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APPROXIMATION BY UNIMODULAR FUNCTIONS

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Introduction. The theorems in this paper are all concerned with either pointwise or uniform approximation by functions which have unit modulus or by convex combinations of such functions. The results are related to, and are outgrowths of, the theorems in [**4**; **5**; **10**].

In § 1, we show that a function bounded by 1, which is analytic in the open unit disc Δ and continuous on $\bar{\Delta}$ may be approximated uniformly on the set where it has modulus 1 (subject to certain restrictions; see Theorem 1) by a finite Blaschke product; that is, by a function of the form

$$(*) \qquad \lambda \prod_{i=1}^{N} (z - \alpha_i)(1 - \bar{\alpha}_i z)^{-1},$$

where $|\lambda| = 1$ and $|\alpha_i| < 1$, $i = 1, \ldots, N$. In § 1 we also discuss pointwise approximation by Blaschke products with restricted zeros. In § 2, we show that continuous functions from the unit circle T into the set of unimodular continuous functions on some compact Hausdorff space X may be "approximately factored" as the quotient of analytic functions, with values in $C(X)$, which closely resemble (*). In § 3, we prove that a function analytic in a neighbourhood of $\bar{\Delta}$ and bounded by 1 may be approximated in a number of very strong norms by convex combinations of finite Blaschke products.

In § 1 we will frequently make use of the factorization theorem for bounded holomorphic functions. This theorem states that each bounded holomorphic function h may be expressed uniquely as $h = BSF$, where B is a (possibly infinite) Blaschke product, S is a singular function, that is, $S(z) = \exp[-g(z)]$, where

$$g(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\mu(e^{i\theta})$$

and μ is a non-negative measure on the unit circle which is singular with respect to Lebesgue measure σ (we shall call S the singular function determined by μ), and finally, $F(z) = \exp[f(z)]$, where

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log|h(e^{i\theta})| \, d\sigma(e^{i\theta}).$$

The proof of this factorization theorem and further discussion of Blaschke products, singular functions, and outer functions may be found in [**8**, Chapter 5].

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1. Approximation of analytic functions.

THEOREM 1. *Let K be a compact subset of the unit circle T. Suppose that f is analytic in the unit disc Δ, continuous on the closed unit disc, and bounded by* 1.

(a) *If $|f| = 1$ on a neighbourhood of K, then f may be uniformly approximated on K by finite Blaschke products.*

(b) *If K has Lebesgue measure zero, then the conclusion holds if only $|f| = 1$ on K.*

Proof. (a) Let $f = BSF$ be the canonical factorization of f into a Blaschke product B, a singular function S, and an outer function F. We shall deal with these terms one at a time.

If f has only finitely many zeros, then B is a finite Blaschke product and we may pass onto consideration of S and F. Otherwise, note that since $f \neq 0$ on a neighbourhood $\mathcal{O}$ of K, the zeros of B do not collect in $\mathcal{O}$. The finite subproducts of B converge to B uniformly on any compact subset of the plane which does not meet the points $\{\bar{\alpha}_j^{-1}\}$ or their points of accumulation in T, where $\{\alpha_j\}$ is the set of zeros of B; in particular, they converge uniformly on K to B.

The measure that determines S is supported off $\mathcal{O}$, again by the continuity of f and the fact that $f \neq 0$ on $\mathcal{O}$. By Frostman's theorem [**6**] we may choose a sequence of points α_n in Δ tending to zero such that for each n,

$$B_n = (S - \alpha_n)(1 - \bar{\alpha}_n S)^{-1}$$

is a Blaschke product. The only singularities of B_n on T are at the points in the closed support of the measure that determines S and these do not lie in $\mathcal{O}$. An argument like the one in the previous paragraph shows that S may be approximated uniformly on K by a finite subproduct of one of the B_n.

Finally, we consider $F = \exp(u + i\,{}^*u)$, where $u \in L^1(T)$, $u \leqq 0$, and ${}^*u$ is the conjugate function of u. Since $|F| = |f| = 1$ on an open set U, $u = 0$ on U. I claim that there is a singular function S_1 whose determining measure is supported off K such that S_1 approximates F to within a given ϵ on K. Once this is established, an argument like that in the preceding paragraph will complete the proof of (a). To prove the existence of such a measure, we proceed as follows.

