition 1 and Lemma 5. Note that

$$E_3(x_1) = [e_1 - e_2,\ x_1 - e_1 - 2e_2 + e_3,\ 5x_1 + (1 - \sqrt{33})(e_1 + e_2) + (3 - \sqrt{33})e_3]$$
$$\simeq \begin{pmatrix} 2 & 1 & 0 \\ 1 & 9 & 3\sqrt{33} \\ 0 & 3\sqrt{33} & 35 \end{pmatrix}.$$

Putting $x = (\sqrt{33}/3)(e_1 + e_2 + e_3)$, then $E_3' = E_3(x) = [e_1 - e_2,\ x + \frac{1}{2}(3 - \sqrt{33})e_1 + \frac{1}{2}(1 - \sqrt{33})e_2 + e_3,\ x - 2(e_1 + e_2) + 4e_3] \simeq E_3(x_1)$. To find a $\mathfrak{p}$-adjacent lattice to $E_3(x_1)$ we have only to find a $\mathfrak{p}$-adjacent lattice E_3'' to E_3'. Let $E_3'' = E_3'(y)$ with $y \in \mathfrak{p}^{-1}E_3' - E_3'$. By Lemma 2 we can assume that $y = (\sqrt{33}/11)z$ with $z \in E_3' - \mathfrak{p}E_3'$. Thus $\mathfrak{p}y \subset \omega E_3' \subset E_3$. Since $x \in E_3'$ and $B(x, y) = B(e_1 + e_2 + e_3, z) \in B(E_3', E_3') \subset \mathfrak{o}$ we have $x + y \in E_3''$. For a vector $w \in E_3$ such that $B(w, x + y) \in \mathfrak{o}$ we have $\pi B(w, x) = B(w, \pi(x + y)) - B(w, \pi y) \in \mathfrak{o}$, also $\omega B(w, x) \in \mathfrak{o}$. Hence $B(w, x) \in \mathfrak{o}$, so $w \in E_3'$. And hence $B(w, y) = B(w, x + y) - B(w, x) \in \mathfrak{o}$, so $w \in E_3'(y) = E_3''$. Hence $E_3(x + y) \subset E_3''$. Since $y = 2\omega(x + y) - (2\omega x + \pi y)$ with $2\omega x + \pi y \in E_3$ and $B(y, 2\omega x + \pi y) \in \mathfrak{o}$, we have $y \in E_3(x + y)$. For a vector $w \in E_3$ such that $B(w, x) \in \mathfrak{o}$ and $B(w, y) \in \mathfrak{o}$, we have $B(w, x + y) \in \mathfrak{o}$. Thus $E_3'' \subset E_3(x + y)$. Hence $E_3'' = E_3(u)$ with $u = x + y = (1/\sqrt{33})(11e_1 + 11e_2 + 11e_3 + 3z) \in (1/\sqrt{33})E_3$. Clearly $\sqrt{33}u \notin \mathfrak{p}E_3 \cup \omega E_3$. Write $\sqrt{33}u = \sum_{i=1}^{3} \alpha_i e_i$ with $\alpha_i \in \mathfrak{o}$. By Lemma 2 and considering the structure of $O(E_3)$, we have only to consider the following three cases:

(i) $\alpha_1 = \alpha_2 = 4$ and $\alpha_3 = 1$. Then

$$E_3'' = [u,\ 4u + \tfrac{1}{2}(1 - \sqrt{33})e_1 - \tfrac{1}{2}(1 + \sqrt{33})e_2,\ 4u - \tfrac{1}{2}(1 + \sqrt{33})e_1 + \tfrac{1}{2}(1 - \sqrt{33})e_2] \simeq E_3\,.$$

(ii) $\alpha_1 = \alpha_2 = 2$ and $\alpha_3 = 5$. Hence

$$E_3'' = [u,\ 7u + \tfrac{1}{2}(1 - \sqrt{33})e_1 - \tfrac{1}{2}(1 + \sqrt{33})e_2 - \sqrt{33}e_3,\ 7u - \tfrac{1}{2}(1 + \sqrt{33})e_1 + \tfrac{1}{2}(1 - \sqrt{33})e_2 - \sqrt{33}e_3] \simeq E_3\,.$$

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(iii) $\alpha_1 = 1$, $\alpha_2 = 4$ and $\alpha_3 = 7$. Thus
$$E_3'' = [u,\ 4u - \sqrt{33}e_3,\ e_1 + 5e_2 - 3e_3] \simeq E_3' .$$
Hence we have the assertion by Lemmas 7, 8 and 9.

