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MILDLY DISTRIBUTIVE SEMILATTICES

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Abstract

There is no single generalization of distributivity to semilattices. This paper investigates the class of mildly distributive semilattices, which lies between the two most commonly discussed classes in this area—weakly distributive semilattices and distributive semilattices. Particular attention is paid to describing and characterizing congruence distributive mildly distributive semilattices, in contrast to distributive semilattices, whose lattice of join partial congruences is badly behaved and which are difficult to describe.

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1. Introduction

Between the classes of weakly distributive semilattices and distributive semilattices lies the class of *mildly distributive* semilattice. These semilattices are first defined by the distributivity of the lattice of all *strong ideals*, and the second section goes on to find a sequence of first order sentences which defines mild distributivity. Section 3 shows how to construct mildly distributive semilattices, and more importantly, how they can be characterized as weakly distributive semilattices in which each *ideal* is a strong ideal. It is this characterization which motivates the rest of the paper, since the correct notion of morphism and congruence for mildly distributive semilattices are *join partial homomorphism* and *join partial congruence*; that is the appropriate notions for weakly distributive semilattices.

Sections 6, 7 and 8 investigate the question: "When is the lattice of join partial congruences of a weakly distributive semilattice distributive?" This problem is solved for *finitely derivable* semilattices, that is those semilattices which have only a finite number of non-principal finitely generated ideals, and has a particularly pleasing solution for mildly distributive semilattices.

The investigation into the lattice of join partial congruences is continued in the final section which looks at the restriction map from the lattice of join partial congruences of a weakly distributive semilattice to the lattice of join partial congruences of a principal filter. For finitely derivable semilattices, this map is a lattice homomorphism if and only if the semilattice is *strongly connected*, a condition which has earlier been shown to imply congruence distributivity.

2. Definitions

Let S be a (lower) semilattice. For a subset A of S let $A^{\uparrow}$ denote the set of all upper bounds of A, and let $A^{\downarrow}$ be the set of all lower bounds of A. A non-empty subset I of S is called a *strong ideal* if for every finite set $F \subseteq I$,

$$F^{\uparrow\downarrow} \subseteq I.$$

A strong ideal is *hereditary*; that is $x \in I$ and $y \leqslant x$ implies $y \in I$. The concept of strong ideal was introduced by O. Frink in [6]; see also C. M. de Barros [3]. In both cases the emphasis was on partially ordered sets rather than semilattices.

For each $a \in S$, let

$$(a] = \{x \in S \colon x \leqslant a\} = \{a\}^{\downarrow} \quad \text{and}$$
$$[a) = \{x \in S \colon x \geqslant a\} = \{a\}^{\uparrow}.$$

Thus $(a]$ is a strong ideal for each $a \in S$ and I is a strong ideal if and only if for each finite set $\{a_1,\dots,a_n\} \subseteq I$,

$$[x) \supseteq [a) \cap \cdots \cap [a) \quad \text{implies } x \in I.$$

If S is a lattice, the strong ideals are precisely the lattice ideals of S. If ${}^{\omega}\mathcal{I}(S)$ denotes the set of all strong ideals of S, then ${}^{\omega}\mathcal{I}(S) \cup \{\phi\}$ is an algebraic closure system on S, and so it is meaningful to talk about the strong ideal generated by a non-empty set A. It will be denoted by $\langle A \rangle$ or $\langle a_1,\dots,a_n \rangle$ if $A = \{a_1,\dots,a_n\}$ is finite. Clearly $\langle a \rangle = (a]$. Strong ideals of this last type are called *principal*.

A semilattice S is called *mildly distributive* if ${}^{\omega}\mathcal{I}(S)$ is a distributive lattice. Our first objective is to describe mildly distributive semilattices in first order terms.

LEMMA 2.1. *If $a_1,\ldots,a_n$ are elements of a semilattice S then $\langle a_1,\ldots,a_n\rangle = \{x \in S\colon [x) \supseteq [a_1) \cap \cdots \cap [a_n)\}$.*

PROOF. It suffices to show that $\{x \in S\colon [x) \supseteq [a_1) \cap \cdots \cap [a_n)\}$ is a strong ideal, so suppose $x_1,\ldots,x_r$ are in this set and $[y) \supseteq [x_1) \cap \cdots \cap [x_r)$. Since $[x_i) \supseteq [a_1) \cap \cdots \cap [a_n)$ we have $[y) \supseteq [x_1) \cap \cdots \cap [x_r) \supseteq [a_1) \cap \cdots \cap [a_n)$ and so y is in the set.

A non-empty subset F of a semilattice S is called a *filter* if $a \wedge b \in F$ if and only if $a \in F$ and $b \in F$. The set of all filters of S is written $\mathcal{F}(S)$. If S is directed above then $\mathcal{F}(S)$ is a lattice, and if not then $\mathcal{F}(S) \cup \{\varnothing\}$ is a lattice. Clearly $[a)$ is a filter for each $a \in S$; such filters are called *principal*. Let

$$G(S) = \{[a_1) \cap \cdots \cap [a_n)\colon a_1,\ldots,a_n \in S\}.$$

If S is not directed above then $\phi \in \mathcal{G}(S)$ and $\mathcal{G}(S)$ is a lower subsemilattice of $\mathcal{F}(S) \cup \{\varnothing\}$; otherwise $\mathcal{G}(S)$ is a lower subsemilattice of $\mathcal{F}(S)$.

The set of all strong ideals that are generated by non-empty finite sets is denoted by ${}^{\omega}\mathcal{J}_f(S)$. If $+$ denotes the join in the lattice ${}^{\omega}\mathcal{J}(S)$ then

$$\langle a_1,\ldots,a_n\rangle + \langle b_1,\ldots,b_m\rangle = \langle a_1,\ldots,a_n, b_1,\ldots,b_m\rangle$$

so that ${}^{\omega}\mathcal{J}_f(S)$ is an upper subsemilattice of ${}^{\omega}\mathcal{J}(S)$.