For $e^{i\theta} \notin K$, let u_θ be the negative harmonic function on Δ determined by extending the point mass with weight -1 at $e^{i\theta}$ to the disc

$$u_\theta(re^{it}) = -P_r(t-\theta) = -\frac{1-r^2}{1-2r\cos(t-\theta)+r^2},$$

where P_r is the Poisson kernel. The conjugate function, ${}^*u_\theta$, of u_θ, is given by

$${}^*u_\theta(re^{it}) = -\frac{2r\sin(t-\theta)}{1-2r\cos(t-\theta)+r^2}.$$

Both u_θ and ${}^*u_\theta$ extend across any arc of T which does not contain $e^{i\theta}$ and, in particular, $u_\theta(e^{it}) = 0$ for all $e^{it} \in T$, $t \neq \theta$. Consider the convex cone C consisting of all functions of the form

$$v(e^{it}) = \sum_{j=1}^{N} s_j\, {}^*u_{\theta_j}(e^{it}), \qquad e^{it} \in K,$$

where $s_j \geqq 0$ for $j = 1, \ldots, N$ and $e^{i\theta_j} \notin K$. These functions are all continuous on K. I claim that ${}^*u$ lies in the (uniform) closure on K of C. For suppose that λ is any real measure on K with

$$-\int_K \frac{\sin(t-\theta)}{1-\cos(t-\theta)}\, d\lambda(t) \leqq 0 \quad \text{for all } \theta, \qquad e^{i\theta} \notin K.$$

Since $u(\theta) \leqq 0$ and $u = 0$ on a neighbourhood U of K, we then have

$$\begin{aligned} 0 &\geqq \int_T u(\theta) \int_K \frac{\sin(t-\theta)}{1-\cos(t-\theta)}\, d\lambda(t)\, d\theta \\ &= \int_K \int_T u(\theta) \frac{\sin(t-\theta)}{1-\cos(t-\theta)}\, d\theta\, d\lambda(t) \\ &= \int_K {}^*u(t)\, d\lambda(t). \end{aligned}$$

This implies that there is a sequence u_n of positive combinations of the u_θs, with $e^{i\theta} \notin K$, such that $||u_n + i^*u_n - (u + i^*u)||_K \to 0$. Exponentiating, we have the desired singular function.

(b) The proof of (b) is a continuation of that of (a). It is clear that we need only show that the outer part of f is the uniform limit on K of singular functions whose measures have no support in K, and this is done as above but is somewhat more difficult because we do not know that u vanishes on a neighbourhood of K, only on K itself.

However, again consider the cone C defined above and let λ be any real measure on K with

$$(1) \qquad -\int_K \frac{\sin(t-\theta)}{1-\cos(t-\theta)}\, d\lambda(t) \leqq 0 \quad \text{for all } \theta, \qquad e^{i\theta} \notin K.$$

Let $U_1 \supseteq U_2 \supseteq \ldots$ be a decreasing sequence of open sets each containing K with the measure of U_n decreasing to zero. It is easily proved that if f is a real twice differentiable function on T, then

$$(2) \qquad \int_{T-U_n} f(\theta) \frac{\sin(t-\theta)}{1-\cos(t-\theta)}\, d\theta, \qquad t \in K,$$

converges uniformly on K to ${}^*f(t)$, $t \in K$, as $n \to \infty$. In particular, this holds when $f(\theta) = P_r(\psi - \theta)$, the Poisson kernel for $re^{i\psi}$. Hence, using (1) and (2) we find that

$$(3) \qquad 0 \geqq \int_K {}^*P_r(\psi - t)\, d\lambda(t)$$

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for all r, $0 \leqq r < 1$, and all ψ, $0 \leqq \psi < 2\pi$. Let $v(re^{i\theta})$ be the harmonic extension to Δ of the measure λ, and $^*v$ its conjugate. Equation (3) shows that $^*v(z) \leqq 0$ on Δ. Note that v extends continuously to $T - K$ and vanishes there. Finally, put

$$h(z) = \exp[{}^*v(z) - iv(z)].$$

Then h is a bounded analytic function on Δ and the boundary-values of h are real almost everywhere. Hence, h is constant; this implies that λ is zero, and gives the desired conclusion.

Remarks. It is clear from the proofs of (a) and (b) that we did not need f to be continuous on all of T; it would be sufficient for f to be continuous in some neighbourhood of K. Furthermore, the conclusion in (a) holds for any f with just $|f| = 1$ on K provided u (and hence f) satisfies some sort of smoothness condition, for example if u is continuously differentiable or satisfies some Lipschitz condition on a neighbourhood of K. The conclusion in (a) also holds with no special assumptions on f, if the complement of K in T has only a finite number of components. Further, the technique used in [**5**], with result (b) in place of Lemma 2, provides a new proof of the result in [**4**]. Finally, Carleson has proved (a) in [**3**, p. 195]. (I wish to thank the referee for this reference.)

