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ON EUTACTIC FORMS

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Let $(a_{ij}) = A$ be a positive definite $n \times n$ symmetric matrix with real entries. To it corresponds a positive definite quadratic form f on $\mathbf{R}^n$: $f(x) = {}^t xAx = \sum a_{ij}x_ix_j$ for x any column vector in $\mathbf{R}^n$. The set of values $f(y)$ for y in $\mathbf{Z}^n - \{0\}$ has a minimum $m(A) > 0$ and the number of "minimal vectors" $y_1, \ldots, y_r$ in $\mathbf{Z}^n$ for which $f(y_i) = m(A)$ is finite. By definition, f and A are called *eutactic* if and only if there are *positive* numbers $s_1, \ldots, s_r$ such that

$$A^{-1} = \sum_{i=1}^{r} s_i y_i {}^t y_i.$$

Also if $\{y_i {}^t y_i\}$ span the $n(n+1)/2$-dimensional vector space of symmetric matrices, we say A and f are *perfect*.

The notion of eutactic form arose in the study of extreme forms. We call A_0 and f_0 *extreme* if and only if the function $P(A) = \det(A)/m^n(A)$ has a local minimum at A_0. Extreme forms hold interest for the following reason: A lattice is a set of points of the form $B\mathbf{Z}^n$ where B is a nonsingular matrix. The volume of the unit cell of this lattice is $|\det B|$ and the square of the distance from the origin to a typical lattice point is ${}^t x {}^t BBx$, for x in $\mathbf{Z}^n$. If we write A for the positive—definite symmetric matrix ${}^tBB$, then the volume of the unit cell is $(\det A)^{1/2}$ and the shortest distance between two lattice points is $m(A)^{1/2}$. A lattice on which spheres of equal radius can be packed (without overlapping) with maximum density therefore corresponds to a minimum of $P(A)$. (The correspondence between A and B is many—to—many. Since $A = {}^tB_1B_1 = {}^tB_2B_2$ if and only if $B_1 = EB_2$, where E is an orthogonal matrix, the positive symmetric matrix corresponds to all the congruent lattices $EB\mathbf{Z}^n$. On the other hand, if U is a matrix of integers with $\det U = 1$ then $U\mathbf{Z}^n = \mathbf{Z}^n$ and the lattice $B\mathbf{Z}^n$ corresponds to all the arithmetically equivalent positive symmetric matrices ${}^tUAU$).

Voronoi's celebrated theorem states that A is extreme if and only if it is perfect and eutactic. This theorem has been reproven many times (for instance, [**10**; **11**]), and it receives incidentally yet another proof below. For more information and bibliography on the "classical" treatment of these forms, see [**1**; **2**; and **3**].

In this paper I prove that the function P is a topological morse function. This leads to a new interpretation of eutactic forms as the non-degenerate topological critical points of P. These are Theorems 1 and 2 respectively and

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they essentially generalize Voronoi's theorem. Theorem 3 states that the number of arithmetically inequivalent critical points is finite.

Another purpose of this paper is to generalize all these ideas and theorems to what I believe is their natural setting, namely replacing the cone of positive-definite symmetric matrices by an arbitrary self-adjoint homogeneous cone. Of course, the former is a special case of the latter. The basic definitions and facts about self-adjoint homogeneous cones may be found in Section 1. In Section 2 some geometry of numbers of these cones is developed. The generalization of the packing function P is investigated in Section 3 and we prove it is a morse function (Theorem 1), assuming a proposition about the Taylor expansion of a certain function. This proposition is proven in Section 5. In 4 we prove Theorems 2 and 3 and derive Voronoi's theorem as a corollary.

I came to the above in an effort to compute the cohomology groups of arithmetic groups of automorphisms of self-adjoint homogeneous cones, and especially of congruence subgroups of $SL(n, \mathbf{Z})$. This application will appear elsewhere [**12**]. After completing my work, I discovered similar ideas concerning the real quadratic form case in a paper by Stogrin [**4**]. However, Stogrin does not mention eutactic forms or morse functions nor provides proofs. I would like to thank David Mumford for setting me onto this problem, and the referee for his helpful comments.

1. Self-adjoint homogeneous cones. This section presents a quick summary of the facts needed in the sequel. For further information and for proofs, see [**5**] and [**6**].

Let V be an N-dimensional vector space, C an open convex cone in V which contains no entire straight line. Let G be the group of linear automorphisms of V which carry C to itself. We say C is *homogeneous* if G acts transitively on C.

Let $\langle , \rangle$ denote an inner product on V. The *dual cone* $\check{C}$ with respect to this inner product is

$$\check{C} = \text{interior of } \{x \in V: \langle x, y\rangle \geqq 0 \text{ for all } y \in C\}.$$

If there exists an inner product such that $\check{C} = C$, we say that C is *self-adjoint*.

A finite $\mathbf{R}$-algebra J is called a *Jordan algebra* if $xy = yx$ and $x^2(yx) = (x^2y)x$ for all x and y in J. Usually J is not associative. We say J is *formally real* if $x^2 + y^2 = 0$ implies $x = y = 0$. If J is formally real it always possesses an identity. For any z in J we write $L(z)$ for the linear map $J \rightarrow J$ given by multiplication by z.

These definitions are connected as follows: If J is a formally real Jordan algebra with identity p, set $C(J) = \{x^2: x \in J \text{ and for some } y \in J, xy = p\}$. Then $C(J)$ is a self-adjoint homogeneous cone in J, where we consider J as a $\mathbf{R}$-vector space with inner product $\langle x, y\rangle = \operatorname{Tr} L(xy)$.

Conversely, if $C \subset V$ is a self-adjoint homogeneous cone and p is a point in C, then V can be given uniquely a structure of Jordan-algebra such that p

is the identity and $C(V) = C$. To do this, let $\mathfrak{k} \oplus \mathfrak{p}$ be the cartan decomposition of the lie algebra of G such that $\exp \mathfrak{k}$ is the stabilizer of p. Since $G \subset \mathrm{Aut}\ (V)$, $\mathfrak{p} \subset \mathrm{End}\ (V)$ and the map $L\colon \pi \to \pi(p)$ is a bijection from $\mathfrak{p}$ to V. We may define the desired Jordan multiplication $x \circ y$ for x, y in V by $x \circ y = L^{-1}(x)(y)$. That this makes V into a Jordan algebra is not obvious, but the proof appears in Section 2 of [**6**].

