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SEMI-ABELIAN SURFACES AND INTEGRABLE SYSTEMS

IBRAHIMA FAYE

Laboratoire Emile Picard, UMR 5580, Université Paul Sabatier, 118 Route de Narbonne, 31062 Toulouse cedex 04, France
(faye@picard.ups-tlse.fr)

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Abstract We study some weight-homogeneous systems which are not algebraically completely integrable (ACI) in the sense of Adler and van Moerebeke, but whose invariant level surface completes into a semi-abelian variety by adding a set of points (thus ACI in the sense of Mumford).

Keywords: integrable systems; semi-abelian surfaces; differential equations; asymptotic solutions; divisor of poles; weight-homogeneous systems

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1. Introduction

A system of polynomial differential equations

$$\dot{z} = f(z), \quad z \in \mathbb{C}^n, \tag{1.1}$$

is called weight-homogeneous when there exist integers $\nu_i \in \mathbb{N}^*$ such that

$$f_i(t^{\nu_1} z_1, \dots, t^{\nu_n} z_n) = t^{\nu_i + 1} f_i(z_1, \dots, z_n)$$

for all $t \in \mathbb{R}$. A polynomial constant of motion is weight-homogeneous of degree N whenever

$$H(t^{\nu_1} z_1, \dots, t^{\nu_n} z_n) = t^N H(z_1, \dots, z_n).$$

Consider the Hamiltonian vector field

$$\dot{z} = f(z) = J \frac{\partial H}{\partial z}, \quad z \in \mathbb{R}^n;$$

$J = J(z)$ is a skew-symmetric matrix, for which the corresponding Poisson bracket

$$\{H_i, H_j\} = \left\langle \frac{\partial H_i}{\partial z}, J \frac{\partial H_j}{\partial z} \right\rangle$$

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satisfies the Jacobi identity. Such a system is algebraically completely integrable (ACI) (in the sense of Adler and van Moerebeke) when J has polynomial entries and when the following two conditions are satisfied.

(i) Besides the polynomial Casimir functions $H_1, \dots, H_k$ (such that $J(\partial H_i/\partial z) = 0$), the system possesses m (ν-homogeneous polynomial) constants of motion $H_{k+1} = H, \dots, H_{k+m}$ in involution (i.e. $\{H_i, H_j\} = 0$), which give rise to m commuting vector fields; for generic c_i, the level surfaces $\bigcap_{i=1}^{k+m}\{H_i = c_i,\ z \in \mathbb{R}^n\}$ are compact and connected and therefore real tori according to the classical Arnold–Liouville Theorem.

(ii) The level surfaces thought of as lying in $\mathbb{C}^n$, $\bigcap_{i=1}^{k+m}\{H_i(z) = c_i,\ z \in \mathbb{C}^n\}$, are related for generic $c \in \mathbb{C}^{k+m}$ to abelian varieties T^m as follows:

$$A = T^m \setminus D,$$

where D is a divisor in T^m. The coordinates z_i are meromorphic functions on T^m, and D is the minimal divisor on T^m, where the variables z_i blow up. The Hamiltonian flows $\dot{z} = J(\partial H_{k+i}/\partial z)$, $i = 1, \dots, m$, run with complex time, are straight-line motions on T^m.

In (ii) T^m could be replaced by a finite cover ramified along D. The sense of Mumford includes the case where T^m is an extension of an abelian variety by a multiplicative group $(\mathbb{C}^*)^r$, i.e. a semi-abelian variety. In the sequel, when referring to ACI, we will mean in the sense of Adler and van Moerebeke. Let d_i be the weight degree of H_i and $F_i := H_i - c_i z_0^{d_i}$.

$\bar{A} = \cap\{F_i = 0\}$ is the natural completion of A into the n-dimensional weighted projective space with weight $\nu = (\nu_0, \dots, \nu_n)$, $\mathbb{P}^n_\nu := \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^*$, where $\mathbb{C}^*$ acts as follows:

$$t(z_0, \dots, z_n) = (z_0 t^{\nu_0}, \dots, z_n t^{\nu_n}).$$

$\bar{A} = A \cup A_\infty$, where A_∞ is the added divisor at infinity; $A_\infty := \bar{A} \cap \{z_0 = 0\}$.

Adler [**1**] and van Moerebeke [**6**] give a procedure which allows us, in some cases, to obtain a completion of A into an abelian variety (see [**5**] for an application to the system of Kowalewski's top). This procedure has several steps, including the following.

(1) Substitute the formal series

$$z_i(t) = \frac{1}{t^{\nu_i}}(z_i^{(0)} + z_i^{(1)} t + z_i^{(2)} t^2 + \cdots)$$

with a complex time t, $i = 1, \dots, n$, in the differential equation (1.1). The leading term belongs to the locus

$$\bigcap_{i=1}^{k+m} \{\nu_i z_i^0 + f_i(z^0)\}, \tag{1.2}$$

which decomposes into several components C_α (possibly of different dimensions). Let J be the Jacobian matrix of (1.2) evaluated at C_α. Only when the spectrum of J

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contains $(n-1)$ non-negative integers can a formal solution exist which depends on $(n-1)$ free parameters. By the majorant method, this solution, called the principal solution, converges [**2**].

(2) For each component C_α, confine the associated series into $A := \bigcap_{i=1}^{k+m}\{H_i = c_i\}$. This yields polynomial relations between the parameters. The parameters which are expressed in terms of the c_i are called trivial parameters and the others, effective parameters. The polynomial equations between the effective parameters define an affine variety of dimension $(m-k)$ called the Painlevé variety and denoted $D_\alpha^{(m-k)}$. Let $D := \sum_\alpha D_\alpha^{(m-1)}$.

(3) Let L be the linear span of a set of polynomial functions $y_0(z), \dots, y_N(z)$ having at worst a simple pole along all the generic expansions $z(t, p, D_\alpha)$, $p \in D_\alpha$. Namely, $y_i(z(t,p,D_\alpha)) = t^{-1}(y_i^{(0)}(p) + y_i^{(1)}(p) + \cdots)\ \forall\alpha$, where $y_i^{(0)} \neq 0$ for some α.