PROPOSITION 2.2. *For any semilattice S, $G(S)$ is anti-isomorphic to ${}^{\omega}\mathcal{J}_f(S)$.*

PROOF. Define a map $\psi\colon {}^{\omega}\mathcal{J}_f(S) \to \mathcal{G}(S)$ by

$$\psi(\langle a_1,\ldots,a_n\rangle) = [a_1) \cap \cdots \cap [a_n).$$

To see that ψ is well defined, suppose $\langle a_1,\ldots,a_n\rangle = \langle b_1,\ldots,b_m\rangle$. Then, for each i, $a_i \in \langle b_1,\ldots,b_m\rangle$ whence $[a_i) \supseteq [b_1) \cap \cdots \cap [b_m)$ and $[a_1) \cap \cdots \cap [a_n) \supseteq [b_1) \cap \cdots \cap [b_m)$. Since the reverse inequality may be similarly obtained, $\psi(\langle a_1,\ldots,a_n\rangle) = \psi(\langle b_1,\ldots,b_m\rangle)$ and ψ is well defined. Clearly

$$\psi(\langle a_1,\ldots,a_n\rangle + \langle b_1,\ldots,b_m\rangle) = \psi(\langle a_1,\ldots,a_n\rangle) \cap \psi(\langle b_1,\ldots,b_m\rangle)$$

and ψ is obviously onto. Finally ψ is one to one by a direct application of Lemma 2.1.

PROPOSITION 2.3. *The following are equivalent for a semilattice S.*
(i) *S is mildly distributive.*
(ii) *${}^{\omega}\mathcal{J}_f(S)$ is a distributive sublattice of ${}^{\omega}\mathcal{J}(S)$.*
(iii) *${}^{\omega}\mathcal{J}_f(S)$ is a distributive lattice.*

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PROOF. (i) $\Rightarrow$ (ii). Suppose $\langle a_1,\ldots,a_n\rangle$ and $\langle b_1,\ldots,b_m\rangle$ are finitely generated strong ideals. As ${}^\omega\mathcal{J}(S)$ is distributive,

$$\begin{aligned}\langle a_1,\ldots,a_n\rangle\cap\langle b_1,\ldots,b_m\rangle &= ((a_1]+\cdots+(a_n])\cap((b_1]+\cdots+(b_m])\\ &\quad\times((a_1]\cap(b_1])+\cdots+((a_1]\cap(b_m])+\cdots+((a_n]\cap(b_m])\\ &=(a_1\wedge b_1]+\cdots+(a_n\wedge b_m]\\ &=\langle a_1\wedge b_1,\ldots,a_n\wedge b_m\rangle\end{aligned}$$

and so ${}^\omega\mathcal{J}_f(S)$ is a (distributive) sublattice of ${}^\omega\mathcal{J}(S)$.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i). It is well known that if the compact elements of an algebraic lattice are a distributive lattice then the algebraic lattice is distributive. In our case ${}^\omega\mathcal{J}_f(S)\cup\{\varnothing\}$ are the compact elements of ${}^\omega\mathcal{J}(S)\cup\{\varnothing\}$, so this last lattice is distributive and hence ${}^\omega\mathcal{J}(S)$ is distributive.

COROLLARY 2.4. *A semilattice S is mildly distributive if and only if $G(S)$ is a distributive lattice.*

Let $\overset{*}{\vee}$ denote the join in $\mathcal{G}(S)$ (when it exists).

THEOREM 2.5. *A semilattice S is mildly distributive if and only if for each n the following formula is valid in S.*

$$\begin{aligned}\forall x\forall a_1,\ldots,a_n\forall b([b\geqslant a_1,\ldots,a_n\Rightarrow b\geqslant x]\\ \Rightarrow x=(x\wedge a_1)\vee(x\wedge a_2)\vee\cdots\vee(x\wedge a_n)).\end{aligned}$$

(In words, this says if x is below every upper bound of $\{a_1,\ldots,a_n\}$ then $x=(x\wedge a_1)\vee\cdots\vee(x\wedge a_n)$.)

PROOF. Suppose S is mildly distributive and x is below every upper bound of $\{a_1,\ldots,a_n\}$. Then $[x)\supseteq[a_1)\cap\cdots\cap[a_n)$ and so

$$\begin{aligned}[x)&=[x)\overset{*}{\vee}([a_1)\cap\cdots\cap[a_n))=\left([x)\overset{*}{\vee}[a_1)\right)\cap\cdots\cap\left([x)\overset{*}{\vee}[a_n)\right)\\ &=[x\wedge a_1)\cap\cdots\cap[x\wedge a_n)\end{aligned}$$

whence $x=(x\wedge a_1)\vee\cdots\vee(x\wedge a_n)$.

Conversely, suppose S satisfies the given formula for each n. First observe that if $a_1\vee\cdots\vee a_n$ exists in S and x is arbitrary, then

(1) $$x\wedge(a_1\vee\cdots\vee a_n)=(x\wedge a_1)\vee\cdots\vee(x\wedge a_n).$$

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Indeed let $a = a_1 \vee \cdots \vee a_n$. Then

$$x \wedge (a_1 \vee \cdots \vee a_n) = (x \wedge a) \wedge (a_1 \vee \cdots \vee a_n)$$
$$= ((x \wedge a) \wedge a_1) \vee \cdots \vee ((x \wedge a) \wedge a_n) \quad \text{(by our assumption)}$$
$$= (x \wedge a_1) \vee \cdots \vee (x \wedge a_n).$$

Assume $x \in \langle a_1, \ldots, a_n \rangle \cap \langle b_1, \ldots, b_m \rangle$. By Lemma 2.1 $x = (x \wedge a_1) \vee \cdots \vee (x \wedge a_n)$ and $x = (x \wedge b_1) \vee \cdots \vee (x \wedge b_m)$. Taking the meet of these two expressions and applying (1), we have

$$x = (x \wedge a_1 \wedge b_1) \vee (x \wedge a_1 \wedge b_2) \vee \cdots \vee (x \wedge a_n \wedge b_m)$$

since each of the elements in the join is in $\langle a_1 \wedge b_1, \ldots, a_n \wedge b_m \rangle$ and strong ideals are closed under finite joins. Hence

$$\langle a_1, \ldots, a_n \rangle \cap \langle b_1, \ldots, b_m \rangle = \langle a_1 \wedge b_1, \ldots, a_n \wedge b_m \rangle.$$

From here it is an easy matter to show that ${}^{\omega}\mathcal{J}_f(S)$ is a distributive lattice, and consequently S is mildly distributive.