Let $C \subset V$ be a self-adjoint homogeneous cone, and L a lattice in V. We say L is *admissible* for C if for some p in $L \cap C$ and corresponding Jordan-algebra structure on V, $L \otimes \mathbf{Q}$ is a sub-Jordan algebra of V. In this case, the same will be true for any p in $L \cap C$. Admissible lattices exist for any self-adjoint homogeneous cone.

Example 1. Let V be the $\mathbf{R}$-vector space of $n \times n$ symmetric matrices, C the cone of positive definite ones. Then $G \cong GL(n, R)/\pm 1$ acting by $A \mapsto {}^t gAg$ for $A \in V$, $g \in G$. Using the inner product $\langle A, B\rangle = \mathrm{tr}\ (AB)$ one sees easily that C is self-adjoint. Choosing p to be the identity matrix, the resulting Jordan algebra structure on V is given by $A \circ B = (AB + BA)/2$ where the multiplication on the right is ordinary matrix multiplication. An admissible lattice is the set L of semi-integral symmetric matrices $\{(a_{ij})\colon a_{ii} \in \mathbf{Z}, a_{ij} \in \frac{1}{2}\mathbf{Z}\}$.

Example 2. Let C be the light-cone in $\mathbf{R}^4$. Setting the speed of light equal to 1, C is self-adjoint with respect to the usual inner product on $\mathbf{R}^4$. G is the Lorentz group acting homogeneously on C. An admissible lattice is $\mathbf{Z}^4$.

Now we will generalize the packing-density function P of the introduction.

Let $\mathbf{R}_+$ denote the positive reals. If $C \subset V$ is a self-adjoint homogeneous cone and $p \in C$, there exists a unique C^∞ function $\varphi\colon C \to R_+$ such that

(i) $\varphi(gx) = \varphi(x)/\det \rho(g)$ for $x \in C$ $g \in G$ and ρ is the action of G on V.

(ii) $\varphi(p) = 1$.

This φ is the *characteristic function* of the cone C. For instance, in Example 1 above, $\varphi(A) = (\det A)^{-(n+1)/2}$. In general,

$$\varphi(x) = \int_C e^{-\langle x,y\rangle} dy$$

with a suitable haar measure dy.

The hypersurfaces $D_a = \{x \in C\colon \varphi(x) = a\}$ are called *discriminant surfaces* in C. Further properties of φ are as follows.

(i) $\varphi(ax) = a^{-N}\varphi(x)$ for $a \in \mathbf{R}_+$, $x \in C$, $N = \dim V$.

(ii) $\varphi(px_1 + qx_2) < p\varphi(x_1) + q\varphi(x_2)$ for $x_1, x_2 \in C$, $p + q = 1$, $p, q > 0$ (strict convexity of φ).

(iii) $\varphi(x) \to \infty$ as $x \to$ boundary of C.

Now let L be a fixed admissible lattice for C. Let $L' = L - \{0\}$ and $\bar{C} =$ closure of C in V. We also fix $p \in L \cap C$ and the inner product $\langle x, y\rangle = \mathrm{Tr}\ L(xy)$ given by the resulting Jordan algebra structure on V. Pick any $z \in C$. Because C is self-adjoint and L is discrete, the set of numbers $\{\langle z, y\rangle\colon y \in L' \cap \bar{C}\}$

has a positive minimum $m(z)$, and the set $M(z)$ of $y \in L' \cap \bar{C}$ such that $\langle z, y\rangle = m\langle z)$ is finite. As a function of z, $m(z)$ is continuous.

We define the packing function F: $C \to \mathbf{R}_+$ by

$$F(z) = m^{-N}(z)\varphi^{-1}(z).$$

Clearly $F(az) = F(z)$ for $a \in \mathbf{R}_+$, $z \in C$.

Remark. Consider C, V, L as in Example 1. In [**7**] it is proven that if A is in C,

$$m(A) = \min_{y \in \mathbf{Z}^n - \{0\}} {}^t yAy.$$

This shows that the packing function P in the introduction is just $F^{n/N}$.

F induces a piece-wise smooth continuous function on $C/\mathbf{R}_+$.

Let z be in C. We say z is

(i) *critical* if its image in $C/\mathbf{R}_+$ is a critical point of F;

(ii) *eutactic* if z^{-1} is in the interior of the convex cone generated by $M(z)$ (Here z^{-1} denotes the Jordan inverse of z, and the interior is taken relative to the linear span of $M(z)$.);

(iii) *perfect* if $M(z)$ spans V;

(iv) *extreme* is z is a local minimum for F.

We will show in Section 2 that these last three definitions reduce to the traditional ones in the case of Example 1. The remark just above shows this already for (iv).

Finally we must make explicit the group of symmetries of this whole situation. Let there be given C, V, G and L as above. Because C is self-adjoint, given $g \in G$ there exists $h \in G$ such that $\langle hx, y\rangle = \langle x, gy\rangle$ for all $x, y \in V$. We write $h = {}^tg$. Let ρ denote the action of G on V, and set $\Gamma = \{g \in G: {}^tgL = L\}$. Then for $g \in \Gamma$, $\det \rho(g) = \det \rho({}^tg) = 1$ and so

$$\varphi(gx) = \varphi(x) \quad \text{for all } g \in \Gamma, x \in C.$$

Clearly also $m(gx) = m(x)$ and $M(gx) = {}^tg^{-1}M(x)$. Therefore $F(gx) = F(x)$ and x being critical, eutactic, perfect, or extreme implies that agx is also for $g \in \Gamma$, and $a \in \mathbf{R}_+$. (To check this for x eutactic, we must use the fact that $(gx)^{-1} = {}^tg^{-1}x^{-1}$, which follows, for instance, from pp. 76-7 in [**6**].)

