ON EXTENSIONS OF TRIADS

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Dedicated to the memory of Professor TADASI NAKAYAMA

Introduction

As an extension of a result due to W. D. Barcus and J. P. Meyer [4], T. Ganea [8] has recently proved a theorem concerning the fibre of the extension $E \cup CF \to B$ of a fibre map $p : E \to B$ to the cone CF erected over the fibre F. In this paper we shall establish a generalized Ganea theorem which asserts that the homotopy type of the fibre of a canonical extension ξ' of a triad $A \xrightarrow{f} Y \xleftarrow{g} B$ (cf. [13]) is determined by those of f and g (see Theorem 3.4). This generalization yields a proof of a well-known theorem of Serre on relative fibre maps (see Corollary 3.9) and, as done by various authors (cf. [1], [10], [12]), a theorem of Blakers- Massey (see Corollary 4. 4).

Our result can be used to derive a dual EHP sequence which generalizes a conditionally exact sequence established by G. W. Whitehead [15] and Tsuchida-Ando [14]. The dual product introduced by M. Arkowitz ([2], [3]) allows us to describe the third homomorphism in that sequence.

Throughout this paper we will work in the category of spaces with base-points, generally denoted by $*$, and based maps. Homotopies are assumed to respect base-points. The closed unit interval is denoted by I. Given a path $\omega : I \to X$ in X, we denote by $\omega_{u,v}$ the path defined by $\omega_{u,v}(t) = \omega((1-t)u + tv)$, where $0 \leqq u \leqq v \leqq 1$. For paths ω, τ with $\omega(1) = \tau(0)$, the path consisting of ω followed by τ will be denoted by $\omega + \tau$, and the inverse of ω by $-\omega$. As usual, Ω and S are used, respectively, to denote the loop and suspension functors. EX and CX denote the space of paths in X emanating from the base-point and the cone over X respectively.

We are indebted to T. Ganea for sending us a preprint of [8].

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§ 1. Preliminaries

Let $A \xrightarrow{f} Y \xleftarrow{g} B$ be a triad and let $E_{f,g}$ be its mapping track, as defined in [13], i.e.,

$$E_{f,g} = \{(a, \gamma, b) \in A \times Y^I \times B \mid f(a) = \gamma(0),\ g(b) = \gamma(1)\}$$

with projections $P_1 : E_{f,g} \to A$, $P_2 : E_{f,g} \to B$. In particular, let E_f and E_g be, respectively, the mapping track constructed for the triads $A \xrightarrow{f} Y \xleftarrow{g} *$, $* \longrightarrow Y \xleftarrow{g} B$, which are usually called the *fibres* of f, g.

Let $I : \Omega Y \to E_{f,g}$ be the natural injection. Then we have

LEMMA 1.1. (see [13]) *$I_*(\gamma_1) = I_*(\gamma_2)$ for $\gamma_1, \gamma_2 \in \pi(V, \Omega Y)$ if and only if there exist $\alpha \in \pi(V, \Omega A)$, $\beta \in \pi(V, \Omega B)$ such that $\gamma_1 = (\Omega f)_*(\alpha) + \gamma_2 + (\Omega g)_*(\beta)$.*

Now let $\chi_1 : E_{P_2} \to E_f$, $\chi_2 : E_{P_1} \to E_g$ be the maps induced by the following homotopy-commutative diagram

$$\begin{tikzcd} E_{f,g} \arrow[r, "P_1"] \arrow[d, "P_2"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"] & Y. \end{tikzcd}$$

LEMMA 1.2. (Dual excision theorem) *χ_1 and χ_2 are homotopy equivalences.*

Proof. We define $\Gamma_2 : E_g \to E_{P_1}$ by $\Gamma_2(\beta, b) = (e;\ *, \beta, b)$, where e is the constant path at the base-point of A. Then it is easily seen that Γ_2 is a homotopy inverse of χ_2.

Next let the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\psi_1"'] & Y \arrow[d, "\varphi"] & B \arrow[l, "g"'] \arrow[d, "\psi_2"] \\ A' \arrow[r, "f'"] & Y' & B' \arrow[l, "g'"'] \end{tikzcd}$$

be homotopy-commutative. This induces the map $\chi : E_{f,g} \to E_{f',g'}$ and the commutative diagram

$$\begin{tikzcd} E_{\psi_1} \arrow[r, "\Phi_1"] \arrow[d] & E_\varphi \arrow[d] & E_{\psi_2} \arrow[l, "\Phi_2"'] \arrow[d] \\ A \arrow[r, "f"] & Y & B, \arrow[l, "g"'] \end{tikzcd}$$

where the vertical maps are appropriate projections.

LEMMA 1.3. *There exist a homeomorphism $\Xi : E_\chi \to E_{\Phi_1,\Phi_2}$ and an injection $l : \Omega E_{f',g'} \to E_{\Phi_1,\Phi_2}$ such that the following diagram is homotopy commutative:*

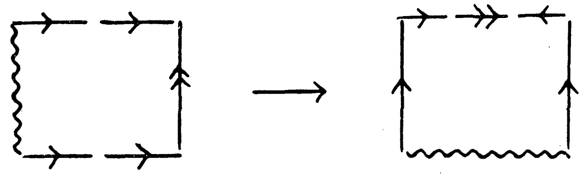
$$\begin{tikzcd}
 & E_\chi \arrow[d, "\Xi"'] \arrow[dr, "P"] & & \\
\Omega E_{f',g'} \arrow[ur, "i"] \arrow[r, "l"] & E_{\Phi_1,\Phi_2} \arrow[r] & E_{f,g} \arrow[r, "\chi"] & E_{f',g'} \\
 & \Omega E_\varphi \arrow[u] \arrow[r] & \Omega Y \arrow[u] \arrow[r, "\Omega_\varphi"] & \Omega Y',
\end{tikzcd}$$

in which i and P are natural injection and projection, respectively. In particular, for a triple $A \xrightarrow{g} B \xrightarrow{h} C$, the fibre of the natural map $\chi : E_{h\circ g} \to E_h$ is of the same homotopy type as E_g.

Proof. It is sufficient to define Ξ by setting

$$\Xi(\alpha, \tilde{\gamma}, \beta;\ a, \gamma, b) = ((\alpha, a), (\tilde{\gamma}\circ\tilde{h}, \gamma), (\beta, b))$$

for $a \in A$, $b \in B$, $\gamma \in Y^I$, $\alpha \in EA'$, $\beta \in EB'$, $\tilde{\gamma} \in E(Y'^I)$, $\gamma(0) = f(a)$, $\gamma(1) = g(b)$, $\alpha(1) = \varphi_1(a)$, $\beta(1) = \varphi_2(b)$, where $\tilde{h} : I^2 \to I^2$ is a homeomorphism indicated by the following picture:



Now, let a cotriad $A \xleftarrow{f} X \xrightarrow{g} B$ be given. We define its mapping cylinder $C_{f,g}$ to be the space obtained from $A \vee (X\times I)/(*\times I) \vee B$ by the identifications $f(x) = (x, 0)$, $g(x) = (x, 1)$, $x \in X$. The injections $I_1 : A \to C_{f,g}$, $I_2 : B \to C_{f,g}$ are obviously defined. The mapping cylinder of a cotriad $* \longleftarrow X \xrightarrow{g} B$ is denoted by C_g, which is usually called the *cofibre* of g. Any point $x \in X$ defines a path $\hat{x}$ in $C_{f,g}$, C_g or SX which is given by

$$\hat{x}(t) = (x, t), \qquad 0 \leqq t \leqq 1.$$

Lemmas 1.1 ~ 1.3 are dualized as follows;

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LEMMA 1.1$'$. *Let $Q : C_{f,g} \to SX$ be the map defined by shrinking A and B to a point. Then $Q^*(\gamma_1) = Q^*(\gamma_2)$ for $\gamma_1, \gamma_2 \in \pi(SX, V)$ if and only if there exist $\alpha \in \pi(SA, V)$, $\beta \in \pi(SB, V)$ such that $\gamma_1 = (Sf)^*(\alpha) + \gamma_2 + (Sg)^*(\beta)$.*

LEMMA 1.2$'$. (Excision theorem) *Let $\chi_1' : C_f \to C_{I_2}$ and $\chi_2' : C_g \to C_{I_1}$ be the maps induced by the homotopy-commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"] & A \arrow[d, "I_1"] \\
B \arrow[r, "I_2"] & C_{f,g}.
\end{tikzcd}$$

Then χ_1' and χ_2' are homotopy equivalences.

LEMMA 1.3$'$. *Let the diagram*

$$\begin{tikzcd}
A \arrow[d] & X \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "\varphi"] & B \arrow[d, "\psi_2"] \\
A' & X' \arrow[l, "f'"] \arrow[r, "g'"] & B'
\end{tikzcd}$$

be homotopy-commutative, and let

$$\begin{tikzcd}
A' \arrow[d] & X' \arrow[l, "f'"'] \arrow[r, "g'"] \arrow[d] & B' \arrow[d] \\
C_{\psi_1} & C_\varphi \arrow[l, "\theta_1"] \arrow[r, "\theta_2"] & C_{\psi_2}
\end{tikzcd}$$

be the associated commutative squares. Then, for the mapping $\chi' : C_{f,g} \to C_{f',g'}$ induced by the above transformation, we have a homeomorphism $\Xi' : C_{\chi'} \to C_{\theta_1,\theta_2}$ such that the following diagram homotopy-commutes:

$$\begin{tikzcd}
C_{f,g} \arrow[r, "\chi'"] \arrow[dd] & C_{f',g'} \arrow[r] \arrow[dd] \arrow[dr] & C_{\chi'} \arrow[r] \arrow[d, "\Xi'"] & SC_{f,g} \arrow[dd] \\
 & & C_{\theta_1,\theta_2} \arrow[ur] \arrow[d] & \\
SX \arrow[r, "S_\varphi"] & SX' \arrow[r] & SC_\varphi \arrow[r] & S^2X.
\end{tikzcd}$$

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In particular, the cofibre of the natural map $C_g \to C_{h\circ g}$ *induced by a triple* $A \xrightarrow{g} B \xrightarrow{h} C$, *is of the same homotopy type as* C_h.

The following lemmas will be needed in the later sections.

