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# CONDITIONS FOR A ZERO SUM MODULO $n$

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In this paper the following result is proved.

THEOREM. *Let $n>0$ and $k\geq 0$ be integers with $n-2k\geq 1$. Given any $n-k$ integers*

(1) $$a_1, a_2, a_3, \ldots, a_{n-k}$$

*there is a non-empty subset of indices $I\subset\{1, 2, \ldots, n-k\}$ such that the sum $\sum_{i\in I} a_i\equiv 0 (\text{mod } n)$ if at most $n-2k$ of the integers* (1) *lie in the same residue class modulo $n$.*

The result is best possible if $n\geq 3k-2$ in the sense that if "at most $n-2k$" is replaced by "at most $n-2k+1$" the result becomes false. This can be seen by taking $a_j=1$ for $1\leq j\leq n-2k+1$ and $a_j=2$ for $n-2k+2\leq j\leq n-k$, noting that the number of 2's here is $n-k-(n-2k+1)=k-1\leq n-2k+1$.

LEMMA 1. *Let $n$ be a positive integer. For $i=1, 2, \ldots, r$ let $A_i$ be a set of $v_i$ positive integers, incongruent modulo $n$, and none$\equiv 0 (\text{mod } n)$. If $\sum_{i=1}^r v_i\geq n$ then the set $\sum_{i=1}^r (\{0\}\cup A_i)$ contains some non-zero multiple of $n$.*

**Proof.** Suppose the result is false. We may presume that $A_r$ is not empty. We use a result of J. H. B. Kemperman and P. Scherk [1] as follows. Let $A$ be the union of $r$ incongruent residue classes $0, a_1, a_2, \ldots, a_{r-1} (\text{mod } n)$ and $B$ the union of $s$ incongruent classes $0, b_1, b_2, \ldots, b_{s-1} (\text{mod } n)$. Suppose that if $a\in A$ and $b\in B$ and $a+b\equiv 0 (\text{mod } n)$ then $a\equiv b\equiv 0 (\text{mod } n)$. Then $A+B$ is the union of at least $\min(n, r+s-1)$ distinct residue classes modulo $n$. If we apply this result to the two sets $\{0\}\cup A_1$ and $\{0\}\cup A_2$ we conclude that the set

$$[\{0\}\cup A_1]+[\{0\}\cup A_2]$$

contains representatives from at least $\min(n, v_1+v_2+1)$ distinct classes modulo $n$. Continuing by induction we conclude that

$$\sum_{i=1}^{r-1} (\{0\}\cup A_i)$$

contains representatives from at least

$$\min\left(n, 1+\sum_{i=1}^{r-1} v_i\right)\geq n-v_r+1$$

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distinct residue classes modulo $n$, and so it contains representatives from at least $n-v_r$ distinct non-zero residue classes modulo $n$. But $-A_r$ has representatives from $v_r$ non-zero residue classes modulo $n$, and so there must be a representative from a non-zero residue class in common because there are only $n-1$ such classes. From this observation the lemma follows.

If $S$ is a set of integers (not necessarily distinct) we let $\sum S$ denote the set of distinct non-zero residue classes (mod $n$) represented by sums of integers in $S$. We call a set of three integers which are incongruent modulo $n$ a *triple*, and any incongruent pair of integers a *double*.

LEMMA 2. *If $S$ is a triple with no subset having a zero sum modulo $n$ then $|\sum S|\geq 5$, and if $S$ does not contain $n/2 \pmod n$ then $|\sum S|\geq 6$. (The notation $|\sum S|$ has the usual meaning, the number of elements in $\sum S$.)*

**Proof.** Let $S=\{a, b, c\}$. If two of the congruences

$$(2) \qquad a+b \equiv c, \qquad a+c \equiv b, \qquad b+c \equiv a \pmod n$$

are false, say $a+b\not\equiv c$, $a+c\not\equiv b$, then $\sum S$ contains the six distinct non-zero residue classes $a, b, c, a+b, a+c, a+b+c \pmod n$, because for example if $a+b+c\equiv a$ then we have the contradiction $b+c\equiv 0 \pmod n$.

Therefore we may assume that at least two of the congruences in (2) hold, say $a+b\equiv c$ and $a+c\equiv b$. In this case, by addition we see that $2a\equiv 0$, i.e. $a\equiv n/2 \pmod n$, and $\sum S$ contains the five distinct non-zero residue classes $a$, $b$, $c$, $b+c$, $a+b+c \pmod n$, because for example if $b+c\equiv a$ we obtain the contradiction $a+b+c\equiv 2a\equiv 0 \pmod n$.