Definition. Let E be a subset of the open unit disc Δ; $B(E)$ will denote the set of all finite Blaschke products formed from points of E.

Carathéodory's theorem [**2**, p. 13] shows that if $E = \Delta$ (and hence $B(E)$ is the set of all finite Blaschke products), then each function which is analytic on Δ and bounded by 1 is the pointwise limit of a sequence of elements of $B(E)$. It is easy to see that if we want to get every analytic function which is bounded by 1 as a pointwise limit of a sequence of elements of $B(E)$, E must be dense in Δ, and Carathéodory's theorem tells us that this condition is also sufficient for such approximation. However, suppose that we choose an arbitrary subset E of Δ; can we describe which functions may be approximated uniformly on compact subsets of Δ by a sequence of elements of $B(E)$? The answer is given in Theorem 2 below.

THEOREM 2. *Let E be a discrete subset of Δ and let K be the set of accumulation points of E on the unit circle T. An analytic function h on Δ is the limit, uniformly on compact subsets of Δ, of a sequence of functions from $B(E)$ if and only if h is bounded by* 1, *is continuous at each point of $T - K$, has modulus* 1 *on $T - K$, and the zeros of h (if h has any zeros) lie in E.*

Proof. Let h be one of the functions with the described properties. If $h = BSF$ is the canonical factorization of h into a (possibly infinite) Blaschke product B, a singular function S, and an outer function F, then the hypotheses imply that the zeros of B lie in E (if h has any zeros), the measure μ which determines S is supported on K, and $\log|F| = 0$ on $T - K$. Let $w = -\int_K \log|F|\, d\sigma + ||\mu||$ and let $\{\sigma_n\}$ be a sequence of positive measures converging to $[-\log|F|]\, d\sigma + d\mu$ in the weak-* topology in the space of measures on T where, for each n, $(1/w)\sigma_n$ is a convex combination of point masses, the points being in K. Such a sequence exists by the Kreĭn-Milman theorem, for example. Clearly the singular functions determined by the measures σ_n converge uniformly on compact subsets of Δ to SF as $n \to \infty$. Thus to prove the theorem we need only show that the singular function f determined by placing a mass of weight $w > 0$ at a single point λ of K is the pointwise limit of elements of $B(E)$. (The partial subproducts of B lie in $B(E)$, of course, and converge uniformly on compact subsets of Δ to B.)

We may assume without loss of generality that $\lambda = 1$. Let $\{a_1, a_2, \ldots\}$ be a sequence of distinct points of E which converge to 1; we may suppose without loss of generality that $|a_j|$ strictly increases to 1. There is a positive integer $N(1)$ and non-negative integers $m_1, \ldots, m_{N(1)}$ such that

$$\left| \prod_{k=1}^{N(1)} |a_k|^{mk} - e^{-w} \right| \leqq 1/2.$$

Let B_1 be the finite Blaschke product

$$B_1(z) = \prod_{k=1}^{N(1)} \left[\frac{a_k - z}{1 - \bar{a}_k z} \cdot \frac{\bar{a}_k}{|a_k|} \right]^{mk}, \qquad z \in \Delta.$$

Then $B_1 \in B(E)$ and $|B_1(0) - e^{-w}| \leqq 1/2$.

There is a positive integer $N(2) > N(1)$ and non-negative integers m_k, $k = N(1) + 1, \ldots, N(2)$, such that

$$\left| \prod_{k=N(1)+1}^{N(2)} |a_k|^{mk} - e^{-w} \right| \leqq 1/4.$$

Let B_2 be the finite Blaschke product

$$B_2(z) = \prod_{k=N(1)+1}^{N(2)} \left[\frac{a_k - z}{1 - \bar{a}_k z} \cdot \frac{\bar{a}_k}{|a_k|} \right]^{mk}, \qquad z \in \Delta.$$

Then $B_2 \in B(E)$ and $|B_2(0) - e^{-w}| \leqq 1/4$.

Continuing this process we see that we may construct for each n, a finite Blaschke product B_n such that

(i) $|B_n(0) - e^{-w}| \leqq 2^{-n}$,

(ii) the zeros of B_n move out to T as n increases, and

(iii) the only point of accumulation of the zeros of all the B_n is 1.

I claim that the finite Blaschke products so constructued converge uniformly on compact subsets of Δ to f. To show this we need to do some computations.