LEMMA 1.4. *Let* $\bar{f} : Y \to \Omega X$ *be the map adjoint to* $f : SY \to X$, *and suppose that* f *and* X *are, respectively,* m- *and* n-*connected. Then* $\bar{f}$ *is* $\min(2n, m)$-*connected.*

Proof. By Lemma (4.1) of Berstein-Hilton [6], we have the commutative diagram

$$\begin{array}{ccc} H_{i-1}(Y) & \xrightarrow{\bar{f}_*} & H_{i-1}(\Omega X) \\ \approx\downarrow & & \downarrow\sigma \\ H_i(SY) & \xrightarrow{f_*} & H_i(X), \end{array}$$

where σ is the homology suspension. Since σ is onto for $i \leqq 2n+1$ and monomorphic for $i \leqq 2n$, we obtain the desired conclusion.

LEMMA 1.5. *Suppose we are given* $f : SY \to X$ *and its adjoint* $\bar{f} : Y \to \Omega X$ *and let* $\bar{f}$, Y *be, respectively,* m-, n-*connected. Then* f *is* $[\min(m, 2n+2)+1]$-*connected.*

Proof. It is sufficient to observe that, in the following commutative diagram, the homotopy suspension E is onto for $i \leqq 2n+2$ and monomorphic for $i \leqq 2n+1$:

$$\begin{array}{ccc} \pi_{i-1}(Y) & \xrightarrow{\bar{f}_*} & \pi_{i-1}(\Omega X) \\ \downarrow E & & \downarrow\approx \\ \pi_i(SY) & \xrightarrow{f_*} & \pi_i(X). \end{array}$$

§ 2. Joins and cojoins

Given a triad $A \xrightarrow{i_1} A \vee B \xleftarrow{i_2} B$ consisting of inclusions, we denote its mapping track E_{i_1, i_2} by $A \hat{*} B$, which is called the *cojoin* of A and B (cf. [2]. Hilton uses the notation $A *' B$ in [9, p. 238]). We have the diagram

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$$
\begin{tikzcd}
\Omega A \arrow[dr, "\Omega i_1"] & & & A \arrow[dr, "i_1"] & \\
& \Omega(A \vee B) \arrow[r, "I"] & A \hat{*} B \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & A \vee B \\
\Omega B \arrow[ur, "\Omega i_2"'] & & & B \arrow[ur, "i_2"'] &
\end{tikzcd}
$$

Let $A\flat B$ be the *flat product* of A and B, i.e., the fibre E_J of the injection $J: A \vee B \to A \times B$. Thus the sequence

$$A\flat B \xrightarrow{L} A \vee B \xrightarrow{J} A \times B$$

is essentially a fibre triple.

Lemma 2.1. *p_1 and p_2 are null-homotopic.*

Proof. Let $r: A \vee B \to A$ be the retraction resulting from shrinking B to base-point. Note that $A \hat{*} B$ is the space of paths in $A \vee B$ which emanate from A and end in B, and that p_1 is the fibre map which assigns to each path the starting point. Then we can readily see that a null-homotopy $p_1 \simeq 0$ is generated by the correspondence $(\gamma, t) \to r\gamma(t)$, $0 \leqq t \leqq 1$, $\gamma \in A \hat{*} B$.

In the light of Lemma 2.1, we have exact rows in the following diagram

$$
\begin{tikzcd}
\pi_k(\Omega A) \oplus \pi_k(\Omega B) \arrow[r, "i_{1*}+i_{2*}"] \arrow[d, "\approx"] & \pi_k(\Omega(A \vee B)) \arrow[r, "I_*"] \arrow[d, equal] & \pi_k(A \hat{*} B) \arrow[r] & 0 \\
\pi_k(\Omega(A \times B)) & \pi_k(\Omega(A \vee B)) \arrow[l, "(\Omega J)_*"'] & \pi_k(\Omega(A\flat B)) \arrow[l, "(\Omega L)_*"'] & 0. \arrow[l]
\end{tikzcd}
$$

Since the composition $(\Omega J)_* \circ (i_{1*} + i_{2*})$ is the direct sum representation, it follows by a routine argument (cf. [8, the proof of Theorem 3.2]) that $I_* \circ (\Omega L)_*$ is bijective. Hence we have established

Proposition 2.2. ([2, p. 22]) *$I \circ (\Omega L): \Omega(A\flat B) \to A \hat{*} B$ is a weak homotopy equivalence.*

Corollary 2.3. *Suppose that A is m-connected and B n-connected. Then $A \hat{*} B$ is $(m+n-1)$-connected.*

M. Arkowitz ([2, 3]) has defined the *dual product* $[\alpha, \beta]$ of $\alpha \in \pi(V, \Omega A)$ and $\beta \in \pi(V, \Omega B)$ to be the unique element $\gamma \in \pi(V, \Omega(A\flat B))$ such that $(\Omega L)_*(\gamma) = -(\Omega i_1)_*(\alpha) - (\Omega i_2)_*(\beta) + (\Omega i_1)_*(\alpha) + (\Omega i_2)_*(\beta)$. Further, we denote the element $I_*(-(\Omega i_2)_*(\beta) + (\Omega i_1)_*(\alpha)) \in \pi(V, A \hat{*} B)$ by $\langle \alpha, \beta \rangle$, and call it the

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cojoin product of α and β. This is nothing but the second dual product $[\alpha, \beta]'$ defined in [2].

PROPOSITION 2.4. ([2, p. 22]) *The weak homotopy equivalence* $I\circ(\Omega L)$ *sends* $[\alpha, \beta]$ *to* $\langle\alpha, \beta\rangle$.

Proof. This is easily seen by noting, in view of Lemma 1.1, that

$$I_*(-(\Omega i_1)_*(\alpha)-(\Omega i_2)_*(\beta)+(\Omega i_1)_*(\alpha)+(\Omega i_2)_*(\beta))$$
$$=I_*(-(\Omega i_2)_*(\beta)+(\Omega i_1)_*(\alpha)).$$

Now let $\bar{f}: V\to\Omega A$, $\bar{g}: V\to\Omega B$ be representatives of α, β and let $f: SV\to A$, $g: SV\to B$ be adjoint to $\bar{f}$, $\bar{g}$ respectively. f and g obviously induce the map $f\hat{*}g: SV\hat{*}SV\to A\hat{*}B$. Let $\varepsilon: V\to\Omega SV$ be the natural injection defined by $\varepsilon(v)=\hat{v}$, $v\in V$, With these notations we have

LEMMA 2.5. $(f\hat{*}g)_*\langle\varepsilon, \varepsilon\rangle=\langle\alpha, \beta\rangle$.

Proof. This follows from the fact that $\alpha=(\Omega f)_*(\varepsilon)$, $\beta=(\Omega g)_*(\varepsilon)$ and from commutativity of the diagram

$$\begin{tikzcd}
\Omega(SV\vee SV) \arrow[r, "I"] \arrow[d, "\Omega(f\vee g)"'] & SV\hat{*}SV \arrow[d, "f\hat{*}g"] \\
\Omega(A\vee B) \arrow[r, "I"] & A\hat{*}B.
\end{tikzcd}$$

We mention here the relationship between the cup-product and the cojoin product. Let A, B be the Eilenberg-MacLane spaces $K(G_1, p+1)$, $K(G_2, q+1)$ respectively. Let

$$\iota\in H^{p+q}(A\hat{*}B;\ G)\approx\mathrm{Hom}(H_{p+q}(\Omega(A\flat B));\ G)\approx\mathrm{Hom}(G, G)$$

be the cohomology class corresponding to the identity homomorphism of G, where G is the tensor product $G_1\otimes G_2$. Then Arkowitz [3] has proved

PROPOSITION 2.6. $\langle\alpha, \beta\rangle^*(\iota)=\alpha\cup\beta$ *for* $\alpha\in H^p(V;\ G_1)$, $\beta\in H^q(V;\ G_2)$.

Dually, the ***join*** $A*B$ of A, B is defined to be the mapping cylinder C_{p_1,p_2} of the cotriad $A\xleftarrow{p_1}A\times B\xrightarrow{p_2}B$, where p_1, p_2 are the projections. Any point of $A*B$ is represented by the symbol $(1-t)a\oplus tb$, $a\in A$, $b\in B$, $0\leqq t\leqq 1$. We have the diagram

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$$\begin{tikzcd} & A \arrow[dr, "j_1"] & & & & SA \\ A\times B \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & A*B \arrow[r, "Q"] & S(A\times B) \arrow[urr, "Sp_1"] \arrow[drr, "Sp_2"'] & & \\ & B \arrow[ur, "j_2"'] & & & & SB \end{tikzcd}$$

in which $j_1\simeq 0$ and $j_2\simeq 0$. Also, if we denote the cofibre of $A\vee B\xrightarrow{J}A\times B$ by $A\#B$, we have a cofibre sequence

$$A\vee B\xrightarrow{J}A\times B\xrightarrow{K}A\#B.$$

Applying the same argument as in the proof of Proposition 2.2 to the diagram

$$\begin{tikzcd} 0 \arrow[r] & H_k(A*B) \arrow[r, "Q_*"] & H_k(S(A\times B)) \arrow[r, "\{(Sp_1)_*,\ (Sp_2)_*\}"] \arrow[d, equal] & H_k(SA)\oplus H_k(SB) \\ 0 & H_k(S(A\#B)) \arrow[l] & H_k(S(A\times B)) \arrow[l, "(SK)_*"'] & H_k(SA\vee SB), \arrow[l, "(SJ)_*"'] \arrow[u, "\approx"'] \end{tikzcd}$$

we obtain.