3. Construction and comparisons

If P_1 and P_2 are disjoint partially ordered sets, then their *ordinal sum* $P_1 \oplus P_2$ is the set $P_1 \cup P_2$ with partial order $p \leqslant q$ if
(i) $p \in P_1$ and $q \in P_2$, or
(ii) $p, q \in P_i$, $i = 1$ or 2 and $p \leqslant q$ in P_i.

PROPOSITION 3.1. *The ordinal sum of two semilattices is mildly distributive if and only if both are mildly distributive.*

REMARK. If the two semilattices in question are not disjoint, take an isomorphic copy of one which is disjoint from the other.

PROOF. Suppose S_1 and S_2 are mildly distributive and $x, a_1, \ldots, a_n \in S_1 \oplus S_2$ are such that $[x) \supseteq [a_1) \cap \cdots \cap [a_n)$. If $\{x, a_1, \ldots, a_n\} \subseteq S_i$ for some i then $x = (x \wedge a_1) \vee \cdots \vee (x \wedge a_n)$ by the mild distributivity of S_i. If $x \in S_2$ and $\{a_1, \ldots, a_n\} \subseteq S_1$ and $a_j \in S_2$ for some $j = 1, \ldots, n$ then $x \wedge a_j = x$ and so $x = (x \wedge a_1) \vee \cdots \vee (x \wedge a_n)$. Finally if $x \in S_2$ and $a_j \in S_2$ for some j, let

$\{a_{j(1)},\ldots,a_{j(r)}\} = \{a_1,\ldots,a_n\} \cap S_2$. Then $[x) \supseteq [a_1) \cap \cdots \cap [a_n) = [a_{j(1)}) \cap \cdots \cap [a_{j(r)})$ and hence

$$x = (x \wedge a_{j(1)}) \vee \cdots \vee (x \wedge a_{j(r)}) = (x \wedge a_1) \vee \cdots \vee (x \wedge a_n).$$

Conversely, if $S_1 \oplus S_2$ is mildly distributive then S_1 and S_2 are mildly distributive, since for each i, S_i is a subsemilattice of $S_1 \oplus S_2$ and if $x, a_1,\ldots,a_n \in S_i$ then $[x) \supseteq [a_1) \cap \cdots \cap [a_n)$ in S_i if and only if $[x) \supseteq [a_1) \cap \cdots \cap [a_n)$ in $S_1 \oplus S_2$.

A semilattice S is said to have the *upper bound property* if whenever a and b have an upper bound in S then $a \vee b$ exists in S. Clearly S has the upper bound property if and only if $S \oplus \mathbf{1}$ is a lattice, where $\mathbf{1}$ is the one element lattice. Observe that each finite semilattice and each lattice has the upper bound property. A more comprehensive survey of semilattices with the upper bound property may be found in R. C. Hickman [10]. For our part we will just provide the following corollary to Proposition 3.1.

COROLLARY 3.2. *A semilattice S with the upper bound property is mildly distributive if and only if $S \oplus \mathbf{1}$ is a distributive lattice.*

THEOREM 3.3. *Let $\{S_i: i \in I\}$ be a family of mildly distributive semilattices, at least two of which are not the single element semilattice. Then the product $\times\{S_i: i \in I\}$ of the S_i's is mildly distributive if and only if, for each i, S_i is directed above.*

PROOF. Suppose each S_i is directed above and $x, a_1,\ldots,a_n \in \times\{S_i: i \in I\}$ are such that $[x) \supseteq [a_1) \cap \cdots \cap [a_n)$. Since each S_i is directed above, $[a_1(i)) \cap \cdots \cap [a_n(i))$ is non-empty for each i. Fix i and select, for each $j \neq i$, an element b_j in $[a_j(j)) \cap \cdots \cap [a_n(j))$. Then for each choice of $b_i \in [a_1(i)) \cap \cdots \cap [a_n(i))$, we have $x \leqslant b$ where $b: I \to \bigcup\{S_i: i \in I\}$ is defined by $b(k) = b_k$ for all $k \in I$. Hence $x(i) \leqslant b_i$ and so $[x(i)) \supseteq [a_1(i)) \cap \cdots \cap [a_r(i))$, whence

$$\begin{aligned} x(i) &= (x(i) \wedge a_1(i)) \vee \cdots \vee (x(i) \wedge a_n(i)) \\ &= ((x \wedge a_1) \vee \cdots \vee (x \wedge a_n))(i). \end{aligned}$$

Since the choice of i was arbitrary, we deduce that $x = (x \wedge a_1) \vee \cdots \vee (x \wedge a_n)$ and $\times\{S_i: i \in I\}$ is mildly distributive.

Now assume that S_i is not directed above and for some $j \neq i$, S_j is non-trivial. Choose $a_i^1, a_i^2 \in S_i$ with $[a_i^1) \cap [a_i^2) = \varnothing$ and $a_j^1, a_j^2 \in S_j$ with $a_j^1 < a_j^2$. For each $k \notin (i, j)$ select an arbitrary element a_k in S_k. Define $a^1, a^2 \in \times\{S_i: i \in I\}$ by

$$a^1(k) = \begin{cases} a_k & \text{if } k \notin \{i, j\}, \\ a_i^1 & \text{if } k = i, \\ a_j^1 & \text{if } k = j, \end{cases} \qquad a^2(k) = \begin{cases} a_k & \text{if } k \notin \{i, j\}, \\ a_i^2 & \text{if } k = i, \\ a_j^1 & \text{if } k = j. \end{cases}$$

Thus $a^1(k) = a^2(k)$ for all $k \neq i$ and $[a^1) \cap [a^2) = \varnothing$ since $[a^1(i)) \cap [a^2(i)) = \varnothing$. Hence $[x) \supseteq [a^1) \cap [a^2)$ for all x, and in particular when

$$x(k) = \begin{cases} a_k & \text{if } k \notin \{i, j\}, \\ a_i^1 & \text{if } k = i, \\ a_j^2 & \text{if } k = j. \end{cases}$$

But for this choice of x,

$$a_j^2 = x(t) \neq (x(j) \wedge a^1(j)) \vee (x(j) \wedge a^2(j)) = a_j^1$$

and so $\times\{S_i\colon i \in I\}$ is not mildly distributive.