Construct the spaces

$$L^{(i)} = \{\text{polynomials } y = y(z_1, \dots, z_n) \text{ of weighted degree less than or equal to } i \text{ with simple pole in } t\}/\{H_j(z) = c_j,\ j = 1, \dots, k+m\}$$

and

$$L^{(i)} = \{y_0 = 1, y_1, \dots, y_{N_i}\},$$

the latter being a basis. Then increase $i = 1, 2, \dots,$ until the first time the image $\Phi_{L^{(i)}}(D)$ in $\mathbb{P}^{N_i}$ satisfies the following requirements:

(i) $\dim \overline{\Phi_{L^{(i)}}(D)} = m-1\ \forall\alpha$;

$\Phi_{L^{(i)}}(D)_| : A \to \Phi_{L^{(i)}}(A)$

(ii) is birational and holomorphic on A with $y \in \overline{\Phi_{L^{(i)}}(A)}$ and finite, implying that $y \in \Phi_{L^{(i)}}(A)$; and

(iii) $\text{genus}(\overline{\Phi_{L^{(i)}}(D)}) = N_i + m$;

where

$$\Phi_{L^{(i)}} : A \to \mathbb{P}^{N_i} : p \mapsto [y_0(z) : \cdots : y_{N_i}(z)],$$
$$\Phi_{L^{(i)}} : D_\alpha \to \mathbb{P}^{N_i} : p \mapsto \lim_{t\to 0} t[y_0(p) : \cdots : y_{N_i}(p)] = [0 : y_1^{(0)} : \cdots : y_{N_i}^{(0)}].$$

If there exist i_0 such that we obtain what is above, then note $N_{i_0} = N$, $L^{i_0} = L$ and $\tilde{D} = \Phi_L(D)$.

(i) and (ii) are almost automatic. Using Riemann–Roch and adjunction formulae [**4**], a divisor D on an abelian variety T^m satisfies $\chi(D) = \dim L(D) = g(D) - m + 1$, where $\chi(D)$ is the Euler characteristic of D, and $g(D)$ the geometric genus of D (the dimension of the space of top holomorphic forms), and if T^m is embedded into $\mathbb{P}^N$, then $\dim L(D) = N + 1$. Combining these two relations yields (iii).

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In this paper, we will study some classical Hamiltonian systems coming from mechanics, and show that they are not ACI, because of condition (iii) above, which cannot be satisfied. But, although they are not ACI, these systems have invariant varieties which could be completed (by adding a set of points) into a commutative algebraic group which is a $\mathbb{C}^*$ extension of an abelian curve and thus a semi-abelian surface.

2. Lagrange's top

Lagrange's top is given by the Hamiltonian vector field:

$$X_1 = \begin{cases} \dot{z}_1 = -z_5 - mz_2z_3, \\ \dot{z}_2 = z_4 + mz_1z_3, \\ \dot{z}_3 = 0, \\ \dot{z}_4 = z_3z_5 - z_2z_6, \\ \dot{z}_5 = z_1z_6 - z_4z_3, \\ \dot{z}_6 = z_2z_4 - z_1z_5, \end{cases}$$

with Hamiltonian $H_1 = \frac{1}{2}(z_1^2 + z_2^2 + (m+1)z_3^2) - z_6$. It possesses the following three other first integrals:

$$\begin{aligned} H_2 &= z_1z_4 + z_2z_5 + (m+1)z_3z_6, \\ H_3 &= z_3, \\ H_4 &= z_4^2 + z_5^2 + z_6^2. \end{aligned}$$

H_2 and H_4 are Casimir polynomials, and H_3 generates a second vector field X_2:

$$X_2 = \begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_2 = -z_1, \\ \dot{z}_3 = 0, \\ \dot{z}_4 = z_5, \\ \dot{z}_5 = -z_4, \\ \dot{z}_6 = 0. \end{cases}$$

X_2 and X_1 commute on the level surface $A := \bigcap_{i=1}^4 \{H_i = h_i\}$.

In [**3**] it is proved that this system linearizes on the generalized Jacobian of an elliptic curve, which is a $\mathbb{C}^*$-extension of the usual Jacobian. In this paper, we conduct its Painlevé analysis, and we apply the procedure cited above as much as possible.

2.1. Asymptotic expansions

Let z_i, $i = 1, \dots, 6$, have the following asymptotic expansion:

$$z_i(t) = \frac{1}{t^{\nu_i}}(z_i^{(0)} + z_i^{(1)}t + z_i^{(2)}t^2 + \cdots), \quad i = 1, \dots, 6. \tag{2.1}$$

Substituting (2.1) in the differential equations of X_1, at the zeroth step, the coefficients of t^{-2} (for z_1, z_2, z_3) and t^{-3} (for z_4, z_5, z_6) yields the following system:

$$\left.\begin{aligned} z_1 - mz_2z_3 - z_5 &= 0,\\ z_2 + mz_1z_3 + z_4 &= 0,\\ z_3 &= 0,\\ 2z_4 + z_3z_5 - z_2z_6 &= 0,\\ 2z_5 + z_1z_6 - z_3z_4 &= 0,\\ 2z_6 + z_2z_4 - z_1z_5 &= 0, \end{aligned}\right\} \tag{L-0}$$

for which we have the following four cases.

Case I: if $z_1 = z_2 = 0$, then the solution of (L-0) is identically zero.

Case II: if $z_1 = 0$, $z_2 \neq 0$, then $z^{(0)} = (0, -2\epsilon, 0, 2\epsilon, 0, -2)$ with $\epsilon^2 = -1$.

Case III: if $z_2 = 0$, $z_1 \neq 0$, then $z^{(0)} = (2\epsilon, 0, 0, 0, 2\epsilon, -2)$ with $\epsilon^2 = -1$.

Case IV: if $z_1 \neq 0$, $z_2 \neq 0$, then $z^{(0)} = (\alpha, -\beta, 0, \beta, \alpha, -2)$, where α is a free parameter and β is related to α by $\alpha^2 + \beta^2 + 4 = 0$.

Step by step we can find the solutions of the system (L-k) corresponding to $z^{(k)}$. We have the following linear algebra result after straightforward computations.