Proposition 2.2′. $(SK)\circ Q: A*B\to S(A\#B)$ *is a weak homotopy equivalence.*

Now recall that the *generalized Samelson product* $\langle\langle\alpha,\ \beta\rangle\rangle\in\pi(S(A\vee B),\ V)$ of $\alpha\in\pi(SA,\ V)$ and $\beta\in\pi(SB,\ V)$ is defined to be the unique element γ such that $q^*(\gamma)=-(Sp_1)^*(\alpha)-(Sp_2)^*(\beta)+(Sp_1)^*(\alpha)+(Sp_2)^*(\beta)$ in the exact sequence

$$0\leftarrow\pi(SA\vee SB,\ V)\leftarrow\pi(S(A\times B),\ V)\xleftarrow{q^*}\pi(S(A\wedge B),\ V)\leftarrow 0,$$

where $A\wedge B$ is the smashed product $A\times B/A\vee B$ and $q: S(A\times B)\to S(A\wedge B)$ is the identification map. Note that, in this argument, A and B are assumed to have non-degenerate base-point. The *generalized Whitehead product* $[\alpha,\ \beta]$ is defined to be the element $Q^*(-(Sp_2)^*(\beta)+(Sp_1)^*(\alpha))\in\pi(A*B,\ V)$. We see from Lemma 1.1′ that the homotopy equivalence $A*B\to S(A\wedge B)$ transforms $\langle\langle\alpha,\ \beta\rangle\rangle$ to $[\alpha,\ \beta]$.

We shall need, in § 5, the map $W: \Omega A*\Omega B\to B\flat A$ which is defined by

$$W((1-t)\alpha\oplus t\beta)=\begin{cases}\beta_{0,2t}\times\alpha, & 0\leqq 2t\leqq 1,\\ \beta\times\alpha_{0,2-2t}, & 1\leqq 2t\leqq 2.\end{cases} \tag{2.7}$$

for $\alpha\in\Omega A$, $\beta\in\Omega B$. Then the following lemma is well-known (cf. [8, § 2]).

Lemma 2.8. *W is a weak homotopy equivalence.*

Dually, we define $W': A\#B\to SA\hat{*}SB$ as follows:

$$
(2.9)\qquad
\begin{aligned}
W'(a,\ b)(t) &= \begin{cases} (b,\ 1-2t), & 0 \leqq 2t \leqq 1, \\ (a,\ 2t-1), & 1 \leqq 2t \leqq 2, \end{cases} \\
W'(a,\ b_0,\ s)(t) &= \begin{cases} (a,\ 1-s), & 0 \leqq 2t \leqq 1, \\ (a,\ 1-2s+2st), & 1 \leqq 2t \leqq 2, \end{cases} \\
W'(a_0,\ b,\ s)(t) &= \begin{cases} (b,\ 1-2st), & 0 \leqq 2t \leqq 1, \\ (b,\ 1-s), & 1 \leqq 2t \leqq 2, \end{cases}
\end{aligned}
$$

for $a \in A$, $b \in B$, $0 \leqq s \leqq 1$. I regret to say that I was unable to show the dual of Lemma 2.8, but we will content ourselves with a partial result (see Corollary 5.10).

§ 3. Extensions of triads

Let the diagram

$$
(3.1)\qquad
\begin{tikzcd}
 & A \arrow[dr, "f"] & \\
E_{f,g} \arrow[ur, "P_1"] \arrow[dr, "P_2"'] & & Y \\
 & B \arrow[ur, "g"'] &
\end{tikzcd}
$$

be associated with a triad $A \xrightarrow{f} Y \xleftarrow{g} B$, and consider the mapping cylinder C_{P_1, P_2} of the cotriad $A \xleftarrow{P_1} E_{f,g} \xrightarrow{P_2} B$. We define the natural extension

$$\xi' : C_{P_1, P_2} \to Y$$

of the triad $(f : g)$ over C_{P_1, P_2} by setting

$$\xi'(a,\ \gamma,\ b;\ t) = \gamma(t),\ \xi'(a) = f(a),\ \xi'(b) = g(b)$$

for $a \in A$, $b \in B$, $\gamma \in Y^I$, $0 \leqq t \leqq 1$.

Next, let $f \nabla g : A \vee B \to Y$ be the map determined by f and g, i.e., the composite $A \vee B \xrightarrow{f \vee g} Y \vee Y \xrightarrow{\nabla} Y$, where ∇ is the folding map. We define

$$\eta' : SE_{f,g} \to C_{f \nabla g}$$

by setting, for $(a,\ \gamma,\ b) \in E_{f,g}$, $0 \leqq s \leqq 1$,

$$
\eta'(a,\ \gamma,\ b;\ s) = \begin{cases} (a,\ 4s) \in CA, & 0 \leqq 4s \leqq 1, \\ \gamma\left(\dfrac{4s-1}{2}\right) \in Y, & 1 \leqq 4s \leqq 3, \\ (b,\ 4-4s) \in CB, & 3 \leqq 4s \leqq 4. \end{cases}
$$

Introduce the homotopy-commutative diagram

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$$
\begin{tikzcd}
 & & C_{P_1,P_2} \arrow[r, "Q"] \arrow[d, "\xi'"] & SE_{f,g} \arrow[d, "\eta'"] & \\
(3.2) & A\vee B \arrow[r, "f\nabla g"] & Y \arrow[r, "k"] \arrow[d] & C_{f\nabla g} \arrow[r] \arrow[d] & S(A\vee B) \\
 & & C_{\xi'} \arrow[r, "\zeta'"] & C_{\eta'}, &
\end{tikzcd}
$$

in which ζ' is the map induced by the upper square and the unlabelled arrows denote the appropriate injections and identification.

The following proposition is an extension of Proposition 1.6 of Ganea [8].

PROPOSITION 3.3. *$\zeta' : C_{\xi'} \to C_{\eta'}$ is a homotopy equivalence.*

Proof. ζ' is given explicitly as follows: if $2s \leqq 1$, then

$$\zeta'(y) = y \in Y,\ \zeta'(a,\ s) = *,\ \zeta'(b,\ s) = *,$$
$$\zeta'(a,\ \gamma,\ b,\ t;\ s) = (a,\ \gamma,\ b,\ t;\ 2s);$$

if $2s \geqq 1$, then

$$\zeta'(y) = y \in Y,\ \zeta'(a,\ s) = (a,\ 2s-1) \in C_{f\nabla g},\ \zeta'(b,\ s) = (b,\ 2s-1),$$

$$\zeta'(a, \gamma, b, t;\ s) = \begin{cases} (a,\ 4t+2s-1) \in C_{f\nabla g}, & 0 \leqq t \leqq \dfrac{1-s}{2} \\ \gamma\left(\dfrac{2t+s-1}{2s}\right), & \dfrac{1-s}{2} \leqq t \leqq \dfrac{1+s}{2} \\ (b,\ 3+2s-4t) \in C_{f\nabla g}, & \dfrac{1+s}{2} \leqq t \leqq 1. \end{cases}$$

for cone parameter s and cylinder one t. We consider $\varepsilon' : C_{\eta'} \to C_{\xi'}$ given by

$$\varepsilon'(y) = y,\ \varepsilon'(a,\ s) = (a,\ s),\ \varepsilon'(b,\ s) = (b,\ s),$$

$$\varepsilon'(a,\ \gamma,\ b,\ u;\ s) = \begin{cases} \left(a,\ \gamma,\ b,\ \dfrac{1-s}{4};\ 4u\right), & 0 \leqq 4u \leqq s, \\ \left(a,\ \gamma,\ b,\ \dfrac{2u(1+s)+1-2s}{4-2s};\ s\right), & s \leqq 4u \leqq 4-s, \\ \left(a,\ \gamma,\ b,\ \dfrac{s+3}{4};\ 4-4u\right), & 4-s \leqq 4u \leqq 4 \end{cases}$$

for suspension parameter u. It is a troublesome but routine matter to verify that ε' is a homotopy inverse of ζ'.

One of the main objects in this section is to prove the following theorem which generalizes Theorem 1.1 in [8].

THEOREM 3.4. *The fibre $E_{\xi'}$ of $\xi' : C_{P_1,P_2} \to Y$ has the same homotopy type as the join $E_{\bar f} * E_g$ of the fibres of f and g.*

Proof. We define $F : E_{\bar f} * E_g \to E_{\xi'}$ and $G : E_{\xi'} \to E_{\bar f} * E_g$ by setting, for $a \in A$, $b \in B$, $\alpha, \beta, \gamma, \tau \in Y^I$, $0 \leqq t \leqq 1$,

$$(3.5) \quad \begin{cases} F(a, \alpha) = (-\alpha; a), \; F(\beta; b) = (\beta; b) \\ F((1-t)(a, \alpha) \oplus t(\beta, b)) = \begin{cases} (-\alpha_{2t,1}; a, \alpha+\beta, b, t), & 0 \leqq 2t \leqq 1 \\ (\beta_{0,2t-1}; a, \alpha+\beta, b, t), & 1 \leqq 2t \leqq 2, \end{cases} \\ G(\tau; a) = (a, e_{\tau(1)} - \tau), \; G(\tau; b) = (\tau + e_{\tau(1)}, b), \\ G(\tau; a, \gamma, b, t) = (1-t)(a, \gamma_{0,t} - \tau) \oplus t(\tau + \gamma_{t,1}, b), \end{cases}$$

where e_x denotes the constant path at x.

$G \circ F$ can be deformed into the identity via a homotopy Φ_u, $0 \leqq u \leqq 2$, whose value $\Phi_u((1-t)(a, \alpha) \oplus t(\beta, b))$ is given by setting, if $0 \leqq 2t \leqq 1$, $0 \leqq u \leqq 1$,

$$(1-t)(a, \alpha_{0,2t} + \alpha_{2t,1}) \oplus t(-\alpha_{2(1-u)t+u,1} + (\alpha+\beta)_{(1-u)t+\frac{u}{2},1}, b);$$

if $1 \leqq 2t \leqq 2$, $0 \leqq u \leqq 1$,

$$(1-t)(a, (\alpha+\beta)_{0,(1-u)t+\frac{u}{2}} - \beta_{0,(1-u)(2t-1)}) \oplus t(\beta_{0,2t-1} + \beta_{2t-1,1}, b);$$

if $0 \leqq 2t \leqq 1$, $1 \leqq u \leqq 2$,

$$(1-t)(a, \alpha_{0,2t(2-u)+\frac{u-1}{2}} + \alpha_{2t(2-u)+\frac{u-1}{2},1}) \oplus t(\beta_{0,\frac{u-1}{2}} + \beta_{\frac{u-1}{2},1}, b);$$

if $1 \leqq 2t \leqq 2$, $1 \leqq u \leqq 2$,

$$(1-t)(a, \alpha_{0,\frac{3-u}{2}} + \alpha_{\frac{3-u}{2},1}) \oplus t(\beta_{0,(2t-1)(2-u)+\frac{u-1}{2}} + \beta_{(2t-1)(2-u)+\frac{u-1}{2}}, b).$$

$F \circ G \simeq 1$ is verified by taking a homotopy Ψ_u, $0 \leqq u \leqq 2$, whose value $\Psi_u(\tau; a, \gamma, b, t)$ is, if $0 \leqq u \leqq 1$,

$$(\delta; a, (\gamma_{0,t} - \tau)_{0,1-\frac{u}{2}} + (\tau + \gamma_{t,1})_{\frac{u}{2},1}, b, t),$$

where

$$\delta = \begin{cases} -(\gamma_{0,t} - \tau)_{2t-ut,1} & 0 \leqq 2t \leqq 1, \\ (\tau + \gamma_{t,1})_{0,(1-t)u+2t-1} & 1 \leqq 2t \leqq 2; \end{cases}$$

if $1 \leqq u \leqq 2$,

$$(\varepsilon;\ a,\ \gamma_{0,(2-u)t+\frac{u-1}{2}} + \gamma_{(2-u)t+\frac{u-1}{2},1},\ b,\ t),$$

where

$$\varepsilon = \begin{cases} -(\gamma_{0,t}-\tau)_{(2-u)t+\frac{u-1}{2},1} & 0 \leqq 2t \leqq 1, \\ (\tau+\gamma_{t,1})_{0,(2-u)t+\frac{u-1}{2}} & 1 \leqq 2t \leqq 2. \end{cases}$$

Thus the proof of Theorem 3.4 is complete.