PROPOSITION 16. *Let $D = 41$. Then the class number of E_n is one if $n = 1$, two if $n = 2, 3$ and more than two if $n \geqq 4$.*

Proof. By Proposition 6 there is a lattice G' in gen E_2 such that $1 \notin Q(G')$. Hence there are three lattices E_4, $G' \perp E_2$ and $G' \perp G'$ in gen E_4. Thus the class number of E_4 is more than two. Let $n = 3$ and $\mathfrak{p} = (\sqrt{41})$. A $\mathfrak{p}$-adjacent lattice to E_3 is isometric to E_3 or $E_3(x)$ with $x = (1/\sqrt{41})(e_1 + 2e_2 + 6e_3)$ by Propositions 1 and 4 and Lemma 3. Thus
$$E_3' = E_3(x) = [x, y, z] \simeq \langle 1\rangle \perp \begin{pmatrix} 5 & 2\sqrt{41} \\ 2\sqrt{41} & 5 \end{pmatrix}$$
with $y = 2e_1 - e_2$ and $z = \sqrt{41}(e_1 + e_3) - 7x$. Take a $\mathfrak{p}$-adjacent lattice $E_3'' = E_3'(u)$ to E_3' such that $u \notin E_3'$. Write $\sqrt{41}u = \alpha x + \beta y + \gamma z$ with $\alpha, \beta, \gamma \in \mathfrak{o}$. If $\alpha \in \mathfrak{p}$, then we may assume that $\alpha = 0$ and $\gamma = 20$ by Lemma 2. Since $Q(u) \in \mathfrak{o}$, we may assume that $\beta = \pm 5 - 8\sqrt{41}$ by Lemma 2. Thus $E_3'' = [x] \perp [-u, \pm 10u + (2\sqrt{41} \pm 80)y - (8 \pm 5\sqrt{41})z] \simeq E_3'$. If $\beta \in \mathfrak{p}$, then we may assume that $\alpha = 7$, $\beta = 0$ and $\gamma = 1$. Hence $u = e_1 + e_3$ and $E_3'' = E_3$. If $\gamma \in \mathfrak{p}$, we may assume that $\alpha = 6$, $\beta = 1$ and $\gamma = 0$. Thus $E_3'' = [u] \perp [14u - 2\sqrt{41}x - z] \perp [7u - \sqrt{41}x - \sqrt{41}y + 2z] \simeq E_3$. If $\alpha\beta\gamma \notin \mathfrak{p}$, then we may assume that $\gamma = 1$ and $\beta \in \boldsymbol{Z}$ by Lemma 2. Note that $O(E_3')$ contains the isometries
$$\text{“}x \to \pm x,\ y \to 2\sqrt{41}y - 5z,\ z \to 33y - 2\sqrt{41}z\text{”},$$
$$\text{“}x \to \pm x,\ y \to \tfrac{1}{2}(17 - \sqrt{41})y + \tfrac{1}{2}(3 - \sqrt{41})z,$$
$$z \to \tfrac{1}{2}(7\sqrt{41} - 13)y - \tfrac{1}{2}(17 - \sqrt{41})z\text{”}$$
and
$$\text{“}x \to \pm x,\ y \to \tfrac{1}{2}(17 + \sqrt{41})y - \tfrac{1}{2}(3 + \sqrt{41})z,$$
$$y \to \tfrac{1}{2}(13 + 7\sqrt{41})y - \tfrac{1}{2}(17 + \sqrt{41})z\text{”}.$$
Hence by Lemmas 2 and 3 we have only to consider the following two cases:

(i) $\sqrt{41}u = -(-2 \pm 3\sqrt{41})x \pm 3y + z$ and

(ii) $\sqrt{41}u = (10 \pm 6\sqrt{41})x \mp 30y + z$.

In the case of (i) we have

$$E_3'' = [u, \sqrt{41}x, y \pm 13x] = [v] \perp [v_1, v_2] \simeq E_3' ,$$

where

$$2v = (-19 \pm \sqrt{41})u - (9 \pm 3\sqrt{41})\sqrt{41}x + (5 \pm \sqrt{41})(y \pm 13x) .$$
$$2v_1 = (11 \pm \sqrt{41})u + (15 \pm 3\sqrt{41})\sqrt{41}x - (7 \pm \sqrt{41})(y \pm 13x)$$

and

$$v_2 = 2(\pm 2 + \sqrt{41})u + (\pm 19 + 2\sqrt{41})\sqrt{41}x - (\pm 6 + \sqrt{41})(y \pm 13x) .$$

Then $Q(v) = 1$. In the case of (ii) we have

$$E_3'' = [u, \sqrt{41}x, y \pm 15x] = [v'] \perp [v_1', v_2'] \simeq E_3' ,$$

where

$$2v' = (-9 \pm \sqrt{41})u - (101 \pm 11\sqrt{41})\sqrt{41}x - (-33 \pm 7\sqrt{41})(y \pm 15x) ,$$
$$v_1' = (21 \pm \sqrt{41})u - (236 \pm 11\sqrt{41})\sqrt{41}x + (21 \pm 15\sqrt{41})(y \pm 15x)$$

and

$$2v_2' = (25 \pm 17\sqrt{41})u - (273 \pm 191\sqrt{41})\sqrt{41}x + (501 \pm 11\sqrt{41})(y \pm 15x) .$$

Then $Q(v') = 1$. Hence the class number of E_3 is two, and that of E_2 is also two, by Lemmas 7, 8 and 9.

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