Thus if we are searching for critical points, for instance, we need only look in a fundamental domain for the group Γ. In Section 4 we will show that there are only a finite number of $\mathbf{R}_+\Gamma$-orbits of critical, eutactic, perfect, and extreme points. For extreme forms this implies the well-known result that there exists an absolutely densest lattice packing of $\mathbf{R}^n$ by spheres of equal radius.

2. The perfect core. We will begin to give a geometrical interpretation of the various kinds of forms defined at the end of Section 1. For this and the remaining sections we assume fixed a self-adjoint homogeneous cone $C \subset V$, an $\mathbf{R}$-vector space of dimension N. We also fix an admissible lattice L, a point $p \in C \cap L$ and we give V the associated Jordan-algebra structure with identity p. Then the inner product on V is $\langle x, y \rangle = \operatorname{Tr} L(xy)$. Set $L' = L - \{0\}$, and $\bar{C}$ = closure of C in V.

We define the sets K and $\check{K}$:

$$K = \{x \in \bar{C}: \langle x, y \rangle \geqq 1 \text{ for all } y \in \bar{C} \cap L'\}.$$
$$\check{K} = \text{closed convex hull of } \bar{C} \cap L'.$$

In the terminology of [**6**], K and $\check{K}$ are called the *perfect core* and *perfect co-core* respectively. We check easily, as in proposition 1, p. 128 of [**6**], that

$$(*) \quad \begin{aligned} K &= \{z \in V: \langle z, x \rangle \geqq 1 \text{ for all } x \in \check{K}\} \\ \check{K} &= \{z \in V: \langle z, x \rangle \geqq 1 \text{ for all } x \in K\}. \end{aligned}$$

Both K and $\check{K}$ are closed convex sets contained in $\bar{C}$. However, it is important to remark that in fact K (but not $\check{K}$) is contained in the open cone C. This is because for any x in the boundary of C the hyperplane orthogonal to x meets $\bar{C}$ in a whole "boundary component" and there are points of $\bar{C} \cap L'$ lying arbitrarily closely to that boundary component. To get an idea of what K and $\check{K}$ look like, the reader may draw them for $V = \mathbf{R}^2$, $C = \mathbf{R}_+{}^2$, $L = L' = \mathbf{Z}^2$. As usual, a *face* of a closed convex set Q is the intersection of Q with one of its supporting hyperplanes. A *vertex* of Q is a face that consists of a single point.

Note that $z \in C$ is perfect if and only if the ray $\mathbf{R}_+ z$ passes through a vertex of K.

Let $\Gamma = \{g \in G: {}^t g L = L\}$ as in Section 1. Then K is a Γ-invariant set and $\check{K}$ is a ${}^t\Gamma$-invariant set. By the results of [**6**], in particular the corollary on p. 143, we know that the set of faces of $\check{K}$ has only finitely many ${}^t\Gamma$-orbits.

Also it follows from Proposition 11, p. 142 of [**6**], that the faces of K and $\check{K}$ have no accumulation point in C.

PROPOSITION 1. *The set of faces of K has only finitely many Γ-orbits.*

Proof. We give the proof, which cannot be found in [**6**]. Any supporting hyperplane of K can be written as $H = \{x \in V: \langle w, x \rangle = a\}$ for some $a \in \mathbf{R}$ and $w \in V$, where we assume that $\langle w, x \rangle \geqq a$ for all $x \in K$ and $\langle w, x_0 \rangle = a$ for some $x_0 \in K$. Since $\mathbf{R}_+ K = C$, this implies first that $a \geqq 0$, then that $\langle w, y \rangle \geqq 0$ for all $y \in C$, hence $w \in \bar{C}$. Now we know $a > 0$, for $\langle w, x_0 \rangle = 0$ would imply $x_0 \notin C$, but $K \subset C$. Replacing w by a positive multiple of itself if necessary, we may assume $a = 1$.

Now we use (*). Since $\langle w, x \rangle \geqq 1$ for all $x \in K$ and $\langle w, x_0 \rangle = 1$ for some $x_0 \in K$, we conclude that w is in the boundary of $\check{K}$, $\partial \check{K}$. Thus we have a surjection $f: \partial \check{K} \to$ set of faces of K where $f(w) = \{x \in K: \langle w, x \rangle = 1\}$.

We will show f is constant on $T - \bigcup$ (faces of T), where T is any face of $\check{K}$. Let w, w' be in the face T but in no face of T. If $x \in f(w)$, then $\{y \in V: \langle x, y \rangle = 1\}$ is a supporting hyperplane of $\check{K}$ which contains w, and therefore also T. Then it also contains w', implying $\langle x, w' \rangle = 1$ and $x \in f(w')$. Thus $f(w) \subset f(w')$, and reversing the roles of w and w' gives $f(w) = f(w')$.

Hence f induces a surjection

$$f'\colon \text{set of faces of } \check{K} \quad \rightarrow \quad \text{set of faces of } K$$

which is obviously equivariant with respect to the map ${}^tg \mapsto g$ of ${}^t\Gamma \to \Gamma$. The proposition follows from this and the fact that the domain of f' is finite modulo ${}^t\Gamma$.

Remark. It is not true that f' is a bijection. The problem is that $\check{K}$ has non-compact faces contained in the boundary of C. However, the proof above may be adapted easily to show that f is an inclusion-reversing bijection from the set of compact faces of $\check{K}$ to the set of faces of K.

Next we want to show that the definitions of extreme, perfect, and eutactic points given in Section 1 coincide with the traditional definitions for positive-definite quadratic forms as given in the introduction. Let C, V, L be as in Example 1 of Section 1, and otherwise let the notations remain as in Section 1. Regard $\mathbf{Z}^n$ as column vectors. As noted already in Section 1, the two definitions of extreme form coincide. Notice also that the Jordan inverse of A coincides with the matrix inverse of A.

PROPOSITION 2. *Let $A \in C$, the cone of positive definite symmetric matrices, and $y_1, \dots, y_r$ be the vectors y in $\mathbf{Z}^n$ with ${}^tyAy = m(A)$. Then $M(A)$ is contained in the convex cone generated by $\{y_i{}^ty_i : i = 1, \dots, r\}$.*

COROLLARY. *The two definitions of perfect (respectively, eutactic) forms for C coincide.*

Proof. The corollary follows easily since $y{}^ty \in M(A)$ if ${}^tyAy = m(A)$.