The composition $E_{\bar{f}} * E_g \xrightarrow{F} E_{\xi'} \longrightarrow C_{P_1, P_2}$ will be denoted by $j : E_{\bar{f}} * E_g \to C_{P_1, P_2}$. This is given by

$$(3.6) \qquad j((1-t)(a,\ \alpha) \oplus t(\beta,\ b)) = (a,\ \alpha+\beta,\ b;\ t).$$

Consequently, the sequence

$$E_{\bar{f}} * E_g \xrightarrow{j} C_{P_1, P_2} \xrightarrow{\xi'} Y$$

is essentially the fibre triple.

Combining Theorem 3.4 with Proposition 3.3 we obtain

COROLLARY 3.7. *Suppose that f is p-connected and g q-connected. Then ξ' and η' are both $(p+q+1)$-connected.*

Remark As in Proposition 1.5 of [8], there exists a map $\Gamma : \Omega Y \to \Omega C_{P_1, P_2}$ such that $\Omega\xi' \circ \Gamma =$ identity. It is sufficient to define Γ by $\Gamma(\omega)(t) = (*, \omega, *; t)$. Note that the diagram

$$\begin{array}{ccc} \Omega C_{P_1, P_2} & \xrightarrow{\Omega Q} & \Omega S E_{f,g} \\ \uparrow & & \uparrow{\scriptstyle \bar{\sigma}} \\ \Omega Y & \xrightarrow{I} & E_{f,g} \end{array}$$

is commutative, in which $\bar{\sigma}$ is the canonical injection

Now we shall deduce the well known theorem of Serre on relative fibre maps from Corollary 3.7. For this purpose we prove.

THEOREM 3.8. *Let $\Phi_1 : C_{P_1} \to C_g$ and $\Phi_2 : C_{P_2} \to C_f$ be the maps induced by the homotopy-commutative diagram* (3.1). *Then the cofibres of Φ_1 and Φ_2 have the same homotopy type as those of ξ'.*

Proof. Let the diagram

$$\begin{tikzcd}[column sep=small] & A \arrow[dr, "i_1"] & \\ E_{f,g} \arrow[ur, "P_1"] \arrow[dr, "P_2"'] & & C_{P_1,P_2} \\ & B \arrow[ur, "i_2"'] & \end{tikzcd}$$

be associated with the cotriad P_1, P_2. Using this, the maps $\chi_1' : C_{P_1} \to C_{i_2}$ and $\chi_2' : C_{P_2} \to C_{i_1}$ are obviously defined. On the other hand, $\xi' : C_{P_1,P_2} \to Y$ determines the maps $k_1 : C_{i_2} \to C_g$ and $k_2 : C_{i_1} \to C_f$. We see easily that the compositions $k_1 \circ \chi_1'$ and $k_2 \circ \chi_2'$ coincide with Φ_1 and Φ_2, respectively. Since both C_{k_1} and C_{k_2} are equivalent to $C_{\xi'}$ by Lemma 1.3′, and since χ_1' and χ_2' are homotopy equivalences by Lemma 1.2′, we conclude that C_{Φ_1} and C_{Φ_2} are equivalent to $C_{\xi'}$, which completes the proof.

Corollary 3.9. (Serre theorem on relative fibre maps) *Suppose that f is p-connected and g q-connected, and that g is a fibration. Let $\overline{\Phi}_1 : C_{\pi_1} \to C_g$, $\overline{\Phi}_2 : C_{\pi_2} \to C_f$ be the maps determined by the commutative square:*

$$\begin{tikzcd}[column sep=small] & A \arrow[dr, "f"] & \\ \mathrm{Ker}(f ; g) \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & & Y, \\ & B \arrow[ur, "g"'] & \end{tikzcd}$$

where $\mathrm{Ker}(f : g)$ *is the fibre space induced from g by f. Then $\overline{\Phi}_1$ and $\overline{\Phi}_2$ are* $(p+q+1)$*-connected.*

This follows from Corollary 3.7 and Theorem 3.8, observing that $\overline{\Phi}_1$ and $\overline{\Phi}_2$ are, respectively, equivalent to Φ_1 and Φ_2 of Theorem 3.8.

Theorem 3.10. *Suppose that f is p-connected and g q-connected. Let V be a 1-connected space such that $\pi_i(V) = 0$ for $i \geqq p+q+1$. If A, B, Y and V have the homotopy type of CW-complexes, then the following sequence is exact:*

$$\begin{tikzcd}[column sep=small] & \pi(A, V) \arrow[dl, "P_1^*"'] & & & \\ \pi(E_{f,g}, V) & & \pi(Y, V) \arrow[ul, "f^*"'] \arrow[dl, "g^*"] & \pi(SE_{f,g}, V) \arrow[l, "Q^* \circ (\xi')^{*-1}"'] & \arrow[l, "(SP_1)^*"'] \arrow[l, shift left=2, "(SP_2)^*"] \\ & \pi(B, V) \arrow[ul, "P_2^*"] & & & \end{tikzcd}$$

§ 4. Lifting cotriads

Let $A \xleftarrow{f} X \xrightarrow{g} B$ be a cotriad and let

(4.1)
$$\begin{tikzcd} & A \arrow[dr, "I_1"] & \\ X \arrow[ur, "f"] \arrow[dr, "g"'] & & C_{f,g} \\ & B \arrow[ur, "I_2"'] & \end{tikzcd}$$

be the associated diagram. Consider the mapping track E_{I_1,I_2} of the triad I_1, I_2 and let $f\varDelta g: X \to A\times B$ be the composition $X \xrightarrow{\varDelta} X\times X \xrightarrow{f\times g} A\times B$, where $\varDelta$ is the diagonal injection. We define $\xi: X \to E_{I_1,I_2}$ and $\eta: E_{f\varDelta g} \to \varOmega C_{f,g}$ by setting, for $x\in X$, $\alpha\in EA$, $\beta\in EB$,

$$\xi(x) = (f(x),\ \hat{x},\ g(x)), \qquad \hat{x}(s) = (x,\ s)\in X\times I \subset C_{f,g}$$

$$\eta(x,\ \alpha\times\beta)(s) = \begin{cases} \alpha(4s), & 0\leqq 4s\leqq 1, \\ \left(x,\ \dfrac{4s-1}{2}\right), & 1\leqq 4s\leqq 3, \\ \beta(4-4s), & 3\leqq 4s\leqq 4. \end{cases}$$

Introduce the following homotopy-commutative diagram

$$\begin{tikzcd} E_\eta \arrow[r, "\zeta"] \arrow[d] & E_\xi \arrow[d] & \\ E_{f\varDelta g} \arrow[r] \arrow[d, "\eta"] & X \arrow[r, "f\varDelta g"] \arrow[d, "\xi"] & A\times B \\ \varOmega C_{f,g} \arrow[r, "I"] & E_{I_1,I_2}, & \end{tikzcd}$$

in which I is the injection and ζ is the map induced by the lower (homotopy-commutative) square

PROPOSITION 4.2. *$\zeta: E_\eta \to E_\xi$ is a homotopy equivalence.*

As shown in [12], we can deduce the Blakers-Massey theorem on excisive triads from the Serre theorem on relative fibre maps. For this purpose we prove

THEOREM 4.3. *Suppose that f is p-connected and g q-connected. Then ξ and η are $(p+q-1)$-connected.*

Proof. We consider the homotopy-commutative diagram

$$\begin{tikzcd} X \arrow[d, "\xi"'] \arrow[dr, "f"] & \\ E_{I_1, I_2} \arrow[r, "p_1"'] \arrow[d, "p_2"'] & A \arrow[d, "I_1"] \\ B \arrow[r, "I_2"] & C_{f,g}, \end{tikzcd}$$

in which the square is associated with the triad I_1, I_2. By Lemma 1.2′, I_2 and I_1 are, respectively, p- and q-connected. Applying Theorem 3.8 to the above square, the map

$$\chi : C_{p_1} \to C_{I_2}$$

induced by the above homotopy-commutative square, is $(p+q+1)$-connected.

Now it is easily seen that the composition

$$C_f \longrightarrow C_{p_1} \xrightarrow{\chi} C_{I_2},$$

in which the first map is determined by ξ, coincides with the homotopy equivalence $\chi_1' : C_f \to C_{I_2}$ of Lemma 1.2′. Thus, $C_f \to C_{p_1}$ is $(p+q)$-connected, and therefore, by resorting to Proposition 4.2 and the sequence

$$C_\xi \to C_f \to C_{p_1} \to SC_\xi \to SC_f \to SC_{p_1} \to \cdots,$$

we can infer that ξ and η are $(p+q-1)$-connected.