These last two results do not provide many examples of mildly distributive semilattices at this stage, since we have, as yet, no example of an up directed mildly distributive semilattice which is not a lattice. However this situation will be rectified later in the paper.

A lower semilattice S is called *distributive* if for all $a, b, c \in S$, $a \geqslant b \wedge c$, $a \ngeqslant b$ and $a \ngeqslant c$ implies the existence of $b', c' \in S$ with $b' \geqslant b$, $c' \geqslant c$ and $a = b' \wedge c'$. This idea was introduced by T. Katriňák in [11], and is a slight generalization of the usual concept of distributive semilattice. Normally a lower semilattice is called "distributive" if it is distributive in our sense and directed above, or equivalently, if $a \geqslant b \wedge c'$. Semilattices of this last kind are explored in J. B. Rhodes [13].

LEMMA 3.4 (Katriňák [11, 1.6]). *A semilattice S is distributive if and only if $\mathcal{F}(S) \cup \{\varnothing\}$ is a distributive lattice.*

As the name suggests, distributive semilattices are mildly distributive, and it is possible to characterize those mildly distributive semilattices which are distributive.

THEOREM 3.5. *A semilattice S is distributive if and only if it is mildly distributive and $\mathcal{G}(S)$ is a sublattice of $\mathcal{F}(S) \cup \{\varnothing\}$.*

PROOF. Suppose S is distributive and $[x) \supseteq [a_1) \cap \cdots \cap [a_n)$. Then

$$\begin{aligned}[x) &= [w) \vee ([a_1) \cap \cdots \cap [a_n)) = ([x) \vee [a_1)) \cap \cdots \cap ([x) \vee [a_n)) \\ &= [x \wedge a_1) \cap \cdots \cap [x \wedge a_n),\end{aligned}$$

where $\vee$ denotes the join in $\mathcal{F}(S) \cup \{\varnothing\}$. Thus

$$x = (x \wedge a_1) \vee \cdots \vee (x \wedge a_n)$$

and S is mildly distributive. To show that $\mathcal{G}(S)$ is a sublattice of $\mathcal{F}(S) \cup \{\varnothing\}$, suppose $[a_1) \cap \cdots \cap [a_n)$ and $[b_1) \cap \cdots \cap [b_m)$ are in $\mathcal{G}(S)$. Then

$$\begin{aligned}
([a_1) \cap \cdots \cap [a_n)) \overset{*}{\vee} ([b_1) \cap \cdots \cap [b_m)) \\
&= \left([a_1) \overset{*}{\vee} [b_1)\right) \cap \cdots \cap \left([a_n) \overset{*}{\vee} [b_m)\right) \\
&= [a_1 \wedge b_1) \cap \cdots \cap [a_n \wedge b_m) \\
&= ([a_1) \cap \cdots \cap [a_n)) \vee ([b_1) \cap \cdots \cap [b_m)).
\end{aligned}$$

Conversely, suppose S is mildly distributive and $\mathcal{G}(S)$ is a sublattice of $\mathcal{F}(S) \cup \{\varnothing\}$. If $a \geqslant b \wedge c$ and $a \ngeqslant b$, $a \ngeqslant c$ then

$$\begin{aligned}
[a) = [a) \cap [b \wedge c) &= [a) \cap \left([b) \overset{*}{\vee} [c)\right) \\
&= ([a) \cap [b)) \overset{*}{\vee} ([a) \cap [c)) \\
&= ([a) \cap [b)) \vee ([a) \cap [c))
\end{aligned}$$

and so $a \geqslant b' \wedge c'$ for some $b' \in [a) \cap [b)$ and $c' \in [a) \cap [c)$. Since $b' \wedge c' \in [a)$, $b' \wedge c' \geqslant a$ and $a = b' \wedge c'$ as required.

COROLLARY 3.6. *A mildly distributive semilattice with the upper bound property is distributive.*

PROOF. If $[a_1) \cap \cdots \cap [a_n)$ is a non-empty member of $\mathcal{G}(S)$ then there exists $b \in S$ with $a_1, \ldots, a_n \leqslant b$. Since S has the upper bound property, this implies $a_1 \vee \cdots \vee a_n$ exists in S and so $[a_1) \cap \cdots \cap [a_n) = [a_1 \vee \cdots \vee a_n)$. Thus each non-empty member of $\mathcal{G}(S)$ is of the form $[a)$ for some $a \in S$, and consequently $\mathcal{G}(S)$ is a sublattice of $\mathcal{F}(S) \cup \{\varnothing\}$.

A semilattice S is called *weakly distributive* if whenever $a_1 \vee \cdots \vee a_n$ exists in S then $(x \wedge a_1) \vee \cdots \vee (x \wedge a_n)$ exists $x \wedge (a_1 \vee \cdots \vee a_n) = (x \wedge a_1) \vee \cdots \vee (x \wedge a_n)$. Weakly distributive semilattices have been studied in R. Balbes [1], J. Varlet [14] and W. H. Cornish and R. C. Hickman [4].

An *ideal* of a semilattice is a non-empty subset I of S such that

(i) if $x \in I$ and $y \leqslant x$ then $y \in I$;

(ii) if $a_1, \ldots, a_n \in I$ and $a_1 \vee \cdots \vee a_n$ exists in S then $a_1 \vee \cdots \vee a_n \in I$.

The family of all ideals of S is denoted by $\mathcal{J}^{\omega}(S)$; clearly $\mathcal{J}^{\omega}(S) \cup \{\varnothing\}$ is an algebraic closure system. From Cornish and Hickman [4, Theorem 1.1] we use

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Proposition 3.7. *A semilattice is weakly distributive if and only if $\mathcal{J}^{\omega}(S)$ is a distributive lattice.*

Theorem 3.8. *A semilattice is mildly distributive if and only if it is weakly distributive and each ideal is a strong ideal.*

Proof. The proof of Theorem 2.5 showed that a mildly distributive semilattice is weakly distributive. If $a_1, \ldots, a_n$ are elements of an ideal I and $[b) \supseteq [a_1) \cap \cdots \cap [a_n)$ then $b \in I$ since $b = (b \wedge a_1) \vee \cdots \vee (b \wedge a_n)$ and each member of this join is in I. Thus I is a strong ideal, completing the first assertion.