To prove the proposition, we may obviously assume that $m(A) = 1$, so that $A \in K$. By the result in [**7**] already mentioned, $m(A) = \min {}^tyAy$ taken over $y \in \mathbf{Z}^{n\prime}$. Therefore, $K = \{B \in C\colon {}^tyBy \geqq 1 \text{ for all } y \in \mathbf{Z}^{n\prime}\}$. Using Proposition 1, p. 128 of [**6**], we see that

(**) $\quad \check{K} = \text{closed convex hull of } \{y{}^ty\colon y \in \mathbf{Z}^{n\prime}\}$.

Now suppose $B \in M(A)$. In particular, $B \in \bar{C} \cap L'$, so that $\langle B, Q\rangle \geqq 1$ for all $Q \in K$, and thus $B \in \check{K}$ by (*) above.

Choose $\delta > 0$ so that for any $y \notin M(A)$, ${}^tyAy > 1 + \delta$. This is possible since C is self-adjoint so that $\{Q \in \bar{C} \cap L\colon \langle A, Q\rangle < 2\}$ is finite.

Let $\underline{P}$ denote the convex hull of $y_1{}^ty_1, \dots, y_r{}^ty_r$. By (**), there are sequences $a_m, b_m \in [0, 1]$ and $Q_m, R_m \in \bar{C}$ such that $a_m + b_m = 1$, $Q_m \in \underline{P}$, $\langle R_m, A\rangle > 1 + \delta$ and

$$B = \lim_{m\to\infty} a_mQ_m + b_mR_m.$$

Passing to a subsequence if necessary, let $a_m \to a$, $b_m \to b$ and $Q_m \to Q \in \underline{P}$. Then $a + b = 1$ and $aQ + bR_m \to B$. Taking the inner product with A gives

$a + b\langle R_m, A\rangle \to 1$. This implies $b = 0$, $a = 1$ and $B = Q \in \underline{P}$, completing the proof of Proposition 2.

3. The packing function F. We keep the notations of Sections 1 and 2. We fix C, V, L, p as in Section 1. For $z \in C$, we have defined $F(z) = m^{-N}(z)\varphi^{-1}(z)$. This induces

$$F: C/\mathbf{R}_+ \to \mathbf{R}_+.$$

In this section we will investigate the topological critical points of F.

We recall the following definitions from [**8**]. Let M be an m-dimensional topological manifold and f a real-valued continuous function on M. Let $x_1, \ldots, x_m$ be the usual coordinates of the point x in R^m.

Definition. A point $q \in M$ is *topologically ordinary for* f if there exist neighborhoods U of q and V of 0 in $\mathbf{R}^m$ and a homeomorphism $g: V \to U$ such that $g(0) = q$ and $f(g(x)) = x_1 + f(q)$, $x \in V$.

We say q is *topologically critical* if it is not topologically ordinary.

We say q is *topologically non-degenerate of index* r if there exist U, V, g as above such that $g(0) = q$ and $f(g(x)) = -x_1^2 - x_2^2 - \ldots - x_r^2 + x_{r+1}^2 + \ldots + x_m^2 + f(q)$, $x \in V$. This implies that q is topologically critical.

We will omit the adjective "topological". In [**8**] it is shown that a non-degenerate point q is critical and has a unique index r which depends only on f and q. These definitions reduce to the usual ones in case M and f are C^∞.

THEOREM 1. *Let C be a self-adjoint homogeneous cone in the Jordan algebra V. Let $p \in V$ be the Jordan identity and $L \subset V$ be an admissible lattice. We define the inner product on V as above:*

$$\langle x, y\rangle = \textit{trace of multiplication by } xy \textit{ in } V.$$

We set

$$K = \{x \in C: \langle x, y\rangle \geqq 1 \textit{ for all } y \in \bar{C} \cap L - \{0\}\}.$$

Let $F: C/\mathbf{R}_+ \to \mathbf{R}$ be the packing function for this data, as defined in Section 1.

For any point $\bar{z} \in C/\mathbf{R}_+$, let z be the point in the boundary of K whose image in $C/\mathbf{R}_+$ is $\bar{z}$. Write z^{-1} for the Jordan inverse of z and

$$H_z = \{v \in V: \langle z^{-1}, v - z\rangle = 0\}.$$

Let S be the minimal face of K containing z. Then:

(i) *If H_z does not support K, then $\bar{z}$ is ordinary for F.*

(ii) *If H_z supports K and $K \cap H_z = S$, then $\bar{z}$ is non-degenerate of index r for F, where* $r = \dim S$.

(iii) *If H_z supports K but $K \cap H_z$ is a face of K strictly containing S, then $\bar{z}$ is ordinary for F.*

Thus F is a topological morse function on $C/\mathbf{R}_+$.

Case (iii) actually occurs, for instance in Example 1 of Section 1 when $n = 4$ [**9**].

The proof of this theorem depends on the following.

PROPOSITION 3. *Let $C \subset V$ be a self-adjoint homogeneous cone, p the Jordan identity in V, and φ the characteristic function for C, normalized so that $\varphi(p) = 1$. Then the Taylor expansion of φ^{-1} about p is given by*

$$\varphi^{-1}(p + v) = 1 + \langle p, v\rangle + \tfrac{1}{2}(\langle p, v\rangle^2 - \langle v, v\rangle) + O(\langle v, v\rangle^{3/2})$$

for $v \in V$.

We will defer the proof of this proposition to Section 5 and devote the rest of this section to the proof of Theorem 1, assuming the proposition.

LEMMA 1. *The tangent hyperplane to the discriminant surface $D_{\varphi(z)}$ at z is H_z, for $z \in \partial K$.*

Proof. The surface $D_{\varphi(z)} = \{x \in C: \varphi^{-1}(x) = \varphi^{-1}(z)\}$. Let $\mathfrak{p}$ be the orthogonal complement to the Lie algebra of the stabilizer of p in the Lie algebra of G. Recall G is the automorphism group of C. Then $\exp \mathfrak{p}$ consists exactly of the self-adjoint elements of G and it acts simply transitively on C, see [**6**]. Say $z = gp$, with $g \in \exp \mathfrak{p}$. As pointed out at the end of Section 1, $z^{-1} = g^{-1}p$.