Suppose further that g is a cofibration and

$$\begin{tikzcd} & A \arrow[dr, "i_1"] & \\ X \arrow[ur, "f"] \arrow[dr, "g"'] & & \mathrm{Coker}\langle f : g\rangle \\ & B \arrow[ur, "i_2"'] & \end{tikzcd}$$

is the associated commutative diagram, where Coker $\langle f : g\rangle$ is the space obtained from $A \vee B$ by the identification $f(x) = g(x)$, $x \in X$. Let

$$\bar{\eta} : E_{f \Delta g} \to \Omega \,\mathrm{Coker}\langle f : g\rangle$$

be the map given by $\bar{\eta} = \Omega q \circ \eta$, where $q : C_{f,g} \to \mathrm{Coker}\langle f : g\rangle$ is the canonical equivalence. Note that, since g is an inclusion, $E_{f\Delta g}$ can be identified with the space consisting of $(\alpha, \beta) \in EA \times EB$ such that $i_1\alpha(1) = i_2\beta(1)$, i.e., the space S_{i_1, i_2} as defined in [13].

Since $\bar{\eta}$ is homotopic to $m : S_{i_1, i_2} \to \Omega \operatorname{Coker} \langle f : g \rangle$ which is given by

$$m(\alpha, \beta) = (\Omega i_1)(\alpha) - (\Omega i_2)(\beta),$$

and since the sequence

$$\pi_k(\Omega^2 \operatorname{Coker} \langle f : g \rangle) \to \pi_k(T_{i_1, i_2}) \to \pi_k(S_{i_1, i_2}) \xrightarrow{m_*} \pi_k(\Omega \operatorname{Coker} \langle f : g \rangle)$$

is exact by Proposition 3.3 of [13], where T_{i_1, i_2} is the subspace of $EA \times EB \times EE \operatorname{Coker} \langle f : g \rangle$ consisting of $(\alpha, \beta, \tilde{\gamma})$ such that $\tilde{\gamma}(s, 1) = i_1\alpha(s)$, $\tilde{\gamma}(1, t) = i_2\beta(t)$, it follows

Corollary 4.4. (Blakers-Massey) *If f and g are, respectively, p- and q-connected, and if g is a cofibration, then T_{i_1, i_2} is $(p+q-2)$-connected.*

Corollary 4.5. *Suppose that f is p-connected and g q-connected. Then, for any CW complex V with* $\dim V \leqq p+q-2$, *the following sequence is exact:*

$$\begin{tikzcd} & & \pi(V, A) \arrow[dr, "I_{1*}"] & \\ \pi(V, \Omega C_{f,g}) \arrow[r] & \pi(V, X) \arrow[ur, "f_*"] \arrow[dr, "g_*"'] & & \pi(V, C_{f,g}). \\ & & \pi(V, B) \arrow[ur, "I_{2*}"'] & \end{tikzcd}$$

The dual of Theorem 3.8 is stated as follows:

Theorem 4.6. *Let $\Phi_1' : E_g \to E_{I_1}$ and $\Phi_2' : E_f \to E_{I_2}$ be the maps induced by the homotopy-commutative square* (4.1). *Then the fibres of Φ_1' and Φ_2' are homotopy-equivalent to those of ξ.*

§ 5. The dual EHP sequence

In this section we construct, for a triad $A \xrightarrow{f} Y \xleftarrow{g} B$, a dual of the EHP sequence and examine its behaviour. The dual EHP cohomology sequence was first defined by G. W. Whitehead [15] and has been extended by Tsuchida-Ando [14].

First, consider the map $\mu : E_f^- \times E_g \to E_{f,g}$ defined by

$$\mu((a, \alpha), (\beta, b)) = (a, \alpha + \beta, b)$$

for $a \in A$, $b \in B$, $-\alpha$, $\beta \in EY$, and the "projections" $\Pi_1 : E_f^- \times E_g \to E_{f,g}$, $\Pi_2 : E_f^- \times E_g \to E_{f,g}$ defined by

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$$\Pi_1((a,\ \alpha),\ (\beta,\ b)) = (a,\ \alpha,\ *),$$
$$\Pi_2((a,\ \alpha),\ (\beta,\ b)) = (*,\ \beta,\ b).$$

We say that an element $\rho \in \pi(SE_{f,g},\ V)$ is *primitive with respect to* μ if and only if $(S\mu)^*(\rho) = (S\Pi_1)^*(\rho) + (S\Pi_2)^*(\rho)$.

Now let

$$q\ :\ E_f^- * E_g \to S(E_f^- \times E_g)$$

be the map which shrinks to a point the ends of the join. We have a map

$$\mathscr{H} = Q \circ j\ :\ E_f^- * E_g \to SE_{f,g},$$

where $j\ :\ E_f^- * E_g \to C_{P_1,P_2}$ and $Q\ :\ C_{P_1,P_2} \to SE_{f,g}$ are defined in (3.6), (3.2). Then we see at once that $\mathscr{H} = (S\mu) \circ q$. Note that $\mathscr{H}$ is equivalent to the map obtained from μ by the Hopf construction. The following lemma allows us to call $\mathscr{H}^*$ the ***dual Hopf invariant*** associated with the triad f, g.

LEMMA 5.1. (cf. [10, Theorem 1]) *$\rho \in \pi(SE_{f,g},\ V)$ is primitive with respect to μ if and only if $\mathscr{H}^*(\rho) = 0$.*

Proof. We consider the diagram associated with the join $E_f^- * E_g$:

$$\pi(E_f^- * E_g,\ V) \xleftarrow{\ q^*\ } \pi(S(E_f^- \times E_g),\ V) \begin{array}{l} \xleftarrow{(Sp_1)^*} \pi(SE_f^-,\ V) \\ \xleftarrow{(Sp_2)^*} \pi(SE_g,\ V). \end{array}$$

Then, by Lemma 1.1′, $q^* \circ (S\mu)^*(\rho) = 0$ if and only if there exist $\alpha \in \pi(SE_f^-,\ V)$, $\beta \in \pi(SE_g,\ V)$ such that

$$(S\mu)^*(\rho) = (Sp_1)^*(\alpha) + (Sp_2)^*(\beta).$$

Suppose first that the latter equality holds. We denote the injections $E_f^- \to E_f^- \times E_g$, $E_g \to E_f^- \times E_g$ by i_1, i_2 respectively. Applying $(Sp_1)^*(Si_1)^*$ to both sides, we obtain $(S\Pi_1)^*(\rho) = (Sp_1)^*(\alpha)$. Similarly, $(S\Pi_2)^*(\rho) = (Sp_2)^*(\beta)$. This proves that ρ is primitive.

Conversely, since $\Pi_k = (\Pi_k \circ i_k) \circ p_k$, $k = 1, 2$, "primitive" implies the existence of α, β such that $(S\mu)^*(\rho) = (Sp_1)^*(\alpha) + (Sp_2)^*(\beta)$. q.e.d.

We now describe an approximation to the fibre and cofibre of ξ' by means of the cofibres of f, g. Let

$$\mu' : C_{P_1,P_2} \to C_{P_1} \vee C_{P_2}$$

be the map obtained by shrinking the "center" $E_{f,g} \times \frac{1}{2}$ of the cylinder part of C_{P_1,P_2}, and let $\Phi_1 : C_{P_1} \to C_g$, $\Phi_2 : C_{P_2} \to C_f$ be as in Theorem 3.8. Let

$$k_1 : Y \to C_f \quad \text{and} \quad k_2 : Y \to C_g$$

denote natural injections and let

$$\sigma_1 : E_{\bar{f}} \to \Omega C_f \quad \text{and} \quad \sigma_2 : E_g \to \Omega C_g$$

denote the (Freudenthal) suspension maps given by

$$\sigma_1(a, \alpha) = -\alpha - \hat{a}, \quad \sigma_2(\beta, b) = \beta - \hat{b} \tag{5.2}$$

for $a \in A$, $b \in B$, $\alpha, \beta \in Y^I$.

Introduce the diagram

$$\begin{tikzcd}
E_{\bar{f}} * E_g \arrow[r, "j"] \arrow[d, "\sigma_1 * \sigma_2"] & C_{P_1,P_2} \arrow[r, "\xi'"] \arrow[d, "\mu'"] & Y \arrow[r] \arrow[d, "\Delta"] & C_{\xi'} \arrow[dd, "\theta"] \\
\Omega C_f * \Omega C_g \arrow[d, "W"] & C_{P_1} \vee C_{P_2} \arrow[d, "\Phi_1 \vee \Phi_2"] & Y \times Y \arrow[d, "k_2 \times k_1"] & \\
C_g \flat C_f \arrow[r, "L"] & C_g \vee C_f \arrow[r, "J"] & C_g \times C_f \arrow[r, "K"] & C_g \# C_f,
\end{tikzcd} \tag{5.3}$$

where W is the map defined in (2.7) and Δ is the diagonal injection. That homotopy-commutativity holds in the middle square, i.e., $(k_2 \times k_1) \circ \Delta \circ \xi' \simeq J \circ (\Phi_1 \vee \Phi_2) \circ \mu'$, can be verified by taking the following homotopy:

$$\begin{aligned}
&(a, \gamma, b\,;\, t) \to [(\gamma - \hat{b}) + *]\Big(\frac{t + 3ut}{4}\Big) \times [* + (\hat{a} + \gamma)]\Big(\frac{t + 3ut + 3 - 3u}{4}\Big) \\
&a \to f(a) \times [* + (\hat{a} + \gamma)]\Big(\frac{3 - 3u}{4}\Big), \quad b \to [(\gamma - \hat{b}) + *]\Big(\frac{1 + 3u}{4}\Big) \times g(b)
\end{aligned} \tag{5.4}$$

where $0 \leqq u \leqq 1$, $a \in A$, $b \in B$, $\gamma \in Y^I$, $0 \leqq t \leqq 1$. Therefore the map θ is induced so that the right square be commutative. Moreover, using (3.6), (5.2) and (2.7), we can verify the following:

$$[(\Phi_1 \vee \Phi_2) \circ \mu' \circ j]((1-s)(a, \alpha) \oplus s(\beta, b)) = \begin{cases} (\alpha + \beta)(4s) \in C_g & 0 \leqq 4s \leqq 1, \\ (b, 2 - 4s) \in C_g & 1 \leqq 4s \leqq 2, \\ (a, 4s - 2) \in C_f & 2 \leqq 4s \leqq 3, \\ (\alpha + \beta)(4s - 3) \in C_f & 3 \leqq 4s \leqq 4, \end{cases}$$

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$$[L\circ W\circ(\sigma_1*\sigma_2)]((1-s)(a,\ \alpha)\oplus s(\beta,\ b))=\begin{cases}\beta(4s)\in C_g & 0\leqq 4s\leqq 1,\\ \hat{b}(2-4s)\in C_g & 1\leqq 4s\leqq 2,\\ \hat{a}(4s-2)\in C_f & 2\leqq 4s\leqq 3,\\ \alpha(4s-3)\in C_f & 3\leqq 4s\leqq 4.\end{cases}$$

It follows that homotopy-commutativity holds in the left square.