When n is sufficiently large, all the zeros of B_n lie within δ of 1; hence, if $|z - 1| \geqq 2\delta$, then we have

$$\sum\left(1 - \left|\frac{b_i - z}{1 - \bar{b}_i z}\right|^2\right) \leqq \delta^{-2}(1 - |z|^2)\sum(1 - |b_i|^2). \tag{4}$$

Here the b_is are a listing of the a_ks, where each a_k is repeated m_k times and the

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summation is extended over those indices so that every a_k in B_n is included. Put

$$s_i = |b_i - z|^2|1 - \bar{b}_i z|^{-2} \quad \text{and} \quad w_i = s_i^{-1}(1 - s_i).$$

Then

$$|B_n(z)|^2 = \prod s_i \geqq \exp[-\sum w_i].$$

However, by (4), $\sum w_i \leqq C(1 - |z|^2)\sum(1 - |b_i|^2)$, where C is a constant depending only on δ. The term $\sum(1 - |b_i|^2)$ remains uniformly bounded for all n because the products $\prod|b_i|$ converge to e^{-w}. Thus, $|B_n(z)|$ is bounded away from zero when $|z - 1| \geqq 2\delta$.

Now let t be a point of $T - \{1\}$ and let D be a small closed disc centred at t which does not contain 1. If $z \in D$ and $|z| > 1$, then

$$|B_n(z)| = |B_n(\bar{z}^{-1})|^{-1} \leqq M,$$

where M is a constant depending only on the distance from D to 1. Hence, some subsequence of $\{B_n\}$ converges uniformly on compact subsets of the interior of D to a holomorphic function. Repeating this procedure and then taking a diagonal sequence we find that there is a subsequence of $\{B_n\}$ which converges uniformly on compact subsets of $\bar{\Delta} - \{1\}$ to a holomorphic function h. It follows that h is continuous at each point of $T - \{1\}$ and has modulus 1 there and, since the zeros of the B_ns move out to T as $n \to \infty$, h does not vanish in Δ. Examining the factorization of h, we conclude that h is a singular function obtained by placing a mass of weight w_1 at 1. However, $e^{-w} = \lim B_n(0) = h(0) = e^{-w_1}$. Thus $w = w_1$ and hence $h = f$, as desired.

The converse statement concerning the continuity and other properties of a sequential limit of elements of $B(E)$ is proved just as in the two previous paragraphs.

COROLLARY 1. *If E clusters at every point of T, then every holomorphic function on Δ which is bounded by* 1 *and whose zero set lies in E* (*including all the zero-free functions*) *is the limit, uniformly on compact subsets of Δ, of a sequence of elements of $B(E)$.*

Remarks. The corollary implies the somewhat surprising conclusion that no condition on E other than the obviously necessary one that E cluster at every point of T is needed to obtain every zero-free holomorphic function in the sequential closure of $B(E)$. Further, the hypothesis that E be discrete is not needed; the only additional functions that lie in the sequential closure of $B(E)$ are the ones with zeros in $\Delta \cap \bar{E}$ (and which, of course, satisfy the continuity and modulus 1 conditions on $T - K$).

Ahern and Clark [**1**, Lemma 4.1 and the first lines of the proof of Lemma 4.2] show that if a mass of weight $w > 0$ is placed at $\lambda \in T$, then it is possible to find finite Blaschke products with zeros in the set E which converge uniformly on compact subsets of Δ to the singular function determined by this mass, provided that E clusters at λ. Their proof is quite different from that of Theorem 2, and does, of course, give a proof of the reverse implication of

Theorem 2 and thus of the corollary, too, when coupled with the opening paragraph of the proof of Theorem 2.

2. Approximation by quotients of unimodular functions. The results of this section are concerned with conditions under which a unimodular function on some space S may be approximated (uniformly) by a quotient of two other unimodular functions having some other prescribed property, for example, belonging to some specified function algebra on S. The following elementary proposition will be needed several times.

PROPOSITION 1. *Suppose that*

$$p(e^{i\theta}) = \sum_{j=0}^{N} a_j(e^{ij\theta} + e^{-ij\theta}) \quad \textit{for } e^{i\theta} \in T,\ a_N \neq 0,$$

N a positive integer, and suppose that $p(e^{i\theta}) \neq 0$ for all $e^{i\theta} \in T$. Then there are N points $\alpha_1, \ldots, \alpha_N$, not necessarily distinct, in the open unit disc Δ such that

$$p(e^{i\theta}) = c\prod_{j=1}^{N} (e^{i\theta} - \alpha_j)\prod_{j=1}^{N} (e^{-i\theta} - \alpha_j),$$

where $c = (-1)^N a_N \prod_{j=1}^{N} (\alpha_j)^{-1}$.