By Proposition 3 we have

$$\begin{aligned}\varphi^{-1}(z + h) &= \varphi^{-1}(gp + h)\\ &= (\det g)\varphi^{-1}(p + g^{-1}h)\\ &= (\det g)(1 + \langle p, g^{-1}h\rangle + O(\langle h, h\rangle)).\end{aligned}$$

Thus the tangent hyperplane in question is $\{x \in V: \langle p, g^{-1}(x - z)\rangle = 0\}$. Since $\langle p, g^{-1}y\rangle = \langle {}^tg^{-1}p, y\rangle = \langle g^{-1}p, y\rangle = \langle z^{-1}, y\rangle$ for any $y \in V$, this hyperplane is just H_z.

COROLLARY. *Let $gp = z$ with $g \in \exp \mathfrak{p}$. Then*

$$\varphi^{-1}(z + h) = (\det g)(1 + \langle z^{-1}, h\rangle + \tfrac{1}{2}(\langle z^{-1}, h\rangle^2 - \langle g^{-1}h, g^{-1}h\rangle) + O(\langle h, h\rangle^{3/2}).$$

LEMMA 2. *Given z in the boundary of K, there exists a neighborhood U of z in C and finite set $Q \subset L$ such that $U \cap K = \{x \in U: \langle x, l\rangle \geqq 1$ for $l \in Q\}$.*

Proof. Let $Q = \{l \in \bar{C} \cap L: \langle l, z\rangle = 1\}$. As in the proof of Proposition 2, Section 2, we see that Q is finite and there exists a neighborhood U of z in C and $\delta > 0$ such that $\langle l, x\rangle > 1 + \delta$ for $l \notin Q$ and $x \in U$. The lemma follows immediately now from the definition of K.

Proof of Theorem 1. *Part* (i). As $z^{-1} \in C$, H_z is transverse to the ray $\mathbf{R}_+z$, and we may identify H_z with $C/\mathbf{R}_+$ in a neighborhood of z. Thus it is enough to show that z is ordinary for F restricted to H_z.

Let Q and U be as in Lemma 2. Since we are assuming H_z does not support K, there exists $v \in H_z \cap U$ and $\delta > 0$ such that $\langle l, v\rangle > 1 + \delta$ for $l \in Q$. Let W be the orthogonal complement to the plane spanned by $v - z$ and z^{-1}. Then $H_z = W + \mathbf{R}(v - z) + z$.

Let U_1 be a neighborhood of 0 in W and $\epsilon > 0$ such that

$$U_2 = \{w + t(v - z) + z: w \in U_1, -\epsilon < t < \epsilon\}$$

is contained in U. By Lemma 2, for any $x \in U, m(x) = \inf_{x \in Q} \langle x, l\rangle$. Therefore, if $w \in U_1$, m restricted to the interval $I_w = \{w + t(v - z) + z: -\epsilon < t < \epsilon\}$ is piecewise linear. Its derivative on each linear piece is one of the numbers $\langle v - z, l\rangle$ for $l \in Q$ and is therefore greater than $\delta > 0$, for every $w \in U_1$.

By the corollary to Lemma 1 and the definition of H_z, if $z + h \in H_z$, $\varphi^{-1}(z + h)$ is proportional to $1 + O(\langle h, h\rangle)$. Now $F = m^{-N}\varphi^{-1}$. Restrict both sides to I_w and let prime denote derivative with respect to t. Then if $w \in U_1$ and $t \in (-\epsilon, \epsilon)$, for almost every t, F' is defined and

$$\begin{aligned} F'(z + w + t(v - z)) &= \\ &= -cm^{-N}(m^{-1}m' - O(\langle v - z, w\rangle + t\langle v - z, v - z\rangle)). \end{aligned}$$

Here c is a positive constant.

Now $m(z) = 1$ and $m' > \delta > 0$ for every $w \in U_1$. Thus shrinking U_1 and ϵ if necessary, we have that F is strictly monotonic on I_w for every $w \in U_1$.

Define the map q: $U_2 \to \mathbf{R} \times W$ by

$$q(w + t(v - z) + z) = (F(w + t(v - z) + z), w).$$

Then q is continuous, and we have shown above that q is one-to-one. By invariance of domain, q is a homeomorphism onto an open neighborhood of $(F(z), z)$ in $\mathbf{R} \times W$. If x is the coordinate in $\mathbf{R}$, $x \circ q = F$. This shows that $\bar{z}$ is an ordinary point for F in $C/\mathbf{R}_+$.

The proofs for parts (ii) and (iii) will be presented in general for the cases when $N = \dim V \geqq 3$. If $N = 1$, $C = \mathbf{R}_+$ and if $N = 2$, $C = \mathbf{R}_+ \times \mathbf{R}_+$. If the coordinates are x_1, x_2 then φ is x_1^{-1} or $(x_1x_2)^{-1}$ if $N = 1$ or 2. Here we present a sketch of the proof for $N = 2$. The case $N = 1$ is trivial.

If $C = \mathbf{R}_+ \times \mathbf{R}_+ \subset \mathbf{R}^2$ with coordinates (x_1, x_2), the discriminant surfaces are hyperbolae. Also the boundary of K, ∂K, is one-dimensional and therefore consists of a chain of line segments joined successively at their vertices whose slopes decrease monotonically as one moves along the chain. If the line H_z supports K, three things can happen.

(a) z is a vertex of ∂K and one of the two 1-dimensional faces of K lies in H_z. Since H_z is tangent to the discriminant curve D at z, this means that D lies above ∂K on one side of z and below ∂K on the other. Since $m \equiv 1$ on ∂K, $F = m^{-N}\varphi^{-1}$ is monotonic along H_z and z is an ordinary point for F.

(b) z is a vertex of ∂K and $H_z \cap K = \{z\}$. Then D lies below ∂K on both sides of z, F is nondegenerate critical of index 0 and F has a local minimum at z.