The middle square of (5.3) induces the map $\chi: E_{\xi'}\to C_g\flat C_f$. We see at once from (5.4) that the composite $E_f^-*E_g\xrightarrow{F}E_{\xi'}\xrightarrow{\chi}C_g\flat C_f$ is given as follows:

$$(\chi\circ F)((1-s)(a,\ \alpha)\oplus s(\beta,\ b))=\begin{cases}(-\alpha_{2s,1}+\tau)\times(-\alpha_{2s,1}+\rho) & 0\leqq 2s\leqq 1,\\ (\beta_{0,2s-1}+\tau)\times(\beta_{0,2s-1}+\rho) & 1\leqq 2s\leqq 2,\end{cases}$$

where

$$\tau=[((\alpha+\beta)-\hat{b})+*]_{\frac{s}{4},s},\quad \rho=[((-\beta-\alpha)-\hat{a})+*]_{\frac{1-s}{4},1-s}.$$

Further we have

$$[W\circ(\sigma_1*\sigma_2)]((1-s)(a,\ \alpha)\oplus s(\beta,\ b))=\begin{cases}(\beta-\hat{b})_{0,2s}\times(-\alpha-\hat{a}) & 0\leqq 2s\leqq 1,\\ (\beta-\hat{b})\times(-\alpha-\hat{a})_{0,2-2s} & 1\leqq 2s\leqq 2.\end{cases}$$

From these results we infer

LEMMA 5.5. *$W\circ(\sigma_1*\sigma_2)$ is homotopic to $\chi\circ F$.*

LEMMA 5.6. *Suppose that f and g are, respectively, p- and q-connected and, further, let Y be r-connected. Then $W\circ(\sigma_1*\sigma_2)$ is $[p+q+\min(p,q,r+1)+1]$-connected and θ is $[p+q+\min(p,\ q,\ r)+2]$-connected.*

Proof. Since the adjoints of σ_1, σ_2 are respectively $(p+r+1)$- and $(q+r+1)$-connected, it follows from Lemma 1.4 that σ_1 and σ_2 are respectively min $(2p,\ p+r+1)$-connected and $\min(2q,\ q+r+1)$-connected. Thus, by Lemma 2.8, $W\circ(\sigma_1*\sigma_2)$ is $[p+q+\min(p,q,r+1)+1]$-connected. To prove the second half, note that, by Lemma 5.5, χ is $[p+q+\min(p,\ q,\ r+1)+1]$-connected. Introduce the homotopy commutative diagram

$$\begin{tikzcd} SE_{\xi'} \arrow[r, "S\chi"] \arrow[d, "\Sigma"'] & S(C_g\flat C_f) \arrow[d, "\sigma"] \\ C_{\xi'} \arrow[r, "\theta"] & C_g\# C_f, \end{tikzcd}$$

in which the suspension maps Σ, σ are respectively $(p+q+r+2)$-, $(p+q+\min(p, q)+3)$-connected. This completes the proof of the second half.

LEMMA 5.7. *The composition*

$$S(E_f^- * E_g) \xrightarrow{SF} SE_{\xi'} \xrightarrow{\Sigma} C_{\xi'} \xrightarrow{\zeta'} C_{\eta'} \xrightarrow{\partial} S^2 E_{f,g}$$

is homotopic to $S\mathscr{H} : S(E_f^- * E_g) \to S^2 E_{f,g}$, *where* ζ' *is the equivalence in* (3.2), ∂ *the map which results from shrinking* $C_{f\nabla g}$ *and* Σ *the suspension map given by*

$$\Sigma(\tau\ ;\ a, \gamma, b, s\ ;\ t) = \begin{cases} \tau(2-2t) \in Y & 1 \leqq 2t \leqq 2, \\ (a, \gamma, b, s\ ;\ 2t) \in CC_{P_1, P_2} & 0 \leqq 2t \leqq 1, \end{cases}$$

$$\Sigma(\tau\ ;\ a\ ;\ t) = (a, 2t) \text{ if } 2t \leqq 1, \quad = \tau(2-2t) \text{ if } 2t \geqq 1,$$

$$\Sigma(\tau\ ;\ b\ ;\ t) = (b, 2t) \text{ if } 2t \leqq 1, \quad = \tau(2-2t) \text{ if } 2t \geqq 1$$

for $a \in A$, $b \in B$, $\gamma \in Y^I$, $0 \leqq s \leqq 1$, $0 \leqq t \leqq 1$, $\tau \in EY$.

Proof. In the following diagram, the squares are homotopy-commutative:

$$\begin{tikzcd}
S(E_f^- * E_g) \arrow[r, "SF"] \arrow[ddr, "Sj"'] & SE_{\xi'} \arrow[r] \arrow[d, "\Sigma"] & SE_{\eta'} \arrow[d, "\Sigma_1"] \\
& C_{\xi'} \arrow[r, "\zeta'"] \arrow[d, "\partial_1"] & C_{\eta'} \arrow[d, "\partial"] \\
& SC_{P_1, P_2} \arrow[r, "SQ"] & S^2 E_{f,g}.
\end{tikzcd}$$

Since $\partial_1 \circ \Sigma \circ SF$ is given, explicitly, by

$$((1-s)(a, \alpha) \oplus s(\beta, b), t) \to \begin{cases} (a, \alpha+\beta, b, s\ ;\ 2t) & 0 \leqq 2t \leqq 1 \\ * & 1 \leqq 2t \leqq 2, \end{cases}$$

we see that homotopy-commutativity holds in the left triangle by (3.6). From $\mathscr{H} = Q \circ j$, follows the conclusion of the lemma.

Let $e : C_{f\nabla g} \to C_{\eta'}$ and $e' : Y \to C_{\xi'}$ denote canonical embeddings. Combining Lemmas 5.6, 5.7 with Puppe's sequence associated with η', we obtain the following reuslt.

THEOREM 5.8. *If* f, g *and* Y *are respectively*, p-, q- *and* r-*connected, and if* A, B *and* Y *have the homotopy type of CW-complexes, then for any* 1-*connected space* V *such that* $\pi_i(V) = 0$ *for* $i \geqq p+q+r+2$, *the following sequence is exact:*

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$$\pi(SE_{f,g},\ V)\xleftarrow{\mathscr{E}^*}\pi(C_{f\nabla g},\ V)\xleftarrow{\mathscr{Q}^*}\pi(S(E_f^- * E_g),\ V)\xleftarrow{(S\mathscr{A})^*}\pi(S^2E_{f,g},\ V)$$
$$\xleftarrow{(S\mathscr{E})^*}\pi(SC_{f\nabla g},\ V)\xleftarrow{(S\mathscr{Q})^*}\pi(S^2(E_f^- * E_g),\ V)\longleftarrow\cdots,$$

where $\mathscr{E}^*$ *is* $(\eta')^*$ *and* $\mathscr{Q}^*$ *denotes* $e^*\circ(\zeta'\circ\Sigma\circ SF)^{*-1}$. *Further, if* $\pi_i(V)=0$ *for* $i\geqq p+q+r+3$, *then the sequence*

$$\pi(E_f^- * E_g,\ \Omega V)\xleftarrow{\mathscr{A}^*}\pi(SE_{f,g},\ \Omega V)\xleftarrow{\mathscr{E}^*}\pi(C_{f\nabla g},\ \Omega V)\xleftarrow{\mathscr{Q}^*}\cdots$$

is exact.

Note that $\mathscr{Q}^*(\rho_1)=\mathscr{Q}^*(\rho_2)$ for $\rho_1,\ \rho_2\in\pi(S(E_f^- * E_g),\ V)$ if and only if $\rho_2=\mathscr{A}^*(\tau)+\rho_1$ for some $\tau\in\pi(S^2E_{f,g},\ V)$.

As an application of Theorem 5.8 we get.

PROPOSITION 5.9. *Let* A *and* B *be, respectively,* p- *and* q-*connected. Then the map* $\Lambda:\Omega A*\Omega B\to S(B\hat{*}A)$ *defined by*

$$\Lambda((1-t)\alpha\oplus t\beta)=(\alpha+\beta,\ t),$$

is $(p+q+\min(p,\ q))$-*connected.*

Proof. Consider the triad $B\xrightarrow{i_1}B\vee A\xleftarrow{i_2}A$. It follows from the theorem of Blakers-Massey that the maps $\Phi_1':\Omega A\to E_{i_1}^-$, $\Phi_2':\Omega B\to E_{i_2}$ are $(p+q-1)$-connected (cf. Theorem 4.6), where Φ_1', Φ_2' are both induced by the commutative diagram

$$\begin{tikzcd}
 & B \arrow[dr, "i_1"] & \\
* \arrow[ur] \arrow[dr] & & B\vee A. \\
 & A \arrow[ur, "i_2"'] &
\end{tikzcd}$$

Since $B\vee A$ is $\min(p,\ q)$-connected and $C_{i_1\nabla i_2}$ is contractible, it follows from Theorem 5.8 that $\mathscr{A}:E_{i_1}^- * E_{i_2}\to SE_{i_1,i_2}=S(B\hat{*}A)$ is $(p+q+\min(p,\ q)+1)$-connected.

We see that the composite

$$\Omega A*\Omega B\xrightarrow{\Phi_1'*\Phi_2'}E_{i_1}^- * E_{i_2}\xrightarrow{\mathscr{A}}S(B\hat{*}A)$$

is just Λ. This completes the proof, noticing that $\Phi_1'*\Phi_2'$ is $(p+q+\min(p,\ q))$-connected.

The above proposition enables us to obtain the following result mentioned at the end of § 2.