Lemma 2.1. *In cases II, III and IV, (L-k) has one degree of freedom for $k = 1, 2, 3, 4$.*

Since we are interested in the five-parameter Laurent solution, we only consider case IV. Let us denote the free parameters obtained at steps 1, 2, 3 and 4 by γ, δ, η and τ, respectively. We get

$$z^{(1)} = \left(\frac{\gamma\beta}{\alpha}, \gamma, \frac{2\gamma}{\gamma(m+1)}, \frac{-2m\gamma}{m-1}, \frac{2m\beta\gamma}{\alpha(m-1)}, 0\right),$$

$$\begin{aligned} z^{(2)} = \Bigg(&\frac{4m\gamma^2 + \delta\alpha^2(m-1)^2}{2\alpha(m-1)^2}, -\frac{\beta(4m\gamma^2 + \delta\alpha^2(m-1)^2)}{2\alpha^2(m-1)^2}, 0,\\ &-\frac{\beta(4m^2\gamma^2 + \delta\alpha 2(m-1)^2)}{2\alpha^2(m-1)^2}, -\frac{4m^2\gamma^2 + \alpha^2\delta(m-1)^2}{2\alpha(m-1)^2}, \delta\Bigg), \end{aligned}$$

$$\begin{aligned} z^{(3)} = \Bigg(&\frac{\eta\beta}{\alpha}, \eta, 0, -\frac{2\eta\alpha^2(m-1)^3 + 4m^2\gamma^3 + \gamma\delta\alpha^2 m(m-1)^2}{\alpha^2(m-1)^3},\\ &\frac{\beta(-2\eta\alpha^2(m-1)^3 + 4m^2\gamma^3 + \gamma\delta\alpha^2 m(m-1)^2)}{\alpha^3(m-1)^3}, 0\Bigg), \end{aligned}$$

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$$
\begin{aligned}
z^{(4)} = \Bigg(& -\frac{8\gamma\eta\alpha^2+\alpha^4(\delta^2+2\tau))(m-1)^4+8\delta\alpha^2\gamma^2m(m-1)^2+16m^2\gamma^4}{8\alpha^3(m-1)^4}, \\
& \beta\frac{8\gamma\eta\alpha^2+\alpha^4(\delta^2+2\tau))(m-1)^4+8\delta\alpha^2\gamma^2m(m-1)^2+16m^2\gamma^4}{8\alpha^4(m-1)^4}, 0, \\
& \beta\frac{\begin{array}{l}8\gamma\eta\alpha^2(m-1)^3(m-3)+3\alpha^4(\delta^2+2\tau))(m-1)^4\\ \qquad +24\delta\alpha^2\gamma^2m(m-1)^2+48m^2\gamma^4\end{array}}{8\alpha^4(m-1)^4}, \\
& \frac{\begin{array}{l}8\gamma\eta\alpha^2(m-1)^3(m-3)+3\alpha^4(\delta^2+2\tau))(m-1)^4\\ \qquad +24\delta\alpha^2\gamma^2m(m-1)^2+48m^2\gamma^4\end{array}}{8\alpha^3(m-1)^4}, \tau\Bigg).
\end{aligned}
$$

Thus we have the following lemma.

Lemma 2.2. *The system of differential equations of X_1 possesses Laurent series which depend on five parameters* (dim(phase space) − 1), *with leading terms given by case IV above.*

2.2. Divisors of poles

We now search for the set of Laurent solutions which stay in a fixed affine invariant surface, related to specific values of h_1, h_2, h_3, h_4, i.e. the Laurent solutions $z(t) = (z_(t), z_2(t), \ldots, z_6(t))$ such that $H_k(z(t)) = h_k$, $k = 1, 2, 3, 4$.

Substituting the Laurent solutions into H_k yields the following relations:

$$
\begin{aligned}
h_3 &= 2\frac{\gamma}{\alpha(m-1)}, \\
h_1 &= -3\delta - 2\frac{m(m+1)\gamma^2}{\alpha^2(m-1)^2}, \\
h_2 &= -12\frac{\eta}{\beta} + 12\frac{m\gamma\delta}{\alpha(m-1)} + 8\frac{m^2\gamma^3(m+3)}{\alpha^3(m+1)^3}, \\
h_4 &= -10\tau + 3\delta^2 + 24\frac{\gamma\eta(m+1)}{\alpha^2(m-1)^2} - 16\frac{m^2(m+1)(m+3)\gamma^4}{\alpha^4(m-1)^4},
\end{aligned}
$$

which show, respectively, that the parameters γ, δ, η and τ are trivial. The only relation between the remaining parameters is $\alpha^2 + \beta^2 + 4 = 0$. So we obtain the following proposition.

Proposition 2.3. *The divisor of poles of the functions z_j, $j = 1, 2, \ldots, 6$, is a genus 0 Riemann surface.*

Now, we wish to obtain a smooth embedding of $\bar{A}$ into $\mathbb{P}^N$, by the meromorphic functions of $L(kD)$, for some $k \geqslant 0$. If the surface A could be completed (by adding a set of points) into an abelian variety, such an embedding could exist with

$$
\text{genus}(D) = N + 2. \tag{2.2}
$$

In our case, since D is of genus 0, (2.2) cannot be satisfied. This provides us with the following proposition.

Proposition 2.4. *The system of Lagrange's top is not ACI.*

2.3. Completion into a semi-abelian variety

The H_i are ν'-homogeneous of degree d_i with $\nu' = (1,1,1,2,2,2)$, $d_1 = 2$, $d_2 = 3$, $d_3 = 1$, $d_4 = 4$,

$$F_i := H_i - c_i z_0^{d_i}, \qquad \bar{A} := \bigcap_{i=1}^{4} \{F_i = 0\}.$$

$\bar{A}$ is the completion of A in $\mathbb{P}^6_\nu$ with $\nu = (1,1,1,2,2,2)$, $A_\infty = \bar{A} \cap \{z_0 = 0\} := E$,

$$E = \left\{ [z_0 : z_1 : z_2 : z_3 : z_4 : z_5 : z_6] \in \mathbb{P}^6_\nu, \quad \begin{aligned} z_0 &= 0 \\ -z_6 + \tfrac{1}{2}(z_1^2 + z_2^2 + (m+1)z_3^2) &= 0 \\ z_1 z_4 + z_2 z_5 &= 0 \\ z_3 &= 0 \\ z_4^2 + z_5^2 + z_6^2 &= 0 \end{aligned} \right\}.$$

From the above equations, we obtain:

$$\begin{aligned} z_6(z_1^2 z_6 + 2z_5^2) &= 0, \\ z_6(z_2^2 z_6 + 2z_4^2) &= 0. \end{aligned}$$