(c) z lies in the interior of a 1-dimensional face of K and $H_z \cap K$ is that face. Then D lies above ∂K on both sides of z, F is nondegenerate critical of index 1 and F has a local maximum at z.

Henceforth in this section we assume $N \geqq 3$.

Part (ii). Similarly to Part (i), we need to show that z is critical nondegenerate of index $r = \dim S$ for F restricted to H_z. As in the corollary to Lemma 1, write $z = gp$ with $g \in \exp \mathfrak{p}$. Let Q and U be as in Lemma 2.

Define $W = \{w \in V: \langle g^{-1}w, g^{-1}(v - z)\rangle = 0 \text{ for all } v \in S \text{ and } \langle w, z^{-1}\rangle = 0\}$. Thus W is a hyperplane of dimension $N - r - 1$.

Let Y be the r-dimensional hyperplane spanned by $v - z$ for $v \in S$. Choose $\epsilon > 0$ so that first, $\langle y, y\rangle < \epsilon$ for $y \in Y$ implies $z + y \in S$ and second such that

$$U_1 = \{z + w + y: w \in W, y \in Y, \langle w, w\rangle < \epsilon \text{ and } \langle y, y\rangle < \epsilon\}$$

is contained in U. So U_1 is an open neighborhood of z in H_z.

Let $Y_0 = \{y \in Y: \langle y, y\rangle = \epsilon\}$ and $W_1 = \{w \in W: \langle w, w\rangle < \epsilon\}$. Consider F restricted to the half-open interval $I(w, y) = \{z + w + ty: 0 \leqq t < \epsilon\}$, $w \in W_1$. By Lemma 2 and as explained in the proof of Part (i), m restricted to I is piece-wise linear with a finite number of pieces, and the slope of m with respect to t is one of the numbers $\langle y, l\rangle$, $l \in Q$. Since $\langle z, l\rangle = 1$ and $\langle z + y, l\rangle \geqq 1$ for $l \in Q$, we have that m is weakly monotonically increasing as t increases.

By the corollary to Lemma 1, we see that

$$\begin{aligned}\varphi^{-1}(z + w + ty) &= (\det g)(1 - \tfrac{1}{2}\langle g^{-1}(w + ty), g^{-1}(w + ty)\rangle + O(t^3))\\ &= (\det g)(1 - \tfrac{1}{2}\langle g^{-1}w, g^{-1}w\rangle - \tfrac{1}{2}t^2\langle g^{-1}y, g^{-1}y\rangle + O(t^3)),\end{aligned}$$

taking into account the definition of W. Shrinking ϵ if necessary, we may conclude that φ^{-1} is strictly monotonic decreasing as t increases on $I(w, y)$.

Since $F = m^{-N}\varphi^{-1}$, we have that F is strictly monotonic decreasing on $I(w, y)$ as t increases.

Now we must pay attention to F restricted to W_1. Let $W_0 = \{w \in W: \langle w, w\rangle = \epsilon\}$. For $w \in W_0$, let $J(w)$ be the half-open interval $\{z + sw: 0 \leqq s < 1\}$. In this case, $m(z + sw) = \inf_{l \in Q} (1 + s\langle w, l\rangle)$, so that m is linear on $J(w)$, not merely piece-wise linear. By definition of W, $z + sw \notin S = H_z \cap K$ for $s > 0$, so that since H_z supports K, $m(z + sw) < 1$ for $s > 0$. Therefore m is strictly monotonically decreasing on $J(w)$. The derivative of m with respect to s in less than some $\delta < 0$ for all $w \in W_0$ since it is one of the numbers $\langle w, l\rangle$, $l \in Q$, $w \in W_0$ and $W_0 \times Q$ is compact.

As in Part (i), we may conclude after shrinking ϵ if necessary, that F is strictly monotonically increasing on $J(w)$ with respect to s for all $w \in W_0$.

Now we will define a map $q: U_1 \to W \times Y$ in two stages. First, define $q_0: z + W_1 \to W$ as follows:

$$q_0(z + sw) = (1/\epsilon)(F(z + sw) - F(z))^{1/2}w \quad \text{for } w \in W_0, 0 \leqq s < 1.$$

Here, $F(z + sw) - F(z)$ is always non-negative and we always take the

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positive square-root. Since F is continuous, so is q_0, and since F is strictly monotonic on $J(w)$, q_0 is one-to-one. By invariance of domain if dim $W \geqq 2$ and by direct inspection if dim $W < 2$, we conclude that q_0 is a homeomorphism of $z + W_1$ onto an open neighborhood of 0 in W.

Next we extend q_0 to $q\colon U_1 \to W \times Y$ as follows:

$$q(z + w + ty) = q_0(z + w) + (1/\epsilon)[F(z + w) - F(z + w + ty)]^{1/2}y$$

for $w \in W_1$, $y \in Y_0$, and $0 \leqq t < \epsilon$. Again the expression in square brackets is always non-negative and the positive square root is taken. In fact, we've shown this expression is strictly monotonic on $I(w, y)$ and therefore q is one-to-one. Clearly, q is continuous. Since dim $U_1 = N - 1 \geqq 2$, we conclude by invariance of domain that q is a homeomorphism of U_1 onto an open neighborhood of 0 in $W \times Y$.

Finally, let $h\colon W \times Y \to \mathbf{R}$ be given by $h(w, y) = \langle w, w\rangle - \langle y, y\rangle$. Then we have for $w \in W_0$ and $y \in Y_0$, $0 \leqq s, t < \epsilon$,

$$\begin{aligned} h \circ q(z + sw + ty) &= (1/\epsilon)(F(z + sw) - F(z))\langle w, w\rangle \\ &\qquad - (1/\epsilon)[F(z + sw) - F(z + sw + ty)]\langle y, y\rangle \\ &= F(z + sw + ty) - F(z). \end{aligned}$$

Thus z is critical non-degenerate of index dim $Y = r$ for F restricted to H_z.

Part (iii). The proof depends on the following geometrical lemma.

Let V be a finite dimensional Hilbert space. For any $v \in V$, $v \neq 0$, we write H_v for the hyperplane orthogonal to the line through v, and $\langle , \rangle$ for the inner product.