COROLLARY 5.10. *$W' : A \sharp B \to SA \hat{*} SB$, as defined in* (2.9), *is* $(p+q+\min(p, q)+2)$*-connected, if A and B are respectively p- and q-connected.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
A*B \arrow[r, "Q"] \arrow[d, "T"] & S(A\times B) \arrow[r, "SK"] & S(A\sharp B) \arrow[r, "SW'"] & S(SA\hat{*}SB) \arrow[dd, "T'"] \\
B*A \arrow[d, "\sigma_B*\sigma_A"] & & & \\
\Omega SB*\Omega SA \arrow[rrr, "\Lambda"] & & & S(SA\hat{*}SB),
\end{tikzcd}$$

in which T is the switching map, Λ the map as defined in Proposition 5.9, T' the involution resulting from inversing suspension parameter, and σ_A, σ_B are defined by $\sigma_A(a)=\hat{a}$, $\sigma_B(b)=-\hat{b}$. Since $\sigma_B*\sigma_A$ is $(p+q+\min(p, q)+3)$-connected and $SK\circ Q$ is a weak equivalence by Proposition 2.2', we get the desired conclusion.

LEMMA 5.11. *Let $\varepsilon : Y\to\Omega SY$ denote the canonical embedding, $\varepsilon(y)=\hat{y}$. Let $W' : C_g\sharp C_f\to SC_g\hat{*}SC_f$ be the map described in* (2.9). *Then the homotopy class of the composition*

$$Y\xrightarrow{e'}C_{\xi'}\xrightarrow{\theta}C_g\sharp C_f\xrightarrow{W'}SC_g\hat{*}SC_f$$

coincides with the cojoin product $\langle(\Omega Sk_2)\circ\varepsilon, (\Omega Sk_1)\circ\varepsilon\rangle$, where $k_1 : Y\to C_f$, $k_2 : Y\to C_g$ are inclusions.

Proof. This follows from

$$\begin{aligned}
[(W'\circ\theta\circ e')(y)](t) &= [(W'\circ K\circ(k_2\times k_1)\circ\Delta)(y)](t) \\
&= \begin{cases} (y, 1-2t)\in SC_f & 0\leqq 2t\leqq 1, \\ (y, 2t-1)\in SC_g & 1\leqq 2t\leqq 2. \end{cases}
\end{aligned}$$

With the above preliminaries, we can establish the dual EHP sequence for a triad $A\xrightarrow{f}Y\xleftarrow{g}B$.

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THEOREM 5.12. *Let $A \xrightarrow{f} Y \xleftarrow{g} B$ be a triad in which A, B, Y have the homotopy type of CW-complexes. If f, g and Y are respectively p-, q- and r-connected, then the diagram*

$$\begin{tikzcd}
\pi(SE_{f,g}, V) \arrow[d, "Q^*"] & \pi(C_{f\vee g}, V) \arrow[l, "\mathscr{E}^*"'] \arrow[d, "k^*"] & \pi(S(E_f * E_g), V) \arrow[l, "\mathscr{Q}^*"'] \arrow[d, "R^*"] & \pi(S^2E_{f,g}, V) \arrow[l, "(S\mathscr{K})^*"'] \arrow[d, "(SQ)^*"] \\
\pi(C_{P_1,P_2}, V) & \pi(Y, V) \arrow[l, "\xi'^*"'] & \pi(SC_g \hat{*} SC_f, V) \arrow[l, "\mathscr{P}^*"'] & \pi(SC_{P_1,P_2}, V) \arrow[l, "d^*"']
\end{tikzcd}$$

commutes and exact rows for 1*-connected space V such that $\pi_i(V) = 0$ for $i \geqq p + q + \min(p, q, r) + 2$, where $\mathscr{P}^*$ is the map induced by $\langle (\Omega Sk_2) \circ \varepsilon, (\Omega Sk_1) \circ \varepsilon \rangle$ and $R^* = (W' \circ \theta \circ \Sigma \circ SF)^{*-1}$ is bijective.*

Proof. Note that $W' : C_g \# C_f \to SC_g \hat{*} SC_f$ is $(p+q+\min(p, q)+2)$-connected. Then we see that the theorem follows from (3.2), Lemmas 5 6, 5.11.

COROLLARY 5.13. *If Y is r-connected, then, for a* 1*-connected space V such that $\pi_i(V) = 0$ for $i \geqq 3r+2$, we have an exact sequence:*

$$\begin{tikzcd}
\pi(S\Omega Y, V) & \pi(Y, V) \arrow[l, "\mathscr{E}^*"'] & \pi(S(\Omega Y * \Omega Y), V) \arrow[l] \arrow[d, "R^*"', "\approx"] & \pi(S^2\Omega Y, V) \arrow[l, "(S\mathscr{K})^*"'] \\
 & & \pi(SY \hat{*} SY, V) \arrow[ul, "\mathscr{P}^* = \langle \varepsilon, \varepsilon \rangle^*"] \arrow[d, "W'^*"', "\approx"] & \\
 & & \pi(Y \# Y, V). &
\end{tikzcd}$$

This follows by applying Theorem 5.12 to the triad $* \to Y \leftarrow *$.

In case where V is the Eilenberg-MacLane space in Corollary 5.13, $\mathscr{P}$ can be described in terms of cup-products in the light of Lemma 2.5 and Proposition 2.6.

Finally, we shall furnish $\mathscr{E}^*$ with some meaning. Consider the situation (3.1). Let $v : C_{f\vee g} \to V$ be given and write $u : Y \to V$ for the composite $Y \xrightarrow{k} C_{f\vee g} \xrightarrow{v} V$. v gives rise to liftings $\tilde{f} : A \to E_u$, $\tilde{g} : B \to E_u$. Let us denote the action of ΩV on E_u by $m : \Omega V \times E_u \to E_u$. Then we get.

PROPOSITION 5.14. *Let τ denote the adjoint of $\eta'^*(v)$. Then*

$$m_*\{\tau, P_2^*(\tilde{g})\} = P_1^*(\tilde{f}).$$

Moreover, given $h: K \to A$, $k: K \to B$ with $f \circ h \simeq g \circ k$, we can find $l: K \to E_{f,g}$ such that $P_1 \circ l \simeq h$, $P_2 \circ l \simeq k$. We see easily that the composite

$$SK \xrightarrow{Sl} SE_{f,g} \xrightarrow{\eta'} C_{f \nabla g} \longrightarrow SA \vee SB,$$

where the last arrow is the identification map resulting by shrinking Y to a point, is homotopic to the difference $j_1 \circ (Sh) - j_2 \circ (Sk)$, where $j_1: SA \to SA \vee SB$, $j_2: SB \to SA \vee SB$ are inclusions. Thus, in case K is a suspension, $v \circ \eta' \circ (Sl)$ represents the generalized Toda bracket $\left\{ u \begin{smallmatrix} f & h \\ g - k \end{smallmatrix} \right\}$ (see [5]).

Further, we assume $f \circ h \simeq g \circ k \simeq 0$. Then h, k can be lifted to $\tilde{h}: K \to E_f^-$, $\tilde{k}: K \to E_g$. We may choose the composite

$$K \xrightarrow{\{\tilde{h}, \tilde{k}\}} E_f^- \times E_g \xrightarrow{\mu} E_{f,g}$$

for l. As $\eta' \circ \mathscr{A} \simeq 0$, $v \circ \eta'$ is primitive with respect to μ. Therefore we get

$$v \circ \eta' \circ (Sl) \simeq v \circ \eta' \circ (S\Pi_1) \circ S\{\tilde{h}, \tilde{k}\} + v \circ \eta' \circ (S\Pi_2) \circ S\{\tilde{h}, \tilde{k}\}.$$

A simple calculation shows

PROPOSITION 5.15. $v \circ \eta' \circ (Sl): SK \to V$ *represents the difference* $-u_f(h) + u_g(k)$ *of functional u-operations.*

§ 6. The EHP sequence

This section studies the situation dual to that considered in § 5. Namely, by generalizing a result of Ganea [8] to a cotriad, we will regain "symmetry".

Let $A \xleftarrow{f} X \xrightarrow{g} B$ be a cotriad and consider the associated diagram (4.1). The notations of § 4 will be used without specific mention.

First, we try to seek an approximation to the fibre and to the cofibre of ξ. Introduce the diagram

$$\text{(6.1)} \qquad \begin{tikzcd} E_g^- \flat E_f \arrow[r, "L"] \arrow[dd, "\nu"] & E_g^- \vee E_f \arrow[r, "J"] \arrow[d, "q_2 \vee q_1"] & E_g^- \times E_f \arrow[r, "K"] \arrow[d, "\Phi_1' \times \Phi_2'"] & E_g^- \# E_f \arrow[dd, "\rho"] \\ & X \vee X \arrow[d, "\nabla"] & E_{l_1}^- \times E_{l_2} \arrow[d, "\mu"] & \\ E_\xi \arrow[r] & X \arrow[r, "\xi"] & E_{l_1, l_2} \arrow[r, "i"] & C_\xi \end{tikzcd}$$

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in which μ is the "multiplication" defined at the beginning of §5, ∇ the folding map, Φ_1' and Φ_2' the maps as defined in Theorem 4.6, and q_2, q_1 are the projections. It is easily seen that the middle square homotopy-commutes, and hence induces the maps ρ, ν.