So $E = E_1 \cup E_2$ with

$$\begin{aligned} z_6 &= 0, \\ z_1^2 + z_2^2 &= 0, \\ z_4^2 + z_5^2 &= 0, \\ z_1 z_4 + z_2 z_5 &= 0, \end{aligned}$$

for E_1, and

$$\begin{aligned} z_1^2 z_6 + 2z_5^2 &= 0, \\ z_2^2 z_6 + 2z_4^2 &= 0, \\ z_1 z_4 + z_2 z_5 &= 0, \\ z_6 - \tfrac{1}{2}(z_1^2 + z_2^2) &= 0, \end{aligned}$$

for E_2. $E_1 \cap E_2 = \{M_1, M_2\}$, with $M_1 = [0:1:\mathrm{i}:0:0:0:0]$, $M_2 = [0:1:-\mathrm{i}:0:0:0:0]$, $E_1 = E_1' \cup E_1''$, $E_1' = \{M_1, M_2\} \cup K_1$, $E_1'' = \{M_1, M_2\} \cup K_2$, where

$$K_1 := \begin{cases} z_2 = \mathrm{i} z_1, \\ z_4 = -\mathrm{i} z_5, \end{cases}$$

$$K_2 := \begin{cases} z_2 = -\mathrm{i} z_1, \\ z_4 = \mathrm{i} z_5, \end{cases}$$

$$E_2 = G_1 \cup G_2,$$

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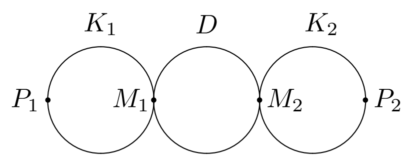
$$G_1 = \{[0 : z_1 : z_2 : 0 : \mathrm{i}z_2\sqrt{z_6/2} : -\mathrm{i}z_1\sqrt{z_6/2} : z_6] \in \mathbb{P}^6_\nu, \quad z_6 = \tfrac{1}{2}(z_1^2 + z_2^2)\},$$
$$G_2 = \{[0 : z_1 : z_2 : 0 : -\mathrm{i}z_2\sqrt{z_6/2} : \mathrm{i}z_1\sqrt{z_6/2} : z_6] \in \mathbb{P}^6_\nu, \quad z_6 = \tfrac{1}{2}(z_1^2 + z_2^2)\}.$$

However, in $\mathbb{P}^6_\nu$:

$$[0 : z_1 : z_2 : 0 : \mathrm{i}z_2\sqrt{z_6/2} : -\mathrm{i}z_1\sqrt{z_6/2} : z_6] = [0 : -z_1 : -z_2 : 0 : \mathrm{i}z_2\sqrt{z_6/2} : -\mathrm{i}z_1\sqrt{z_6/2} : z_6],$$

and by replacing $(-z_1, -z_2)$ by (z_1, z_2), we get that $G_1 = G_2 =: D$.

So, A_∞ has the following form:



Now let us look at components of A_∞ on which we can extend the fields X_1 and X_2 so that they remain independent. All components of A_∞ are integral curves for the linear field X_2. It suffices therefore to look at which of them are crossed by integral curves of X_1.

$\bar{D} = D \cup \{M_1, M_2\}$. By the majorant method, the series obtained converge in a neighbourhood of every point of D. For the other points of A_∞ we see in charts:

$$\text{if } p \in A_\infty \setminus D, \text{ then } (z_1(p), z_2(p)) \neq (0, 0).$$

Suppose that $z_1(p) \neq 0$ and put $U_1 := \{z_1 \neq 0\}$.

Remark 2.5. Beside P_1 and P_2, all points of $A_\infty \setminus D$ are in this chart:

$$v_0 := \frac{z_0}{z_1}, \quad v_2 := \frac{z_2}{z_1}, \quad v_3 := \frac{z_3}{z_1}, \quad v_4 := \frac{z_4}{z_1^2}, \quad v_5 := \frac{z_5}{z_1^2}, \quad v_6 := \frac{z_6}{z_1^2}.$$

Using the system of X_1, and then taking $z_1 = 1$, we obtain

$$\begin{aligned}
\dot{v}_0 &= v_0(v_5 + mv_2v_3), \\
\dot{v}_2 &= v_4 + 2v_2v_5 + mv_3(1 + v_2^2), \\
\dot{v}_3 &= v_3(v_5 + mv_2v_3), \\
\dot{v}_4 &= 2v_4v_5 - v_2v_6 + v_3(v_5 + 2mv_2v_4), \\
\dot{v}_5 &= v_6 + 2v_5^2 + v_3(2mv_2v_5 - v_4), \\
\dot{v}_6 &= v_2v_4 - v_5 + 2v_5v_6 + 2mv_2v_3v_6.
\end{aligned}$$

After the above coordinate changes, the equations which define $\bar{A}$ become

$$\begin{aligned}
\tfrac{1}{2}(1 + v_2^2 + (m+1)v_3^2) - v_6 - c_1v_0^2 &= 0, \\
v_4 + v_2v_5 + (m+1)v_3v_6 - c_2v_0^3 &= 0, \\
v_3 - c_3v_0 &= 0, \\
v_4^2 + v_5^2 + v_6^2 - c_4v_0^3 &= 0,
\end{aligned}$$

$$v_3 = c_3 v_0, \qquad v_6 = a v_0^2 + \tfrac{1}{2}(v_2^2 + 1),$$

with $a = \frac{1}{2}((m+1)c_3^2 - 2c_1)$.

We are reduced to the following system of three equations:

$$\begin{aligned}
\dot{v}_0 &= v_0(v_5 + m c_3 v_0 v_2),\\
\dot{v}_2 &= v_0(\alpha(1 + v_2^2) + \tfrac{1}{2}v_0^2),\\
\dot{v}_5 &= \tfrac{1}{2}(1 + v_2^2) + \alpha v_0^2 + 2v_5^2 + c_3 v_0[(2m+1)v_2 v_5 + a v_0(1 + v_2^2) - \tfrac{1}{2}v_0^3]
\end{aligned}$$

on

$$\begin{aligned}
\bar{A} = \{(v_0, v_2, v_5),\ &(1 + v_2^2)v_5^2 + 2v_0 v_2 v_5[a(1 + v_2^2) - \tfrac{1}{2}v_0^2]\\
&+ v_0^2[a(1 + v_2^2) - \tfrac{1}{2}v_0^2]^2 + [\tfrac{1}{2}(1 + v_2^2) + \alpha v_0^2]^2 - c_4 v_0^3 = 0\}.
\end{aligned}$$

$A_\infty = \bar{A} \cap \{v_0 = 0\}$. K_1 and K_2 are given by $v_0 = v_2 - \mathrm{i} = 0$ and $v_0 = v_2 + \mathrm{i} = 0$, respectively. M_1 corresponds to $v_0 = v_5 = v_2 - \mathrm{i} = 0$, and M_2 to $v_0 = v_5 = v_2 + \mathrm{i} = 0$. For $v = (0, \mathrm{i}, v_5)$, the differential equation is holomorphic and vanishes only if $v_5 = 0$. Then for every point of $K_1 \setminus M_1$, there exists a unique solution to the differential equation with initial conditions at the given point.