LEMMA. *Let X be a closed convex set in V whose boundary contains 0, and let E be the smallest face of X containing 0. We assume $E \neq X$. Then there exists $t \in X$ such that H_t supports X and $H_t \cap X = E$.*

Proof. Let

$$\begin{aligned} Y &= \{y \in V\colon \langle y, x\rangle \geqq 0 \text{ for all } x \in X \text{ and } H_y \cap X = E\} \\ &= \{y \in V\colon \langle y, x\rangle \geqq 0 \text{ for all } x \in X, \text{ and } \langle y, x\rangle = 0 \\ &\qquad\qquad \text{for } x \in X \text{ implies } x \in E\}. \end{aligned}$$

Then it is easy to see that Y is an open convex cone in V, it is non-empty, and $0 \notin Y$.

Now set $Y_0 = \{y\colon \langle y, x\rangle \geqq 0 \text{ for all } x \in X\}$. If $y_0 \in Y_0$ and $y \in Y$ it is easy to see that $y_0 + y \in Y$. Thus, Y_0 is the closure of Y in V.

If $X \cap Y = \emptyset$, then by the separating hyperplane theorem, there is $q \in V$, $q \neq 0$, and $b \in \mathbf{R}$ such that $\langle q, x\rangle \geqq b$ for $x \in X$ and $\langle q, y\rangle \leqq b$ for $y \in Y_0$. Now $0 \in X$ implies $b \leqq 0$ and $0 \in Y_0$ implies $b \geqq 0$, so $b = 0$. This means $q \in Y_0$. But $\langle q, q\rangle > 0$, yielding a contradiction.

Thus $X \cap Y \neq \emptyset$ and the lemma is proved.

To apply this to (iii), write $T = K \cap H_z$. Consider H_z as a vector space with origin at z, and with inner product $B(x, y) = \langle g^{-1}(x - z), g^{-1}(y - z)\rangle$ where $gp = z$ with $g \in \exp \mathfrak{p}$. Apply the lemma to $z \in S \subset T \subset H_z$. We obtain a point $t \in T$ such that

$$\{x \in H_z: 0 = \langle g^{-1}(x - z), g^{-1}(t - z)\rangle\} \cap T = S.$$

Clearly $t \notin S$.

The proof of Part (ii) of Theorem 1 can now be applied to each of the half-spaces:

$$h_t^+ = \{x \in H_z: 0 \leqq \langle g^{-1}(x - z), g^{-1}(t - z)\rangle\}$$

and

$$h_t^- = \{x \in H_z: 0 \geqq \langle g^{-1}(x - z), g^{-1}(t - z)\rangle\}$$

Using the notation of that proof, for h_t^+ we would set Y equal to the $r + 1$ dimensional half space spanned by $v - z$ for $v \in S$ and $\mathbf{R}_+(t - z)$ and W equal to the $N - r - 2$ dimensional hyperplane orthogonal to Y with respect to the inner product $B'(x, y) = \langle g^{-1}x, g^{-1}y\rangle$. For h_t^- we set Y equal to the r-dimensional hyperplane spanned by $v - z$ for $v \in S$ and letting W_0 be the B'-orthogonal complement to Y, we set W equal to the $N - r - 1$ dimensional half space spanned by W_0 and $\mathbf{R}_+(z - t)$.

Here $r = \dim S$. Let $k = N - r - 2$. Let $(x, y_1 \ldots y_r, w_1 \ldots w_k)$ be coordinates on $\mathbf{R}^{N-1}$ and define $G: \mathbf{R}^{N-1} \to \mathbf{R}$ by

$$G(x, y, w) = \operatorname{sgn}(x)x^2 - \left(\sum_1^r y_j^2\right) + \sum_1^k w_l^2.$$

Because of our choice of t, the proof of Theorem (ii) goes through and yields a 1-1 continuous map φ from a neighborhood U of z in H_z to a neighborhood of 0 in $\mathbf{R}^{N-1}$ such that $F = G \circ \varphi$ on U. In fact $F|h_t^+$ behaves like a function with a nondegenerate critical point of index $r + 1$ at 0 restricted to a half-space, and $F|h_t^-$ similarly but with index r.

By invariance of domain, φ is a homeomorphism. It only remains to show that G is topologically ordinary at 0. But $\operatorname{sgn}(x)x^2$ is topologically equivalent to x, and G is equivalent to $G'(x, y, w) = x + G''(y, w)$ with G'' continuous. This is obviously regular everywhere. This completes the proof.

4. Voronoi's theorem. In this section we apply Theorem 1 to obtain a generalization of Voronoi's theorem that extreme equals perfect plus eutactic. Let C, V, L, p, F, etc. be as in previous sections.

THEOREM 2. *For any z in C, z is critical non-degenerate for F if and only if z is eutactic.*

Proof. These properties of z depend only on $\mathbf{R}_+z$, so we may assume z is in the boundary of K, ∂K. Then by Theorem 1, z is critical, non-degenerate for $F \Leftrightarrow H_z$ supports K and $H_z \cap K$ is the minimal face S containing z.

Let Q and U be as in Lemma 2 of Section 3. Note that $Q = M(z)$ as defined in Section 1. We have that $S = K \cap \{x + z: \langle l, x\rangle = 0 \text{ for all } l \in Q\}$.

Now z is not eutactic $\Leftrightarrow z^{-1}$ is not in the interior relative to its span of the convex hull of Q (definition) $\Leftrightarrow$ there exists an $x \in V$: $\langle z^{-1}, x\rangle = 0$ and $\langle x, l\rangle \geqq 0$ for all $l \in Q$ and $\langle x, l_0\rangle > 0$ for some $l_0 \in Q$ (separating hyperplane theorem) $\Leftrightarrow$ there exists an $x \in V$: $x + z \in H_z$ and $x + z \in K$ but $x + z \notin S$ (definition of H_z and S, and replacing x by ax with small $a > 0$, so that $x + z \in U$) $\Leftrightarrow$ either H_z does not support K or H_z supports K but $H_z \cap K \supsetneqq S$.

Comparing with the first paragraph, we see that the proof is complete.