If S is weakly distributive and each ideal is a strong ideal then $\mathcal{J}^{\omega}(S) = {}^{\omega}\mathcal{J}(S)$ since strong ideals are always ideals. By Proposition 3.7 this implies that S is mildly distributive.

A proper ideal P is called *prime* if $a \wedge b \in P$ implies $a \in P$ or $b \in P$ for all $a, b \in S$, and a proper filter F is called *prime* if $a_1 \vee \cdots \vee a_n \in F$ implies $a_i \in F$ for some i. Clearly a filter is prime if and only if its complement is a prime ideal. Balbes showed [1, Theorem 2.2] that if S is a weakly distributive semilattice, I an ideal of S, F a filter of S and $I \cap F = \varnothing$ then there exists a prime ideal P with $I \subseteq P$ and $P \cap F = \varnothing$. In view of Theorem 3.8 this result readily extends to mildly distributive semilattices.

4. Join partial homomorphisms

Let $f\colon S_1 \to S_2$ be a semilattice homomorphism. Then f is called

(i) *join partial* if whenever $a_1 \wedge \cdots \wedge a_n$ exists in S_1 then $f(a_1) \vee \cdots \vee f(a_n)$ exists in S_2 and $f(a_1) \vee \cdots \vee f(a_n) = f(a_1 \vee \cdots \vee a_n)$;

(ii) *strong join partial* if whenever $[x) \supseteq [a_1) \cap \cdots \cap [a_n)$ in S_1 then $[f(a_1)) \cap \cdots \cap [f(a_n))$ in S_2.

Lemma 4.1. *A strong join partial homomorphism is join partial.*

Proof. Suppose $f\colon S_1 \to S_2$ is strong join partial and $a_1 \vee \cdots \vee a_n$ exists in S_1. Then $[a_1 \vee \cdots \vee a_n) = [a_1) \cap \cdots \cap [a_n)$ and so $[f(a_1 \vee \cdots \vee a_n)) \supseteq [f(a_1)) \cap \cdots \cap [f(a_n))$. Since f is isotone, $f(a_1 \vee \cdots \vee a_n) \geqslant f(a_i)$ for each $i = 1, \ldots, n$, whence $[f(a_1 \vee \cdots \vee a_n)) = [f(a_1)) \cap \cdots \cap [f(a_n))$, giving $f(a_1 \vee \cdots \vee a_n) = f(a_1) \vee \cdots \vee f(a_n)$.

Lemma 4.2. *Let S_1 be a mildly distributive semilattice, S_2 an arbitrary semilattice and $f\colon S_1 \to S_2$ a join partial homomorphism. Then f is strong join partial.*

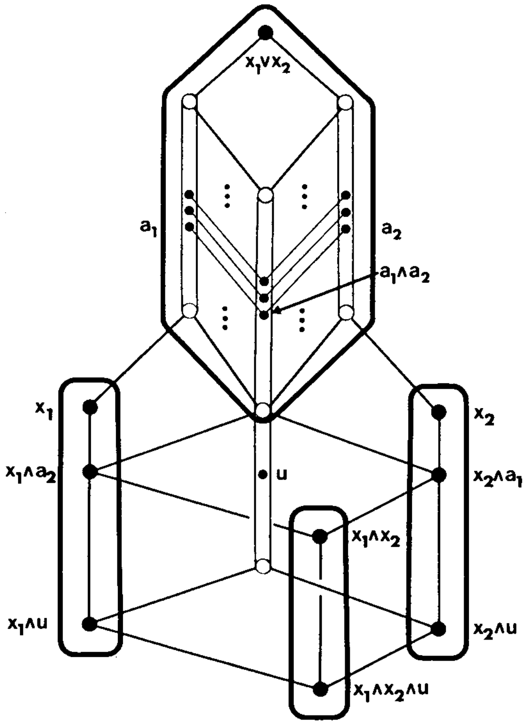

Figure 1. The semilattice S and the join partial congruence Φ. To interpret this diagram, first consider the lattice which has as elements all the large dots •, circles ○ and elements inside the chains ||. (Here two parallel lines between two circles or dots indicate the infinite chain Z of all integers. Some of the elements in these chains are shown by smaller dots.) Remove from this lattice the seven circles, leaving the semilattice S. The congruence classes of Φ, other than singletons, are circles. So, for example, the congruence class of $x_1 \vee x_2$ is isomorphic to ($\bigvee \times Z) \oplus \mathbf{1}$.

PROOF. Suppose $[x) \supseteq [a_1) \cap \cdots \cap [a_n)$ for some $x, a_1, \ldots, a_n \in S$. Then $x = (x \wedge a_1) \vee \cdots \vee (x \wedge a_n)$ and so $f(x) = (f(x) \wedge f(a_1)) \vee \cdots \vee (f(x) \wedge f(a_n))$, whence $[f(x)) = [f(x) \wedge f(a_1)) \cap \cdots \cap [f(x) \wedge f(a_r)) \supseteq [f(a_1)) \cap \cdots \cap [f(a_r))$.

A distributive lattice D is called a *strong free distributive extension* of a semilattice S if

(i) There is a strong join partial embedding $\varepsilon: S \to D$.

(ii) $\varepsilon(S)$ generates D as a lattice.

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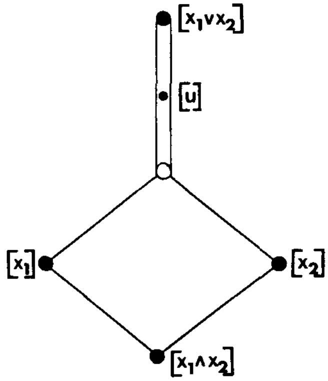

Figure 2. S/Φ. Clearly $[x_1] \vee [x_2]$ does not exist even though $x_1 \vee x_2$ exists.

(iii) If D_1 is a distributive lattice and if f: $S \to D_1$ is a strong join partial homomorphism, then there exists a lattice homomorphism h: $D \to D_1$ such that $f = h \circ \varepsilon$.

Clearly, a strong free distributive extension of a semilattice is unique up to isomorphism (if it exists).