THEOREM 6.2. *Let f, g be respectively p-, q-connected and let X be r-connected. Then ρ is $[p+q+\min(p, q, r+1)-1]$-connected and ν is $[p+q+\min(p, q, r+1)-2]$-connected.*

Proof. Apply the suspension functor to the right square and then augment as follows:

$$
\begin{tikzcd}
E_g^- * E_f \arrow[r, "Q_1"] \arrow[d, "\Phi_1' * \Phi_2'"] & S(E_g^- \times E_f) \arrow[r, "SK"] \arrow[d, "S(\Phi_1' \times \Phi_2')"] & S(E_g^- \# E_f) \arrow[dd, "S\rho"] \\
E_{I_1}^- * E_{I_2} \arrow[r, "Q_2"] & S(E_{I_1}^- \times E_{I_2}) & \\
SX \arrow[r, "S\xi"] & SE_{I_1, I_2} \arrow[r, "Si"] & SC_\xi \\
C_{f,g} \arrow[u, "Q"] \arrow[r, "\gamma"] & C_{p_1, p_2} \arrow[u, "Q_3"] \arrow[r, "h"] & C_\gamma \arrow[u, "l"]
\end{tikzcd}
\tag{6.3}
$$

in which $p_1 : E_{I_1, I_2} \to A$, $p_2 : E_{I_1, I_2} \to B$ are projections, γ the map determined by the commutative diagram:

$$
\begin{tikzcd}
A \arrow[d, equal] & X \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "\xi"] & B \arrow[d, equal] \\
A & E_{I_1, I_2} \arrow[l, "p_1"'] \arrow[r, "p_2"] & B,
\end{tikzcd}
$$

and l the map induced by the identification maps Q, Q_3. It follows from 1.3′ that l is a weak homotopy equivalence, since C_γ is homotopy-equivalent to the mapping cylinder of a cotriad $* \leftarrow C_\xi \to *$. Also, by Theorems 4.3 and 4.6, $\Phi_1' * \Phi_2'$ is $(p+q+\min(p, q))$-connected.

Define a map $\xi' : C_{p_1, p_2} \to C_{f,g}$ as the canonical extension of a triad $A \xrightarrow{I_1} C_{f,g} \xleftarrow{I_2} B$ (see §3). We see that $\xi' \circ \gamma = \text{identity}$. Since the fibre of ξ' is equivalent to $E_{l_1}^- * E_{l_2}$, by Theorem 3.4, we get the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H_k(E_{l_1}^- * E_{l_2}) \arrow[r, "j_*"] & H_k(C_{p_1,p_2}) \arrow[r, "\xi'_*"] \arrow[d, equal] & H_k(C_{f,g}) \arrow[r] & 0 \\ 0 & H_k(C_\gamma) \arrow[l] & H_k(C_{p_1,p_2}) \arrow[l, "h_*"'] & H_k(G_{f,g}) \arrow[l, "\gamma_*"'] \arrow[u] & 0, \arrow[l] \end{tikzcd}$$

in which the rows are exact for $k \leqq p+q+\min(p, q, r+1)+1$. Chasing this diagram, we conclude thet $h \circ j$ is $(p+q+\min(p, q, r+1)+1)$-connected.

Now, since $Q_3 \circ j = S\mu \circ Q_2$ by (3.6), homotopy-commutativity of (6.3) implies

$$\begin{aligned} S\rho \circ SK \circ Q_1 &\simeq Si \circ S\mu \circ Q_2 \circ (\Phi_1' * \Phi_2) \\ &= Si \circ Q_3 \circ j \circ (\Phi_1' * \Phi_2') \simeq l \circ h \circ j \circ (\Phi_1' * \Phi_2'). \end{aligned}$$

Upon noticing that $SK \circ Q_1$ is a weak equivalence by Proposition 2.2′, we infer that $S\rho$ is $[p+q+\min(p, q, r+1)]$-connected.

Finally, the connectivity of ν follows from the homotopy-commutative diagram

$$\begin{tikzcd} S(E_g^- \flat E_f) \arrow[r, "S\nu"] \arrow[d] & SE_\xi \arrow[d] \\ E_g^- \# E_f \arrow[r, "\rho"] & C_\xi \end{tikzcd}$$

where the vertical maps are "suspension maps", the left of which is $[p+q+\min(p, q)]$-connected, whereas, the right is $[p+q+\min(r, p+q-1)]$-connected.

q.e.d.

Next, using the map $\mu' : C_{f,g} \to C_f \vee C_g$ which results from shrinking the center of cylinder to a point, we define the *Hopf invariant*

$$H : \Omega C_{f,g} \to C_f \hat{*} C_g$$

associated with a cotriad f, g as the composition

$$\Omega C_{f,g} \xrightarrow{\Omega\mu'} \Omega(C_f \vee C_g) \xrightarrow{I} C_f \hat{*} C_g.$$

The following is dual to Lemma 5.1.

LEMMA 6.4. *Let $r_1 : C_{f,g} \to C_f \vee C_g$, $r_2 : C_{f,g} \to C_f \vee C_g$ be the "injections" which are respectively the compositions of $C_{f,g} \to C_f$, C_g (projections) with C_f,*

$C_g \to C_f \vee C_g$. *Then* $H_*(\tau) = 0$ *for* $\tau \in \pi(V, \Omega C_{f,g})$ *if and only if the equality* $(\Omega_{\mu'})_*(\tau) = (\Omega r_1)_*(\tau) + (\Omega r_2)_*(\tau)$ *holds.*

Now we shall define $F' : C_\xi \to C_f \hat{*} C_g$, dual to the map F defined in (3.5). Put

$$F'(x, s) = (\mu' x)_{\frac{1-s}{2}, \frac{1+s}{2}} = -\hat{x}_{0,s} + \hat{x}_{0,s} \qquad x \in X,\ 0 \leqq s \leqq 1,$$
$$F'(\beta) = \mu'\beta \qquad \beta \in E_{l_1, l_2} \subset (C_{f,g})^I,$$

where $-\hat{x}_{0,s} \in (C_f)^I$, $\hat{x}_{0,s} \in (C_g)^I$. This corresponds to the map $\mathscr{T}$ defined by Ganea [8]. We see easily that the following diagram is commutative:

$$\begin{tikzcd} \Omega C_{f,g} \arrow[r, "I"] \arrow[d, "H"'] & E_{l_1, l_2} \arrow[d, "k"] \\ C_f \hat{*} C_g & C_\xi . \arrow[l, "F'"'] \end{tikzcd}$$

Observe that it seems difficult to define a dual of G' given in Theorem 3.4.

LEMMA 6.5. *The composite map*

$$\Omega^2 C_{f,g} \xrightarrow{\text{injection}} E_\eta \xrightarrow{\zeta} E_\xi \xrightarrow{\bar{\sigma}} \Omega C_\xi \xrightarrow{\Omega F'} \Omega(C_f \hat{*} C_g)$$

is homotopic to ΩH, *where* $\bar{\sigma}$ *is the suspension map.*

LEMMA 6.6. *The diagram homotopy-commutes:*

$$\begin{tikzcd} E_g^- \# E_f \arrow[r, "\rho"] \arrow[d, "W'"'] & C_\xi \arrow[dd, "F'"] \\ SE_g^- \hat{*} SE_f \arrow[d, "\omega"'] & \\ SE_f \hat{*} SE_g^- \arrow[r, "\sigma_1 \hat{*} \sigma_2"] & C_f \hat{*} C_g , \end{tikzcd}$$

where ω *is the involution switching factors and* σ_1, σ_2 *are given as follows:*

$$\sigma_1(\alpha, x ; s) = \begin{cases} \alpha(2s) & 2s \leqq 1 \\ (x, 2-2s) & 2s \geqq 1 \end{cases}$$

$$\sigma_2(x, \beta ; s) = \begin{cases} \beta(1-2s) & 2s \leqq 1 \\ (x, 2-2s) & 2s \geqq 1. \end{cases}$$

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Hence, if f, g and X are respectively p-, q- and r-connected, then F' is $[p+q+\min(r+1, p, q)-1]$*-connected.*

Proof. This follows by combining the following facts:

ρ is $[p+q+\min(p, q, r+1)-1]$-connected by Theorem 6.2,
W' is $[p+q+\min(p, q)-1]$-connected by Corollary 5.10,
σ_1, σ_2 are respectively $[p+\min(p, r)+1]$-, $[q+\min(q, r)+1]$-connected by Lemma 1.5.

LEMMA 6.7. *Let $l_1 : S\Omega E_f \to X$, $l_2 : S\Omega E_g^- \to X$ be respectively the composite maps of canonical ones:*

$$S\Omega E_f \longrightarrow E_f \xrightarrow{q_1} X, \qquad S\Omega E_g^- \longrightarrow E_g^- \xrightarrow{q_2} X.$$

Then the homotopy class of the composition

$$\Omega E_f * \Omega E_g^- \xrightarrow{t*t} \Omega E_f * \Omega E_g^- \xrightarrow{W} E_g^- \flat E_f \xrightarrow{\nu} E_\xi \xrightarrow{\text{projection}} X$$

coincides with the generalized Whitehead product $[l_1, l_2]$, where t denote inversions.

This follows from the fact that the above composition is equal to $\nabla \circ (q_2 \vee q_1) \circ L \circ W \circ (t*t)$.

Combining Lemmas 6.5, 6.6, 6.7 with Theorem 6.2 and noting that $\bar{\sigma}$ is $(p+q+r-1)$-connected, we get

THEOREM 6.8. *Let f, g and X be p-, q- and r-connected respectively, and let k be a positive integer. Then, for any CW-complex K with* $\dim K + k \leqq p+q+\min(r+1, p, q)-3$, *we have the following exact sequence*

$$\begin{array}{ccccc}
\pi(K, \Omega^{k+1}E_{f\Delta g}) \to \cdots & \pi(K, \Omega E_{f\Delta g}) & \xrightarrow{(\Omega\eta)_*} & \pi(K, \Omega^2 C_{f,g}) & \xrightarrow{(\Omega H)_*} \\
\downarrow & \downarrow & & \downarrow{\scriptstyle (\Omega I)_*} & \\
\pi(K, \Omega^{k+1}X) \longrightarrow \cdots & \pi(K, \Omega X) & \xrightarrow{(\Omega\xi)_*} & \pi(K, \Omega E_{l_1, l_2}) & \longrightarrow
\end{array}$$

$$\begin{array}{ccccc}
\pi(K, \Omega(C_f \hat{*} C_g)) & \longrightarrow & \pi(K, E_{f\Delta g}) & \xrightarrow{\eta_*} & \pi(K, \Omega C_{f,g}) \\
{\scriptstyle R}\downarrow\approx & & \downarrow & & {\scriptstyle I_*}\downarrow \\
\pi(K, \Omega E_f * \Omega E_g^-) & \xrightarrow{P_*} & \pi(K, X) & \xrightarrow{\xi_*} & \pi(K, E_{l_1, l_2}),
\end{array}$$

in which P_* is the map induced by $[l_1, l_2]$ and R the bijection $(t*t)_* \circ (\Omega F' \circ \bar{\sigma} \circ \nu \circ W)_*^{-1}$.

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