If $v_0 = v_2 - \mathrm{i} = 0$, v_5 is a solution of $\dot{v}_5 = 2v_5^2$. v_5 then gives $v_5 = 1/(ct + d)$, where c and d are constants. $K_1 \setminus M_1$ is an integral curve for X_1. It is also an integral curve for X_2, so the two fields are dependent on $K_1 \setminus M_1$, and by continuity, they remain dependent at the point M_1. For P_1, we can do the same thing in the chart $U_4 := \{z_4 \neq 0\}$, but, by the remark above, it is not necessary; by continuity, X_1 and X_2 remain dependent on K_1. On K_2, the conclusion is the same. We then obtain the following proposition.

Proposition 2.6. *$A \cup D$ is a commutative algebraic group.*

Proof. $\{y_0 = 1,\ y_1 = z_1,\ y_2 = z_2,\ y_3 = z_1^2,\ y_4 = z_4,\ y_5 = z_5,\ y_6 = z_2^2,\ y_7 = z_1 z_2\}$ is a basis of $L(2D)$ (cf. (2.2)) and provides a smooth embedding into $\mathbb{P}^7$. The vector fields X_1 and X_2 extend holomorphically and remain independent on D. The associated flows ϕ_1 and ϕ_2 are complete on $A \cup D = \bar{A} \setminus \{K_1, K_2\}$. As in the demonstration of Liouville–Arnold theorem, let $p_0 \in A \cup D$ be a point base and

$$\begin{aligned}
\Gamma : \mathbb{C}^2 &\to A \cup D,\\
(s, t) &\mapsto \phi_1^s(\phi_2^t(p_0)).
\end{aligned}$$

Define $\Lambda := \{(s, t) \in \mathbb{C}^2,\ \Gamma(s, t) = p_0\}$. $A \cup D$ is biholomorphic to $\mathbb{C}^2/\Lambda$, so it is equipped with a commutative group structure. Moreover, Λ is defined by ϕ_1 and ϕ_2, which are holomorphic on $A \cup D$; $\mathbb{C}^2/\Lambda$ is then an analytic variety; $A \cup D$ is also an analytic one by the biholomorphism between them. Since the last one is projective (its image in $\mathbb{P}^7$), it is an algebraic variety by Chow's Theorem. $\square$

Proposition 2.7. *$A \cup D$ is a $\mathbb{C}^*$-extension of an algebraic curve.*

Lemma 2.8. *Λ is a rank 3 lattice.*

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Proof. Λ is a lattice of $\mathbb{C}^2$, it is of rank non-superior to 4. If $\mathrm{rank}(\Lambda) = 4$, then by Liouville–Arnold theorem, $A \cup D$ would be compact. Let $\Lambda_1 := \{s \in \mathbb{C},\ \phi_1^s(p_0) = p_0\}$ and $\Lambda_2 := \{t \in \mathbb{C},\ \phi_2^t(p_0) = p_0\}$. The integral curves of the vector fields X_1 and X_2 are cyclic. We then get the following consequences:

(a) there exist $s_0 \neq 0$ such that $\Lambda_1 \simeq s_0\mathbb{Z}$ and $t_0 \neq 0$ such that $\Lambda_2 \simeq t_0\mathbb{Z}$;

(b) by changing the base point p_0 if necessary, there exists $(s_1, t_1) \notin (s_0\mathbb{Z}, t_0\mathbb{Z})$ such that $\phi_1^{s_1}(\phi_2^{t_2}(p_0)) = p_0$. Λ is then of rank 3 and is generated by $\{(s_0, 0), (0, t_0), (s_1, t_1)\}$.

$\square$

Proof of Proposition 2.7. Let $\Psi : \mathbb{C}^2 \to \mathbb{C}$ be the projection on the first variable. Ψ induces a morphism $\bar{\Psi} : \mathbb{C}^2/\Lambda \to \mathbb{C}/\Psi(\Lambda)$. $\Psi(\Lambda)$ is generated by s_0, s_1. So $\mathbb{C}/\Psi(\Lambda)$ is isomorphic to T^2,

$$\mathrm{Ker}(\bar{\Psi}) = \{\overline{(s,t)} \in \mathbb{C}^2/\Lambda,\ s \in \Psi(\Lambda)\} = (\Psi(\Lambda) \times \mathbb{C})/\Lambda \simeq (\mathbb{Z}^2 \times \mathbb{C})/\mathbb{Z}^3 \simeq \mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*.$$

We then have the following exact sequence:

$$0 \xrightarrow{\exp} \mathbb{C}^* \to A \cup D \xrightarrow{\bar{\Psi}} T^2 \to 0.$$

$\square$

3. Kirchhoff's top

The system of Kirchhoff's top is given by

$$X_1 = \begin{cases} \dot{z}_1 &= (a_3 - a_1)z_2 z_3 + (c_3 - c_1)z_5 z_6, \\ \dot{z}_2 &= (a_1 - a_3)z_1 z_3 + (c_1 - c_3)z_4 z_6, \\ \dot{z}_3 &= 0, \\ \dot{z}_4 &= -a_1 z_2 z_6 + a_3 z_3 z_5, \\ \dot{z}_5 &= a_1 z_1 z_6 - a_3 z_3 z_4, \\ \dot{z}_6 &= -a_1 z_1 z_5 + a_1 z_2 z_4, \end{cases}$$

generated by the Hamiltonian

$$H_1 = \tfrac{1}{2}(a_1(z_1^2 + z_2^2) + a_3 z_3^2 + c_1(z_4^2 + z_5^2) + c_3 z_6^2).$$

It possesses the following three other first integrals:

$$\begin{aligned} H_2 &= z_1 z_4 + z_2 z_5 + z_3 z_6, \\ H_3 &= z_3, \\ H_4 &= z_4^2 + z_5^2 + z_6^2. \end{aligned}$$

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H_2 and H_4 are Casimir polynomials; H_3 generates a second vector field X_2:

$$X_2 = \begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_2 = -z_1, \\ \dot{z}_3 = 0, \\ \dot{z}_4 = z_5, \\ \dot{z}_5 = -z_4, \\ \dot{z}_6 = 0. \end{cases}$$

The H_i are homogeneous (without weights) of degree $d_i : d_1 = d_2 = d_4 = 2$, $d_3 = 1$; thus we complete the level surface in $\mathbb{P}^6$.