LEMMA 3. *If $S$ is a double having no zero sum then $|\sum S|=3$.*

The proof is obvious.

Now we turn to the theorem itself, giving a proof by contradiction. Let $a_1, a_2, \ldots, a_{n-k}$ be a sequence of $n-k$ positive integers with no zero sums modulo $n$, and such that at most $n-2k$ terms belong to the same residue class modulo $n$. We will show that there is a partition of the index set $\{1, 2, \ldots, n-k\}$ into disjoint sets $P_1 \cup \cdots \cup P_r$ in such a way that if $i, j\in P_t$ and $i\neq j$ then $a_i\not\equiv a_j \pmod n$, and so that

$$\sum_{t=1}^{r} |\sum S_t| \geq n$$

where $S_t=\{a_i \mid i\in P_t\}$ for $1\leq t\leq r$. Then Lemma 1 gives us the required contradiction.

First suppose that the sequence (1) contains no integer $\equiv n/2 \pmod n$. Select from the sequence (1) in any manner whatsoever, triples of elements if possible until all that is left in the sequence (1) is a single repeated element $a$ modulo $n$, or two repeated elements $a$ and $b$ modulo $n$, with $a\not\equiv b$. Suppose by this process we get $j$ triples, with $j\geq 0$, and the remaining elements $a$ (with say $\lambda$ occurrences)

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and $b$ (with say $\mu$ occurrences), and we may presume $\lambda \geq \mu \geq 0$. Since there are $n-k$ elements in the sequence (1) we see that

$$n-k = 3j+\lambda+\mu, \qquad j = \tfrac{1}{3}(n-k-\lambda-\mu).$$

In addition to the $j$ triples we form $\mu$ doubles of the form $\{a, b\}$ and $\lambda-\mu$ singles of the form $\{a\}$. Call the triples $S_i$ with $1 \leq i \leq j$, the doubles $S_i$ with $j+1 \leq i \leq j+\mu$, and the singles $S_i$ with $j+\mu+1+\leq i \leq j+\lambda$. By Lemmas 2 and 3 we conclude that

$$\sum_{i=1}^{j+\lambda} |\sum S_i| = 6j+3\mu+(\lambda-\mu) = n+(n-2k)-\lambda \geq n,$$

the last inequality holding because $n-2k \geq \lambda$, there being at most $n-2k$ identical elements modulo $n$ in the sequence (1).

Finally, suppose that $n/2$ is in the sequence (1). It can occur only once since $n/2+n/2 \equiv 0 (\text{mod}\, n)$. By the same process as in the first part we choose $j$ triples $S_1, \ldots, S_j$ without using the element $n/2$, so that what remains in the sequence (1) are the element $n/2$ once, the element $a$ occurring $\lambda$ times, and the element $b$ occurring $\mu$ times, again with $\lambda \geq \mu \geq 0$. We deal with three special cases: $\lambda > \mu$; $\lambda = \mu > 0$; $\lambda = \mu = 0$. In all cases we have $n-k = 3j+\lambda+\mu+1$.

In case $\lambda > \mu$ we choose $\mu$ doubles of the form $\{a, b\}$ and an additional double $\{a, n/2\}$, and also $\lambda-\mu-1$ singles of the form $\{a\}$. Thus we get

$$\sum_i |\sum S_i| \geq 6j+3(\mu+1)+(\lambda-\mu-1) = n+(n-2k)-\lambda \geq n$$

as before.

In case $\lambda = \mu > 0$ we have, in addition to the $j$ triples without the element $n/2$ also the triple $\{n/2, a, b\}$ and the $\lambda-1$ doubles of the form $\{a, b\}$. This gives us

$$\sum_i |\sum S_i| \geq 6j+5+3(\lambda-1) = n+(n-2k)-\lambda \geq n.$$

In case $\lambda = \mu = 0$ we have one single $\{n/2\}$ and we get

$$\sum_i |\sum S_i| \geq 6j+1 = n+(n-2k)-1 \geq n.$$

## Reference

1. H. Halberstam and K. F. Roth, *Sequences I*, Oxford, 1966, p. 50, Theorem 15′.

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