THEOREM 4.3. *A semilattice has a strong free distributive extension if and only if it is mildly distributive.*

PROOF. If S has a strong free distributive extension D and $(x] \supseteq (a_1] \cap \cdots \cap (a_n]$ in S then $(\varepsilon(x)] \supseteq (\varepsilon(a_1)] \cap \cdots \cap (\varepsilon(a_n)]$ in D and so $\varepsilon(x) = \varepsilon(x \wedge a_1) \vee \cdots \vee \varepsilon(x \wedge a_1)$ in D, which implies $x = (x \wedge a_1) \vee \cdots \vee (x \wedge a_n)$ in S and S is mildly distributive.

If S is mildly distributive then ${}^{\omega}\mathcal{J}_f(S)$ is its strong free distributive extension. It is possible to prove this by direct computation. However, by Theorem 3.8, ${}^{\omega}\mathcal{J}_f(S) = \mathcal{J}_f^{\omega}(S)$, the lattice of all finitely generated ideals of S, and each strong join partial homomorphism from S is join partial, so a computational proof would mimic the details of Theorem 1.3 from Cornish and Hickman [4], and so we refer the reader to that paper.

Suppose S_1 and S_2 are semilattices, S_1 is weakly distributive and f: $S_1 \to S_2$ is a join partial homomorphism. Then the semilarttice congruence Ker f defined by $a \equiv b$ (Ker f) if and only if $f(a) = f(b)$ satisfies the following condition:

(i) If $a_i \equiv b_i$ (Ker f) for $i = 1, \ldots, n$ and both $a_1 \vee \cdots \vee a_n$ and $b_1 \vee \cdots \vee b_n$ exist in S_1 then $a_1 \vee \cdots \vee a_n \equiv b_1 \vee \cdots \vee b_n$ (Ker f).

A semilattice congruence on S_1 which satisfies (i) will be called *join partial.*

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PROPOSITION 4.4. *Let S be a weakly distributive semilattice and Θ a join partial congruence on S. Then f: $S \to S/\Theta$ defined by $f(a) = [a]\theta$, the congruence class of a under Θ, is a join partial homomorphism.*

This proposition may not seem surprising and it is not hard to prove; however it is interesting to note that it is in general false if S_1 is not weakly distributive.* Indeed consider the semilattice given in Figure 1. The non-trivial congruence classes of Φ are circles. For $i = 1, 2$ we have $x_i \equiv u \wedge x_i(\Phi)$, so that $[x_1], [x_2] < [u]$ in S/Φ. Also $[u] < [x_1 \vee x_2]$ in S/Φ so that $[x_1] \vee [x_2] \neq [x_1 \vee x_2]$ in S/Φ; in fact S/Φ is easily seen to be the semilattice given in Figure 2.

5. Finitely derivable mildly distributive semilattices

We recall a few definitions and results from Hickman [9]. An element x of a semilattice S is called *completely removable* if

(i) x is join reducible, that is $x = x_1 \vee \cdots \vee x_n$ for some $x_i < x$;

(ii) $[x)/\{x\}$ is either empty or a non-principal filter.

PROPOSITION 5.1 (Hickman [9, Theorem 4.1]). *Let S be a weakly distributive semilattice and x a completely removable element of S. If $H = S\backslash\{x\}$, then H is a weakly distributive subsemilattice of S. Furthermore*

(i) *if $a, a_1, \ldots, a_n \in H$ then $a = a_1 \vee \cdots \vee a_n$ in H if and only if $a = a_1 \vee \cdots \vee a_n$ in S,*

(ii) *the map r: $\mathcal{I}_f^\omega(S) \to \mathcal{I}_f^\omega(H)$ defined by $r(I) = I\backslash\{x\}$ is a lattice isomorphism.*

PROOF. The statement of (ii) is not mentioned explicitly in Hickman [9]. Rather it is shown that g: $\mathcal{I}_f^\omega(H) \to \mathcal{I}_f^\omega(S)$ defined by $g(\langle a_1, \ldots, a_n \rangle_\omega^H) = \langle a_1, \ldots, a_n \rangle_\omega^S$ is a lattice isomorphism. (Here $\langle a_1, \ldots, a_n \rangle_\omega^H$ and $\langle a_1, \ldots, a_n \rangle_\omega^S$ denote the ideals generated by $\{a_1, \ldots, a_n\}$ in H and S, respectively.) However it is not hard to check that r is g^{-1}, and so is an isomorphism.

A weakly distributive semilattice S is called *finitely derivable* if there exists a sequence of weakly distributive semilattices $S_0, \ldots, S_r = S$ such that S_0 is a distributive lattice and for each i, $S_i = S_{i-1}\backslash\{x_i\}$ for some completely removable element x_i in S_{i-1}. The sequence $S_0, \ldots, S_r$ is called a deriving sequence for S. It is

*The author would like to thank Dr. Brian Davey for pointing out this possibility.

an easy consequence of Proposition 5.1 that $\mathcal{J}_f^\omega(S) \cong S_0$ and S has r non-principal finitely generated ideals, and these correspond to $x_1, \ldots, x_r$ in S_0.

PROPOSITION 5.2 (Hickman [9, Proposition 4.2]). *A weakly distributive semilattice is finitely derivable if and only if it has a finite number of non-principal finitely generated ideals. Furthermore, a finitely derivable weakly distributive semilattice S has a deriving sequence $S_0, \ldots, S_r$ such that $(x_i]$ is a lattice S_{i-1} for each $i = 1, \ldots, r$.*

A deriving sequence of the type given in this proposition will be called a lower deriving sequence. Many proofs will be given using induction on the number of non-principal finitely generated ideals of S. This (partially ordered) set will be denoted $\mathfrak{N}(S)$ and its cardinality by $|\mathfrak{N}(S)|$.

Finitely derivable mildly distributive semilattices are easy to describe, the reason being the simple description of finitely generated strong ideals. This is exploited to give the following separation theorem.

THEOREM 5.3. *A weakly distributive semilattice is mildly distributive if and only if for each pair of distinct non-principal finitely generated ideals there exists a principal ideal containing one but not the other.*

PROOF. Suppose S is mildly distributive and $\langle a_1, \ldots, a_n \rangle_\omega$, $\langle b_1, \ldots, b_m \rangle_\omega$ are non-principal finitely generated ideals with $\langle a_1, \ldots, a_n \rangle_\omega \not\subseteq \langle b_1, \ldots, b_m \rangle_\omega$. Then for some i, $a_i \notin \langle b_1, \ldots, b_m \rangle_\omega$ and so there exists $x \in [b_1) \cap \cdots \cap [b_m)$ such that $a_i \nleqslant x$. Then $(x]$ is the required principal ideal.