3.1. Asymptotic expansions

Let z_i, $i = 1, \dots, 6$, have the following asymptotic expansion:

$$z_i(t) = \frac{1}{t}(z_i^{(0)} + z_i^{(1)} t + z_i^{(2)} t^2 + \cdots). \tag{3.1}$$

Substituting (3.1) into the differential equations of X_1; at the zeroth step the coefficients of t^{-2} yield the system

$$\left.\begin{aligned} z_1 + (a_3 - a_1) z_2 z_3 + (c_3 - c_1) z_5 z_6 = 0, \\ z_2 + (a_1 - a_3) z_1 z_3 + (c_1 - c_3) z_4 z_6 = 0, \\ z_3 = 0, \\ z_4 - a_1 z_2 z_6 + a_3 4 z_3 z_5 = 0, \\ z_5 + a_1 z_1 z_6 - a_3 z_3 z_4 = 0, \\ z_6 - a_1 z_1 z_5 + a_1 z_2 z_4 = 0, \end{aligned}\right\} \tag{K-0}$$

which has the following three solutions.

Case I: $z_6 = 0$, then the solution is identically zero.

Case II: $z_6 \neq 0$ and $z_5 = 0$, then $z^{(0)} = (0, (\epsilon/a_1), 0, x\epsilon, 0, x)$, where $\epsilon^2 = -1$, $a_1(c_3 - c_1)x^2 = 1$.

Case III: $z_6 \neq 0$, $z_5 \neq 0$, then $z^{(0)} = (\alpha, \beta, 0, a_1 x \beta, -a_1 x \alpha, x)$, where x is as above and $1 + a_1^2(\alpha^2 + \beta^2) = 0$.

Let (K-k) denote the system of equations corresponding to $z^{(k)}$. After computations we obtain the following lemma.

Lemma 3.1. *In cases II and III, (K-k) has one degree of freedom for $k = 1$ and three degrees of freedom for $k = 2$.*

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Let γ be the free parameter obtained for $k=1$, and δ, η and τ the others obtained for $k=2$. In case III we get

$$z^{(1)} = \left(-\frac{(c_3-c_1)\gamma x a_3}{a_3-a_1}, \frac{(c_3-c-1)\gamma x a_3\alpha}{\beta(a_3-a_1)}, -\frac{(c_3-c_1)\gamma x}{\beta(a_3-a_1)}, \frac{\alpha\gamma}{\beta}, \gamma, 0\right),$$

$$\begin{aligned} z^{(2)} = \Bigg((c_3-c_1)\left[x(\eta-a_1\alpha\tau)-\frac{a_3\alpha\gamma^2}{a_1\beta^2(a_3-a_1)}\right], \\ \delta, 0, -a_1(\tau\beta+\delta x)-\frac{a_3 x\gamma^2(c_3-c-1)}{\beta(a_3-a-1)}, \eta, \tau\Bigg). \end{aligned}$$

This yields the following lemma.

Lemma 3.2. *The system of differential equations of X_1 possesses Laurent series, which depend on five parameters* (dim(phase space) − 1), *with leading terms given by case III above.*

3.2. Divisor of poles

Confining the Laurent solutions to the invariant surface

$$\bigcap_{i=1}^{4}\{H_i=h_i\},$$

yields

$$\begin{aligned} \gamma &= h_3, \\ \delta &= \frac{(a_1-a_3)[(a_1^2\beta^2+1)(4h_1-h_4(c_1+c_3))+3a_1^3 x h_2\alpha\beta(c_3-c_1)](3a_3+a_1))}{6a_1(a_1-a_3)\alpha}, \\ \tau &= \frac{(a_1-a_3)(a_1^2\beta^2+1)(2c_1h_4-2h_1-c_3h_4)-a_1h_3^2(c_1-c_3)}{6(c_1-c_3)(a_1-a_3)x(a_1^2\beta^2+1)}, \\ \eta &= \frac{\begin{aligned} a_1[(a_1-a_3)(a_1^2\beta^2+1)((2h_1-2h_4c_3+c_1h_4)\beta+3a_1x\alpha h_2(c_3-c_1)) \\ +h_3^2\beta(c_3-c_1)(2a_1-3a_3)-3\alpha x c_3 h_2 a_1 a_3] \end{aligned}}{6(c_1-c_3)(a_1-a_3)x(a_1^2\beta^2+1)}. \end{aligned}$$

So these parameters (γ, η, δ and τ) are trivial. The relationship between the remaining parameters is the previous one: $1+a_1^2(\alpha^2+\beta^2)=0$. Moreover, the leading terms depend on x, which verifies that $a_1(c_3-c_1)x^2=1$, giving the following proposition.

Proposition 3.3. *The divisors of poles of the functions z_j, $j=1,\dots,6$, are two isomorphic Riemann surfaces of genus* 0.

As in the case of Lagrange's top, since the divisors of poles are Riemann surfaces of genus 0, we cannot satisfy requirement (2.2), giving the following proposition.

Proposition 3.4. *The system of Kirchhoff's top is not ACI.*

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3.3. Completion

As in the case of Lagrange's top, we now wish to complete the invariant variety into a semi-abelian surface:

$$A_\infty = \left\{ [0 : z_1 : z_2 : z_3 : z_4 : z_5 : z_6] \in \mathbb{P}^6, \quad \begin{aligned} (c_3 - c_1)z_6^2 + a_1(z_1^2 + z_2^2) &= 0 \\ z_1 z_4 + z_2 z_5 &= 0 \\ z - 3 &= 0 \\ z_4^2 + z_5^2 + z_6^2 &= 0 \end{aligned} \right\}.$$

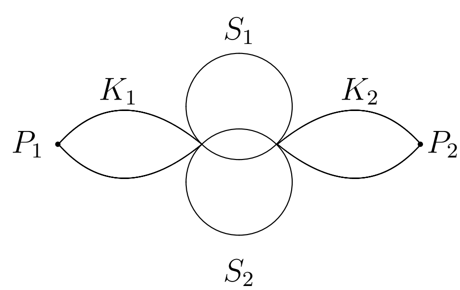
From the above equations, we get

$$z_6^2\left(z_1^2 - \frac{c_3 - c_1}{a_1} z_5^2\right) = 0,$$
$$z_6^2\left(z_2^2 - \frac{c_3 - c_1}{a_1} z_4^2\right) = 0;$$

$M_1 = [0 : 1 : \mathrm{i} : 0 : 0 : 0 : 0]$ and $M_2 = [0 : 1 : -\mathrm{i} : 0 : 0 : 0 : 0]$; $E := A_\infty = E_1 \cup E_2$, with $E_1 \cap E_2 = \{M_1, M_2\}$. E_1 has the same form as above:

$$E_2 = S_1 \cup S_2, \qquad S_1 = \left\{ \left[0 : z_1 : z_2 : 0 : a_1 z_6 z_2 : -a_1 z_6 z_1 : \frac{1}{\sqrt{a_1(c_3 - c_1)}} \right] \right\},$$
$$S_2 = \left\{ \left[0 : z_1 : z_2 : 0 : -a_1 z_6 z_2 : a_1 z_6 z_1 : \frac{1}{\sqrt{a_1(c_3 - c_1)}} \right] \right\}, \qquad S_1 \cap S_2 = \{M_1, M_2\}.$$

A_∞ has the following form:



$D_i := S_i \backslash \{M_1, M_2\}$, $i = 1, 2$. In an analogous manner to the Lagrange's top, we obtain the following proposition.