To derive Voronoi's theorem for self-adjoint homogeneous cones as a corollary, we need the following fact: *For $z \in C$, z is extreme if and only if z is non-degenerate critical for F of index* 0.

This is because by definition, z is extreme if and only if z is a local minimum for F.

COROLLARY. *For $z \in C$, z is extreme $\Leftrightarrow$ z is perfect and eutactic.*

Proof. Using Lemma 3 and Theorems 1 and 2, we have that z in ∂K is extreme if and only if z is eutactic and is a vertex of K. As pointed out in Section 2, the latter condition is equivalent to z being perfect.

THEOREM 3. *Let $\Gamma = \{g \in \text{Aut } (C, V): {}^t gL = L\}$ as in Section 1. Then there are only a finite number of Γ-orbits of extreme, perfect, eutactic, and critical points in C.*

Proof. By Proposition 1, Section 2, the number of Γ-orbits of vertices of K is finite, and these are just the perfect points. Since extreme and eutactic points are critical by what we have shown above, it remains only to prove the theorem for critical points. By Theorem 1, z is critical if and only if H_z supports K. By Lemma 1, Section 3, H_z is tangent to the discriminant hypersurface $D_{\varphi(z)}$ at z. Since φ is strictly convex, this implies that $\varphi(z') > \varphi(z)$ for every z' in $H_z \cap C$, $z' \neq z$.

Now let $T = H_z \cap K$ and suppose w is in the interior of the face T and that H_w supports K, so $T \subset K \cap H_w$. By the same reasoning, we have $\varphi(w) > \varphi(z)$ and $\varphi(z) > \varphi(w)$ if $w \neq z$. Thus $w = z$, and there is at most one critical point per face of K. The theorem now follows from Proposition 1 of Section 2.

Remark. There may exist faces without critical points. For instance, K has unbounded faces in general. But for $z \in C$, $z^{-1} \in C$, so $H_z \cap \bar{C}$ is bounded. Thus no z in the interior of an unbounded face could have H_z support K.

5. Taylor expansion of the reciprocal of the characteristic function. In this section we prove Proposition 3, Section 3. The proof is technical and involves the real root structure of $G = \text{Aut } (C, V)$ as investigated in [**6**].

Let $\mathfrak{p}$ be as defined in the proof of Lemma 1, Section 3. Let $\mathfrak{k}$ be the lie algebra of the stabilizer of p. Then Lie $G = \mathfrak{k} \oplus \mathfrak{p}$ and there are canonical identifications $\mathfrak{p} \cong$ tangent space of C at $p \cong V$.

Pick $v \in V$, $v \neq 0$, and denote the corresponding element of $\mathfrak{p}$ again by v. For t near 0, let $g_t \in \exp \mathfrak{p}$ be such that $g_t p = p + tv$. Then $\varphi^{-1}(p + tv) = \det g_t$.

Now v is contained in some maximal commutative lie algebra $\mathfrak{a}$ of $\mathfrak{p}$. As shown on p. 90 of [**6**], $A = \exp \mathfrak{a}$ is a maximal $\mathbf{R}$-split torus contained in $\exp \mathfrak{p}$.

By p. 104 of [**6**], we have the following facts: There exists an orthonormal set of vectors $e_1, \ldots, e_n \in V \cong \mathfrak{p}$ such that $e_i e_j = \delta_{ij} e_i$ in the Jordan algebra V and the orbit $Ap = \sum_{i=1}^n \mathbf{R}_+ e_i$. Denoting left multiplication by x in the Jordan algebra V by $L(x)$ as usual, write $L(e_i) = E_i$. Then $E_i \circ E_j = \delta_{ij} E_j$ and

$$(*) \quad \left(\exp \sum_{i=1}^n s_i E_i\right) p = \sum_{i=1}^n (\exp s_i) e_i \quad \text{for } s_i \in \mathbf{R}_+.$$

Finally, $p = \sum_{i=1}^n e_i$.

Recall that $\langle x, y \rangle = \operatorname{Tr} L(xy)$ for $x, y \in V$.

Now Lie $A = \mathfrak{a} = \sum_{i=1}^n \mathbf{R}\, e_i$. Since $v \in \mathfrak{a}$, we may write $v = \sum s_i e_i$ with $s_i \in \mathbf{R}$. However, v being in C, $v = (\exp w) p$ for some $w \in$ Lie G and hence in $\mathfrak{a}$. This shows, by (*), that $s_i > 0$ for every i.

Therefore:

$$(1) \quad p + tv = \sum_{i=1}^n (1 + ts_i) e_i;$$

$$(2) \quad g_t = \exp \sum \log (1 + ts_i) E_i \quad \textit{by } (*);$$

$$(3) \quad \det g_t = \exp \left(\operatorname{Tr} \sum \log (1 + ts_i) E_i\right) \quad \text{since } \det \exp = \exp \operatorname{Tr};$$

$$(4) \quad \langle e_i, e_i \rangle = 1 \Rightarrow \operatorname{Tr} L(e_i^2) = \operatorname{Tr} L(e_i) = \operatorname{Tr} E_i = 1.$$

Hence

$$(5) \quad \det g_t = \exp \sum \log (1 + ts_i) \operatorname{Tr} E_i = \prod_{i=1}^n (1 + ts_i)$$

$$= 1 + t \sum_{i=1}^n s_i + t^2 \left(\sum_{\substack{i,j=1 \\ i<j}}^n s_i s_j \right) + O(t^3).$$

$$(6) \quad \langle p, v \rangle = \sum_{i=1}^n s_i \quad \text{and} \quad \langle v, v \rangle = \sum_{i=1}^n s_i^2 \quad \text{since the } e_i \text{ are orthonormal.}$$

Also $\sum_{i<j} s_i s_j = \frac{1}{2}\left(\left(\sum s_i\right)^2 - \sum s_i^2\right)$.

$$(7) \quad \varphi^{-1}(p + tv) = \det g_t = 1 + t\langle p, v \rangle + \tfrac{1}{2} t^2 (\langle p, v \rangle^2 - \langle v, v \rangle) + O(t^3).$$

This being true for all $v \in V$ and t near 0, Proposition 3 follows immediately.

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