Conversely, suppose S is weakly distributive and distinct non-principal finitely generated ideals may be separated by a principal ideal. Let $\langle a_1, \ldots, a_n \rangle_\omega$ be a non-principal finitely generated ideal and suppose $[x) \supseteq [a_1) \cap \cdots \cap [a_n)$. If $x \notin \langle a_1, \ldots, a_n \rangle_\omega$ then consider the ideal $\langle x, a_1, \ldots, a_n \rangle_\omega$. If it is non-principal then there exists $y \in S$ such that $\langle a_1, \ldots, a_n \rangle_\omega \subseteq (y]$ and $\langle x, a_1, \ldots, a_n \rangle \not\subseteq (y]$, so that $x \leqslant y$ and $[x) \not\supseteq [a_1) \cap \cdots \cap [a_n)$. On the other hand, if $\langle x, a_1, \ldots, a_n \rangle_\omega$ is principal then $x \vee a_1 \vee \cdots \vee a_n$ exists in S, and this implies that $a_1 \vee \cdots \vee a_n$ exists in S since $[x) \supseteq [a_1) \cap \cdots \cap [a_n)$. In both cases there is a contradiction, and so $x \in \langle a_1, \ldots, a_n \rangle_\omega$ and S is mildly distributive.

COROLLARY 5.4. *A weakly distributive semilattice with only one nonprincipal finitely generated ideal is mildly distributive.*

Once we reach two non-principal finitely generated ideals the result breaks down. Indeed let $S_0 = (\mathbf{3} \times \mathbf{2}) \oplus Z^-$, where $\mathbf{3}$ and $\mathbf{2}$ are the three and two element

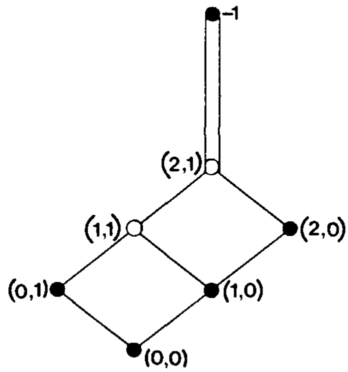

Figure 3. This semilattice is not mildly distributive since $\langle(0,1),(1,0)\rangle_\omega \neq \langle(0,1),(1,0)\rangle$. In this diagram the infinite chain is Z^-, the set of all negative integers.

chains respectively, and Z^- is the chain of negative integers. Define $S_1 = S_0\backslash\{(2,1)\}$ and $S_2 = S_1\backslash\{(1,1)\}$. Then $|\mathfrak{N}(S_2)| = 2$ and S_2 is not mildly distributive—see Figure 3.

THEOREM 5.5. *Let S be a finitely derivable mildly distributive semilattice and let $S_0,\ldots,S_r$ be a deriving sequence for S. Then S_i is mildly distributive for each i.*

PROOF. By induction it suffices to show that S_{r-1} is mildly distributive, and by definition, $S = S_{r-1}\backslash\{x\}$ where x is completely removable in S_{r-1}. Let $\langle a_1,\ldots,a_n\rangle_\omega$ and $\langle b_1,\ldots,b_m\rangle_\omega$ be distinct non-principal finitely generated ideals in S_{r-1}. If $x \notin \{a_1,\ldots,a_n, b_1,\ldots,b_m\}$ then $\langle a_1,\ldots,a_n\rangle^S_\omega$ and $\langle b_1,\ldots,b_m\rangle^S_\omega$ are distinct non-principal finitely generated ideals in S and so there exists $(y]$ in S which separates them. But then $(y]$ separates $\langle a_1,\ldots,a_n\rangle_\omega$ and $\langle b_1,\ldots,b_m\rangle_\omega$ in S_{r-1}.

If $x \in \{a_1,\ldots,a_n, b_1,\ldots,b_m\}$, then since x is completely removable there exists a finite set $C \subseteq S_{r-1}$ with $x = \bigvee C$ and $x \notin C$. Simply replace each occurrence of x in the generating sets $\{a_1,\ldots,a_n\}$ and $\{b_1,\ldots,b_m\}$ with C and proceed as in the first case.

Unfortunately it is not possible to extend this result to distributive semilattices. Let S_0 be the distributive lattice $((\mathbf{2}\times\mathbf{2})\oplus Z^-)\times\mathbf{2}$, $S_1 = S_0\backslash\{(1,1,1)\}$ and $S_2 = S_1\backslash\{(1,1,0)\}$ (see Figure 4). Then S_2 is a distributive semilattice since it is isomorphic to

$$(((\mathbf{2}\times\mathbf{2})\backslash\{(1,1)\})\oplus Z^-)\times\mathbf{2}$$

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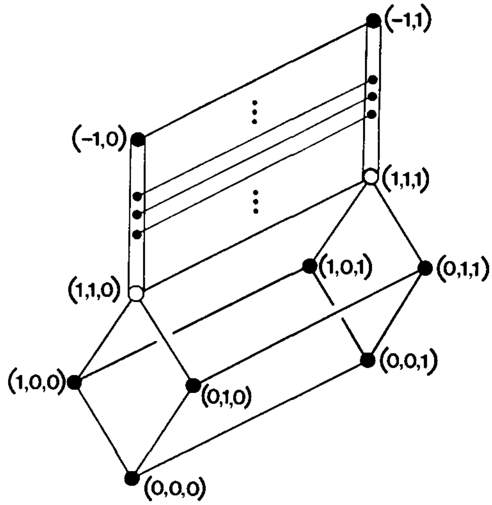

Figure 4. This semilattice is distributive since it is isomorphic to $(((\mathbf{2} \times \mathbf{2})\backslash\{(1,1)\}) \oplus Z^-) \times \mathbf{2}$. The infinite chains are both Z^-.

and each component of the final product is an up directed distributive semilattice (it is an easy matter to show that the product of two such semilattices is distributive). However S_1 is not distributive by the following theorem.