Proposition 3.5. (i) *$A \cup D_1 \cup D_2$ is a commutative algebraic group.* (ii) *It is a $\mathbb{C}^*$-extension of an abelian curve.*

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4. Euler–Poinsot's top

The system of Euler–Poinsot's top has the following form:

$$\left.\begin{aligned}\dot z_1 &= (\lambda_3-\lambda_2)z_2z_3,\\ \dot z_2 &= (\lambda_1-\lambda_3)z_1z_3,\\ \dot z_3 &= (\lambda_2-\lambda_1)z_1z_2,\\ \dot z_4 &= \lambda_3z_3z_5-\lambda_2z_2z_6,\\ \dot z_1 &= \lambda_1z_1z_6-\lambda_3z_3z_4,\\ \dot z_1 &= \lambda_2z_2z_4-\lambda_1z_1z_5.\end{aligned}\right\} \tag{4.1}$$

The Hamiltonian function is $H_1 = H = \frac{1}{2}(\lambda_1 z_1^2 + \lambda_2 z_2^2 + \lambda_3 z_3^2)$. Besides the following Casimir functions $H_3 = z_4^2 + z_5^2 + z_6^2$ and $H_4 = z_1z_4 + z_2z_5 + z_3z_6$, the system possesses a fourth first integral $H_2 = \frac{1}{2}(z_1^2 + z_2^2 + z_3^2)$. H_2 generates the following second vector field:

$$\left.\begin{aligned}\dot z_1 &= 0,\\ \dot z_2 &= 0,\\ \dot z_2 &= 0,\\ \dot z_4 &= z_3z_5-z_2z_6,\\ \dot z_5 &= z_1z_6-z_3z_4,\\ \dot z_6 &= z_2z_4-z_1z_5.\end{aligned}\right\} \tag{4.2}$$

4.1. Asymptotic expansions

As in the preceding cases, we substitute in the differential equation (4.1) $z_i(t)$ by asymptotic expansions $(1/t)(z_i^{(0)} + z_i^{(1)}t + z_i^{(2)}t^2 + \cdots)$. For $z^{(0)}$, we obtain the following system:

$$\left.\begin{aligned}z_1 + (\lambda_3-\lambda_2)z_2z_3 &= 0,\\ z_2 - (\lambda_3-\lambda_1)z_1z_3 &= 0,\\ z_1 + (\lambda_2-\lambda_1)z_1z_2 &= 0,\\ z_4 - \lambda_2z_2z_6 + \lambda_3z_3z_5 &= 0,\\ z_5 + \lambda_1z_1z_6 - \lambda_3z_3z_4 &= 0,\\ z_6 - \lambda_1z_1z_5 + \lambda_2z_2z_4 &= 0.\end{aligned}\right\} \tag{EP-0}$$

Its non-trivial solution is

$$z^{(0)} = (x, y, (\lambda_1-\lambda_2)xy, (\lambda_1-\lambda_2)(\lambda_2-\lambda_3)xy\alpha, \alpha, (\lambda_1-\lambda_2)xy),$$

with

$$(\lambda_1-\lambda_2)(\lambda_3-\lambda_1)x^2 - 1 = 0, \qquad (\lambda_1-\lambda_2)(\lambda_2-\lambda_3)y^2 - 1 = 0,$$

and α is a free parameter.

Let us denote by (EP-k) the differential equation obtained at step k. After computations, we get the following lemma.

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Lemma 4.1. *(EP-k) has one degree of freedom for $k = 1$ and three degrees of freedom for $k = 2$.*

Denote by β the free parameter obtained at step 1 and by γ, δ and η the free parameters obtained at step 2. We obtain

$$z^{(1)} = \left(0, 0, 0, \frac{\lambda_1}{\lambda_2}(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)xy\beta, \beta, \frac{\lambda_3}{\lambda_2}(\lambda_1 - \lambda_2)x\beta\right),$$

$$\begin{aligned} z^{(2)} = \Bigg(&\gamma, \delta, (\lambda_2 - \lambda_1)(x\delta + y\gamma), \\ &\frac{\begin{aligned}(y\eta - \alpha\delta)(\lambda_3^2(\lambda_2 - \lambda_1) - \lambda_2^2(\lambda_3 - \lambda_1)) + xy\alpha\gamma(\lambda_3^2(\lambda_2 - \lambda_1) \\ -\lambda_1^3(\lambda_3 - \lambda_2)^2 + \lambda_2^2(\lambda_1 - \lambda_3)(\lambda_1(\lambda_2 - \lambda_3) + \lambda_3(\lambda_2 - \lambda_1)))\end{aligned}}{x(\lambda_3^2(\lambda_2 - \lambda_1) + \lambda_1^2(\lambda_3 - \lambda_2))}, \\ &\eta, \frac{\begin{aligned}\eta(\lambda_2\lambda_3 - \lambda_1\lambda_2 + \lambda_1\lambda_3) + x\alpha\gamma(\lambda_1 - \lambda_2)(\lambda_3^2(\lambda_1 - \lambda_2) + \lambda_1^2(\lambda_2 - \lambda_3)) \\ +2y\alpha\delta\lambda_2\lambda_3(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1))\end{aligned}}{x(\lambda_3^2(\lambda_2 - \lambda_1) + \lambda_1^2(\lambda_3 - \lambda_2))}\Bigg). \end{aligned}$$

As above we have the following lemma.