Proof. Let $Q(z) = z^N\sum_{j=0}^{N} a_j(z^j + 1/z^j)$. Then Q is a polynomial of degree $2N$ and $e^{-iN\theta}Q(e^{i\theta}) = p(e^{i\theta})$ for all $e^{i\theta} \in T$. Furthermore,

$$z^{2N}Q(1/z) = Q(z) \quad \text{for } z \neq 0.$$

Hence, if α is a zero of Q, then so is α^{-1} (note that $Q(0) = a_N \neq 0$). Hence,

$$Q(z) = a_N\prod_{j=1}^{N} (z - \alpha_j)\prod_{j=1}^{N} (z - \alpha_j^{-1}),$$

where $\alpha_1, \ldots, \alpha_N$ lie in Δ (since p, and hence Q, is not zero on T). This yields the desired conclusion.

THEOREM 3. *Let X be a compact Hausdorff space and let A be a uniformly closed subalgebra of $C(X)$ which contains the constants. Suppose that $g \in A$, $|g(x)| = 1$ for all $x \in X$, and $u = \operatorname{Re} g$. If a is any real number, then the unimodular function $f = \exp[iau]$ is the uniform limit of quotients of unimodular elements of A.*

Proof. Since g is unimodular, $2u = g + \bar{g} = g + g^{-1}$. Let $b = (1/4)a$ and $f_1 = \exp[ib(g + g^{-1})]$. Let N be a large positive integer and define

$$p_N = g^N\sum_{k=0}^{N} \frac{(ib)^k}{k!} (g + g^{-1})^k.$$

Then $p_N \in A$ and $g^{-N}p_N$ approximates f_1 uniformly to within $\epsilon/4$ for large enough N; furthermore, since g has unit modulus we have by Proposition 1,

$$p_N = c\prod_{j=1}^{N} (g - \alpha_j)\prod_{j=1}^{N} (1 - g\alpha_j),$$

where $\alpha_1, \ldots, \alpha_N$ are points of the open unit disc in the complex plane and c is defined by $c(N!)(\Pi\alpha_j) = (-1)^N(ib)^N$.

Put

$$B = \prod_{j=1}^{N} (g - \alpha_j)(1 - \bar{\alpha}_j g)^{-1}$$

and

$$C = \prod_{j=1}^{N} (g - \bar{\alpha}_j)\alpha_j(1 - \alpha_j g)^{-1}(\bar{\alpha}_j)^{-1}.$$

Then B and C are in the algebra A, are unimodular, and

$$\arg B(x) = 2\arg\left[\prod_{j=1}^{N} (g(x) - \alpha_j)\right],$$

$$\arg C(x) = -2\arg\left[\prod_{j=1}^{N} (1 - \alpha_j g(x))\right] + 2\arg\left(\prod_{j=1}^{N} \alpha_j\right).$$

Hence, $\arg[CB^{-1}p_N^2] \equiv 0 \pmod{2\pi}$ when N is a multiple of 4 and thus

$$|p_N^2 g^{-2N} - BC^{-1}g^{-2N}| = |p_N^2 CB^{-1} - 1| = |1 - |p_N^2||$$

and the last term is less than $\epsilon/2$ when N is large. Thus $f_1^2 = f$ may be approximated uniformly by quotients of unimodular elements of A.

Definition. For a positive integer N, $A(\Delta^N)$ will denote the algebra of functions analytic on the polydisc Δ^N and continuous on its closure.

A monomial m is an element of $A(\Delta^N)$ of the form

$$m(z_1, \ldots, z_N) = z_1^{k_1} \ldots z_N^{k_N},$$

where $k_1, \ldots, k_N$ are non-negative integers.

A function $f \in A(\Delta^N)$ is inner if $|f(w)| = 1$ for all $w \in T^N$.

Rudin's proof in [**10**] also yields the following result.

THEOREM 4. *Let G be a compact abelian group with dual Γ and let Σ be a semigroup in Γ which generates Γ. Let C_Σ be the algebra of continuous functions on G whose transforms are supported on Σ. Then the closed convex hull of the quotients of the unimodular elements of C_Σ is the closed unit ball of $C(G)$.*

COROLLARY 2. *The closed convex hull of the quotients of the inner functions in $A(\Delta^N)$ is the closed unit ball of $C(T^N)$.*

It is interesting to note that when $N > 1$, the next theorem shows that there are unimodular functions in $C(T^N)$ which cannot be uniformly approximated by quotients of inner functions in $A(\Delta^N)$ although, of course, the corollary above shows that they may be approximated by convex combinations of such quotients.