Lemma 4.2. *The system of differential equations (4.1) possesses a Laurent series which depends on five parameters with leading terms given by $z^{(0)}$ above.*

4.2. Divisor of poles

By confining the Laurent solutions into the invariant surface

$$\bigcap_{i=1}^{4}\{H_i = h_i\},$$

we obtain

$$h_1 = \lambda_1 x\gamma + \lambda_2 y\delta - \lambda_3 xy(\lambda_2 - \lambda_1)^2(x\delta + y\gamma), \tag{4.3}$$

$$h_2 = x\gamma + y\delta - xy(\lambda_2 - \lambda_1)^2(x\delta + y\gamma), \tag{4.4}$$

$$\begin{aligned} h_3 = {}& -2\alpha^2(\lambda_2 - \lambda_1)^2 x\gamma + 2\alpha\eta + \beta^2 \\ & + 2y\alpha(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)[xy\alpha\gamma(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2) + y\eta - \alpha\delta] \\ & - \frac{\beta^2[\lambda_3^2(\lambda_2 - \lambda_1) + \lambda_1^2(\lambda_3 - \lambda_2)]}{\lambda_2^2(\lambda_3 - \lambda_1)} \\ & - \frac{2\alpha(\lambda_2 - \lambda_1)[2y\alpha\delta(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)\lambda_2\lambda_3 + \eta(\lambda_2\lambda_3 + \lambda_1\lambda_3 - \lambda_1\lambda_2)]}{\lambda_3^2(\lambda_2 - \lambda_1) + \lambda_1^2(\lambda_3 - \lambda_2)}, \end{aligned} \tag{4.5}$$

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$$\begin{aligned}
h_4 = {} & xy\alpha\gamma[-2(\lambda_2-\lambda_1)^2+(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)+(\lambda_3-\lambda_2)] \\
& + y\eta + \alpha\delta + \frac{\alpha\delta(\lambda_2-\lambda_1)}{\lambda_3-\lambda_1} \\
& + \frac{\begin{array}{r} y(\lambda_1-\lambda_2)[2y\alpha\delta\lambda_3\lambda_2(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)+\eta(\lambda_3\lambda_2+\lambda_1\lambda_3-\lambda_1\lambda_2)] \\ +(y\eta-\alpha\delta)[\lambda_3^2(\lambda_2-\lambda_1)+\lambda_2^2(\lambda_1-\lambda_3)] \end{array}}{\lambda_3^2(\lambda_2-\lambda_1)+\lambda_1^2(\lambda_3-\lambda_2)}. \quad (4.6)
\end{aligned}$$

Equations (4.3) and (4.4) show that γ and δ are trivial. Using (4.5) and (4.6), we eliminate η and find a relation between α and β which has the following form

$$c_1\alpha^2 - h_4\alpha + (c_2\beta^2 + c_3) = 0, \tag{4.7}$$

where c_i, $i = 1, 2, 3$, are functions of λ_j, $j = 1, 2, 3, 4$, x and y. Equation (4.7) defines a genus 0 Riemann surface. Recall that we had

$$(\lambda_1-\lambda_2)(\lambda_3-\lambda_1)x^2 - 1 = 0, \qquad (\lambda_1-\lambda_2)(\lambda_2-\lambda_3)y^2 - 1 = 0.$$

So we obtain the following proposition.

Proposition 4.3. *The divisors of poles of the functions* z_j, $j = 1, \ldots, 6$, *are four Riemann surfaces of genus* 0.

As above, we have the following proposition.

Proposition 4.4. *The system of Euler–Poinsot's top is not ACI.*

4.3. Completion

Using the same preceding notation, A_∞ is given here by

$$\begin{aligned}
\lambda_1 z_1^2 + \lambda_2 z_2^2 + \lambda_3 z_3^2 &= 0, \\
z_1^2 + z_2^2 + z_3^2 &= 0, \\
z_4^2 + z_5^2 + z_6^2 &= 0, \\
z_1 z_4 + z_2 z_5 + z_3 z_6 &= 0.
\end{aligned}$$

The first two equations give

$$\begin{aligned}
z_2^2 &= -\frac{\lambda_1-\lambda_3}{\lambda_2-\lambda_3} z_1^2, \\
z_3^2 &= -\frac{\lambda_2-\lambda_1}{\lambda_2-\lambda_3} z_1^2.
\end{aligned}$$

Then the fourth becomes $z_1(z_4 + \zeta z_5 + \xi z_6) = 0$, where

$$\begin{aligned}
\zeta^2 &= \frac{\lambda_1-\lambda_3}{\lambda_2-\lambda_3}, \\
\xi^2 &= -\frac{\lambda_2-\lambda_1}{\lambda_2-\lambda_3}.
\end{aligned}$$

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If $z_1 = 0$, then $z_2 = z_3 = 0$ and $z_4^2 + z_5^2 + z_6^2 = 0$. Note $C := \{[0:0:0:0:z_4:z_5:z_6] \in \mathbb{P}^6,\ z_4^2 + z_5^2 + z_6^2 = 0\}$. If $z_1 \neq 0$, then $z_4 = -(\zeta z_5 + \xi z_6)$, and the third equation becomes $(1+\zeta^2)z_5^2 + (1+\xi^2)z_6^2 + 2\zeta\xi z_5 z_6 = 0$. Dividing by $1+\zeta^2$ yields $z_5^2 + 2\chi z_5 z_6 + \chi^2 z_6^2 = 0$, so $z_5 = \chi z_6$ and $z_4 = (\zeta\chi - \xi)z_6$ with

$$\chi^2 = \frac{\lambda_1 - \lambda_3}{\lambda_1 - \lambda_2}.$$

Note $S_i := \{[0 : z_1 : \zeta z_1 : \xi z_1 : (\zeta\chi - \xi)z_6 : -\chi z_6 : z_6] \in \mathbb{P}^6\}$, $i = 1,2,3,4$, corresponding to the different values of (ζ, ξ); $A_\infty = C \cup (\cup_{i=1}^4 S_i)$.

Remark 4.5. The Jacobian matrix associated to the system of A_∞ is

$$\begin{pmatrix} \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 & 0 & 0 & 0 \\ z_1 & z_2 & z_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_4 & z_5 & z_6 \\ z_4 & z_5 & z_6 & z_1 & z_2 & z_3 \end{pmatrix}.$$

So the singular locus of A_∞ is $C \cup [0:1:\zeta:\xi:0:0:0]$.

Applying the method above, we obtain the following proposition.

Proposition 4.6.

(i) $A \cup (\cup_{i=1}^4 S_i)$ *is a commutative algebraic group.*

(ii) *It is a* $\mathbb{C}^*$*-extension of an abelian curve.*

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