THEOREM 5. *Let $f \in C(T^N)$, f unimodular. Then f may be uniformly approximated on T^N by quotients of inner functions in $A(\Delta^N)$ if and only if there are monomials m_1 and m_2 such that $m_1\bar{m}_2 f$ is the uniform limit on T^N of functions of the form* $\exp(ig)$, *where g is the real part of a function in $A(\Delta^N)$.*

Proof. Suppose that f is the uniform limit of quotients of the form FG^{-1}, where F and G are inner functions in $A(\Delta^N)$. F has the form $m\tilde{Q}(1/z)/Q(z)$, where m is a monomial, Q is a polynomial with no zero in the closed unit polydisc, and $\tilde{Q}$ is the polynomial obtained from Q by replacing the coefficients by their complex conjugates. (Here $1/z$ means $(1/z_1, \ldots, 1/z_N)$ [**11**, Theorem 5.2.5].) However, Q has a logarithm q in $A(\Delta^N)$ and since $\tilde{Q}(1/z) = \bar{Q}(z)$ for $z \in T^N$, we have

$$F(z) = m(z)\exp(-2i \operatorname{Im} q(z)), \qquad z \in T^N,$$

where $q \in A(\Delta^N)$. Likewise, we find that

$$G(z) = n(z)\exp(-2i \operatorname{Im} p(z)), \qquad z \in T^N,$$

where $p \in A(\Delta^N)$ and $n(z)$ is a monomial. This yields the desired conclusion.

The converse is even easier. Suppose that f is of the form $m_1\bar{m}_2 \exp(ig)$, where m_1 and m_2 are monomials and $g = \operatorname{Re} G$, $G \in A(\Delta^N)$. Clearly we need only approximate $\exp(ig)$ by quotients of inner functions in $A(\Delta^N)$. However, $A(\Delta^N)$ is generated by its unimodular elements, and so the conclusion follows from Theorem 3.

THEOREM 6. *Suppose that the unimodular elements of the function algebra A separate the points of X. Then the closed convex hull of the quotients of the unimodular elements of A is the closed unit ball of $C(X)$.*

Proof. The algebra generated by the quotients of the unimodular elements of A is dense in $C(X)$ by the Stone-Weierstrass theorem. Lef $f \in C(X)$, $\|f\|_\infty \leqq 1$. Choose unimodular functions $g_1, \ldots, g_k$, and h in A such that

$$\left\| \sum_{j=1}^{k} c_j g_j - fh \right\| < \epsilon$$

where $c_1, \ldots, c_k$ are complex scalars. We may assume that $\|\sum_{j=1}^k c_j g_j\| \leqq 1$. Define $\Phi: X \to T^{k+1}$ by $\Phi(x) = (g_1(x), \ldots, g_k(x), h(x))$ and let $K = \Phi(X)$. Define $G(\rho) = \bar{h}(\Phi^{-1}(\rho))\sum_{j=1}^k c_j g_j(\Phi^{-1}(\rho))$ for $\rho \in K$. G is a continuous function on K which is bounded by 1 and as such may be uniformly approximated on K by convex combinations of quotients of inner functions in $A(\Delta^N)$ by Corollary 2. Since the composition of a unimodular element in A with an inner function in $A(\Delta^N)$ is again a unimodular element of A, this implies that $\bar{h}\sum_{j=1}^k c_j g_j$ may be approximated uniformly on X by convex combinations of quotients of unimodular elements of A and hence the same is true of f.

THEOREM 7. *Let X be a compact Hausdorff space and let $f: T \to C(X)$ be a continuous function such that $f(e^{i\theta})$ is a unimodular function for each θ, $0 \leqq \theta \leqq 2\pi$. Then there are functions g and h analytic on a neighbourhood of the closed unit disc with values in $C(X)$ such that*

(i) *$g(e^{i\theta})$ and $h(e^{i\theta})$ are unimodular functions for each θ, $0 \leqq \theta \leqq 2\pi$,*

(ii) *$||f(e^{i\theta}) - g(e^{i\theta})\bar{h}(e^{i\theta})|| < \epsilon$ for each $\theta \leqq 0 \leqq 2\pi$, where the vertical bars denote the supremum norm on X.*

Proof. For $x \in X$, let $f_x(e^{i\theta}) = f(e^{i\theta})(x)$; f_x is a continuous function on T with values in T and hence there is an integer $k(x)$ and a continuous real-valued function $g(x, e^{i\theta})$ on T with $f_x(e^{i\theta}) = \exp[ik(x)\theta]\exp[ig(x, e^{i\theta})]$. We may assume without loss of generality that $f_x(1) = 1$ for all $x \in X$ and hence also that $g(x, 1) = 0$ for all $x \in X$. Given $\epsilon > 0$ there is a neighbourhood $N(\epsilon, x)$ of x and a neighbourhood $M(\epsilon, \theta)$ of $e^{i\theta}$ such that $|f_x(e^{i\theta}) - f_y(e^{i\psi})| < \epsilon$ for $y \in N(\epsilon, x)$ and $e^{i\psi} \in M(\epsilon, \theta)$. A finite number of the neighbourhoods $\{M(\epsilon, \theta): 0 \leqq \theta < 2\pi\}$ cover T, and hence x has a neighbourhood $O(\epsilon, x)$ such that

$$|f_x(e^{i\theta}) - f_y(e^{i\theta})| < \epsilon \quad \text{for all } \theta \text{ and } y \in O(\epsilon, x).$$

It follows that $k(x)$ is a continuous function of x and g is continuous on $X \times T$. Hence, g may be approximated uniformly by sums of products of elements from $C_r(T)$ with elements from $C_r(X)$ where the subscript r denotes the real-valued functions. Thus we may assume without loss of generality that $g(x, \theta) = g(x)h(\theta)$, where $g \in C_r(X)$, $h \in C_r(T)$. Since h may be approximated uniformly by a real trigonometric polynomial, we may further assume that $h(\theta) = a \cos n\theta$ or $h(\theta) = a \sin n\theta$, where a is real and n is a non-negative integer. For definiteness we will take $h(\theta) = a \cos n\theta$; the case $h(\theta) = a \sin n\theta$ can be handled analogously. Consequently, we have reduced the theorem to approximating a unimodular function of the form

$$\exp[ik(x)\theta]\exp[ig(x)\cos n\theta],$$

where $k(x)$ is a continuous integer-valued function on X and $g(x)$ is a continuous real-valued function on X. We may assume without loss of generality that $1 \leqq g(x) \leqq 2$.

Let $F(e^{i\theta}) = \exp(ig(x)\cos n\theta)$ and

$$F_N(z) = \sum_{k=0}^{N} \frac{i^k}{k!}\left(\frac{g}{4}\right)^k\left[z^n + \frac{1}{z^n}\right]^k, \qquad N = 1, 2, \ldots,$$

and finally

$$p_N(z) = z^{nN}F_N(z).$$

Then $p_N(z)$ is an analytic function (on the whole plane) with values in $C(X)$ and $z^{2Nn}p_N(1/z) = p_N(z)$. Hence, for each $x \in X$, $p_N(z)(x)$ has Nn roots $\alpha_1(x), \ldots, \alpha_{Nn}(x)$ in the open unit disc and its other roots are at

$$[\alpha_1(x)]^{-1}, \ldots, [\alpha_{nN}(x)]^{-1}.$$

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Furthermore, an application of Cauchy's formula shows that the coefficients of $p_N(z)$ are continuous complex-valued functions on X, and hence the symmetric functions in $\alpha_1, \ldots, \alpha_{Nn}$ are continuous. Note that no α_j vanishes since $p_N(0) \neq 0$. Thus Proposition 1 yields

$$p_N(z) = c(x) \prod_{j=1}^{nN} (z - \alpha_j(x)) \prod_{j=1}^{nN} (1 - \alpha_j(x)z),$$

where

$$c(x) = \frac{(-1)^N}{N!} \left(\frac{ig}{4}\right)^N \prod_{j=1}^{nN} (\alpha_j(x))^{-1} \in C(X).$$

Let

$$B(z) = \prod_{j=1}^{nN} (z - \alpha_j(x))(1 - \bar{\alpha}_j(x)z)^{-1}$$

and

$$C(z) = \prod_{j=1}^{nN} (z - \bar{\alpha}_j(x))\alpha_j(x)(1 - \alpha_j(x)z)^{-1}(\bar{\alpha}_j(x))^{-1}.$$

Then B and C are analytic in a neighbourhood of the closed unit disc, take values in $C(X)$, and both $B(e^{i\theta})$ and $C(e^{i\theta})$ are unimodular elements of $C(X)$. Finally, as in the proof of Theorem 3, we clearly have

$$||F(e^{i\theta})C(e^{i\theta})e^{2iNn\theta} - B(e^{i\theta})|| < \epsilon$$

when N is sufficiently large. Since $\exp(ik(x)\theta)$ is easily seen to be a quotient of functions of the desired form, our proof is complete.

COROLLARY 3. *The quotients of the finite Blaschke products are uniformly dense in the continuous unimodular functions on T.*

Proof. Take X to be a single point.

Remark. The corollary has been proved, by different methods, by Helson and Sarason [**7**, p. 9]. (I am indebted to the referee for this reference.)

3. Banach spaces of analytic functions on the unit disc.

THEOREM 8. *Let A be a Banach space consisting of functions analytic on the unit disc Δ and suppose that A contains every function which is analytic in a neighbourhood of the closed unit disc. If f is analytic in a neighbourhood of $\bar{\Delta}$ and bounded by one on Δ, then f may be approximated in the norm of A by convex combinations of finite Blaschke products.*

Proof. We may assume without loss of generality that the norm of A is at least as strong as the sup norm, for otherwise the theorem is known (see [**4**]). Since f is analytic on a neighbourhood of $\bar{\Delta}$, there is an $F \in A(\Delta)$ and an $r < 1$ with $f(z) = F(rz)$, $z \in \Delta$. We may assume that $||F||_\infty \leqq 1$. By the closed graph theorem, there is a constant $c(r)$ depending only on r such that $||g(rz)||_A \leqq c(r)||g(z)||_\infty$ for all $g \in A(\Delta)$. By [**4**] there is a convex combination

b of finite Blaschke products with $\|b(z) - F(z)\|_\infty < c(r)^{-1}\epsilon$. Hence

$$\|b(rz) - f(z)\|_A < \epsilon.$$

However, by the proof in [**4**], $b(rz)$ is itself actually equal to a convex combination of finite Blaschke products, completing the proof.

COROLLARY 4. *If the functions analytic in a neighbourhood of $\bar{\Delta}$ are dense in A, then every function in A which is bounded by 1 is the limit in the norm of A of a sequence of convex combinations of finite Blaschke products.*

Examples. (1) Let $p \geqq 0$ and let A consist of all those analytic functions $\sum_{n=0}^{\infty} a_n z^n$ in Δ for which $\sum_{n=1}^{\infty} n^p |a_{n-1}|$ is finite, with that sum as norm.

(2) Let A consist of all functions $f \in A(\Delta)$ for which f is in the class $\mathrm{Lip}(\alpha)$ on T with norm: $\|f\|_A = \|f\|_\infty + \|f\|_\alpha$, where

$$\|f\|_\alpha = \sup\{|f(x) - f(y)|\, |x - y|^{-\alpha}: x, y \in T, x \neq y\}.$$

(3) Let A consist of all functions $f \in A(\Delta)$ for which f is in the class $C^{(n)}(T)$ with norm: $\|f\|_A = \sum_{k=0}^{n} \|f^{(k)}\|_\infty$.

Each of these examples satisfies the conditions of the theorem and (1) and (3) those of the corollary so that the convex hull of the finite Blaschke products contains in its A-norm closure each function analytic in a neighbourhood of $\bar{\Delta}$ and bounded by 1.

THEOREM 9. *Let B be a semisimple commutative Banach algebra with unit. Suppose that B is generated by an element x with $|\hat{x}| = 1$ on bB, the Šilov boundary of B. Then the closed convex hull of the unimodular elements of B (that is, those $y \in B$ with $|\hat{y}| = 1$ on bB) is the closed unit ball of B.*

Proof. Let

$$A = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ analytic in } \Delta: \sum_{n=0}^{\infty} |a_n|\, \|x^n\|_B < \infty \right\}$$

and define a norm on A by

$$\|f\|_A = \sum_{n=0}^{\infty} |a_n|\, \|x^n\|_B.$$

Then A fulfills the hypotheses of Theorem 8 and Corollary 4, and hence, if $f \in A$ and $\|f\|_\infty \leqq 1$, then there is a convex combination C of finite Blaschke products with $\|f - C\|_A < \epsilon$.

Let p be a polynomial and suppose that $p(x) = y \in B$ has norm less than 1. Since we may assume without loss of generality that $\hat{x}$ maps the maximal ideal space of B onto the whole unit circle (the only other possibility is that it is mapped onto a proper compact subset K of T and in that case B must be isometrically isomorphic to $C(K)$ and the theorem is known in this case [**9**, Theorem 1]), we have $\|p\|_\infty < 1$. This means that there is a convex combination C of finite Blaschke products with $\|p - C\|_A < \epsilon$ and hence $\epsilon > \|p - C\|_A \geqq \|p(x) - C(x)\|_B$. However, $C(x)$ is a convex combination of unimodular elements of B, proving the theorem.

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