THEOREM 5.6. *Let S be a weakly distributive semilattice with $|\mathcal{N}(S)| = 1$. Then S is distributive if and only if S is of the form $H \oplus L$ where H is a distributive semilattice with the upper bound property, H is not a lattice, and L is a (possibly empty) distributive lattice without a least element.*

PROOF. Suppose S is distributive and let $H = \langle a_1,\ldots,a_n\rangle_\omega$ be the only non-principal finitely generated ideal in S. Either $a_1 \vee a_2$ exists in S or $\langle a_1, a_2\rangle_\omega = \langle a_1,\ldots,a_r\rangle_\omega$, and so we may assume that H is two generated, that is $H = \langle a_1,\ldots,a_2\rangle_\omega$. If $[a_1] \cap [a_2) = \varnothing$ then $S = H$ since for all $x \in S$, $[x) \supseteq [a_1) \cap [a_2)$, and we have the desired result. So suppose $[a_1) \cap [a_2) \neq \varnothing$. Then $S = \langle a_1, a_2\rangle_\omega \cup ([a_2) \cap [a_2))$; for if not then there exists $x \in S$ with $x \notin \langle a_1, a_2\rangle \cup ([a_1) \cap [a_2))$. Since $\langle a_1, a_2\rangle_\omega$ is the only non-principal finitely generated ideal, $x \vee a_1 \vee a_2$ exists in S. Choose a prime ideal P with $\langle a_1, a_2\rangle_\omega \subseteq P$ and $P \cap ([a_1) \cap [a_2)) = \varnothing$. Clearly $x \notin P$. Since S is distributive, $[x) \vee ([a_1) \cap [a_2)) = \{y \wedge a\colon y \geqslant x \text{ and } a \geqslant a_1, a_2\} = [x \wedge a_1) \cap [x \wedge a_2)$, and $(x \wedge a_1) \vee (x \wedge a_2)$ exists in S since $x \notin [a_1) \cap [a_2)$. Thus $x \wedge a = (x \wedge a_1) \vee (x \wedge a_2) \in P$ for some $a \in [a_1) \cap [a_2)$, which contradicts the fact that P is prime. This shows that $S = \langle a_1, a_2\rangle_\omega \cup ([a_1) \cap [a_2))$.

To prove the converse it is only necessary to show that the class of distributive semilattices is closed under ordinal sums, and this result is due to W. H. Cornish (private communication).

6. The lattice of join partial congruence relations

The main purpose of Hickman [9] was to investigate the relationship between the lattice of join partial congruences of a weakly distributive semilattice S (denoted $\mathcal{C}^{\omega}(S)$) and the lattice of lattice congruences of $\mathcal{J}_f^{\omega}(S)$ (denoted $\mathcal{C}(\mathcal{J}_f^{\omega}(S))$). The principal result was

PROPOSITION 6.1. *The restriction map* ρ: $\mathcal{C}(\mathcal{J}_f^{\omega}(S)) \to \mathcal{C}^{\omega}(S)$ *defined by* $a \equiv b(\rho(\Gamma))$ *if and only if* $(a] \equiv (b](\Gamma)$ *is an isomorphism if and only if* $C^{\omega}(S)$ *is a distributive lattice.*

After this proposition, it is natural to question under what circumstances is $\mathcal{C}^{\omega}(S)$ distributive, and below two results regarding this question are detailed.

PROPOSITION 6.2 (Hickman [9, Theorem 3.4]). *Let S be a distributive semilattice. Then $\mathcal{C}^{\omega}(S)$ is distributive if and only if S has the upper bound property.*

Let S be a semilattice, $a, b \in S$ with $a \leqslant b$. Then a is said to be strong below b if $a \vee y$ exists for all $y \leqslant b$. We say that *a is connected to b with complexity* 0 if there exists $a = z_0 \leqslant z_1 \leqslant \cdots \leqslant z_r = b$ such that z_{i-1} is strong below z_i for $i = 1,\ldots,r$. For example in Figure 5, a is not strong below 1, since $a \vee c$ does not exist, but a is strong below b, b is strong below 1 and so a is connected to 1 with complexity 0.

PROPOSITION 6.3 (Hickman [9, Theorem 3.8]). *Let S be a weakly distributive semilattice and suppose that a is connected to b with complexity* 0 *for all $a, b \in S$ with $a \leqslant b$. Then $C^{\omega}(S)$ is distributive.*

The condition given in this last proposition is obviously some way short of being a characterization for congruence distributivity. For example the semilattice in Figure 5 is congruence distributive, (as will be shown below) and yet c is not connected to 1 with complexity 0. The aim of this and the next section is to provide a condition which is a weakening connectivity with complexity 0, and

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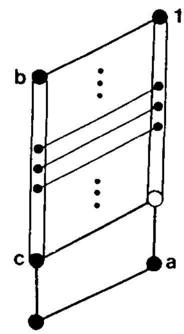

Figure 5. This semilattice is isomorphic to $((\mathbf{2} \oplus Z^-) \times \mathbf{2}) \backslash \{(1,1)\}$. It is connected but not strongly connected.

which characterizes congruence distributivity for finitely derivable weakly distributive semilattice. The most important join partial congruences have been described Katrinák, and Cornish and Hickman. Let S be a weakly distributive semilattice, F a filter, I an ideal and $a, b \in S$ with $a \leqslant b$.

PROPOSITION 6.4. (a) (Katrinák [12, page 167]). *$\Psi(F)$ defined by $x \equiv y(\Psi(F))$ if and only if $x \wedge f = y \wedge f$ for some $f \in F$ is a join partial congruence which has F as a congruence class.*

(b) (Cornish and Hickman [4, Theorems 3.1, 3.2]). *$\Theta(I)$ defined by $x \equiv y(\Theta(I))$ if and only if $\langle x, i\rangle_\omega = \langle y, i\rangle_\omega$ is the smallest join partial congruence which has I as a congruence class, and $T(a, b)$ defined by $x \equiv y(T(a, b))$ if and only if $x \wedge a = y \wedge a$ and $\langle x, b\rangle_\omega = \langle y, b\rangle_\omega$ is the smallest join partial congruence identifying a and b.*

The notion of projectivity for semilattices is an easy generalization of the well known concept of projectivity in lattices: see for example R. P. Dilworth and P. Crawley [5] or G. Grätzer [7]. The pair of elements (a, b) is called a *quotient* if $a \leqslant b$, and in this case (a, b) is written b/a. The quotient d/c is a *lower transpose* of b/a and b/a is an upper transpose of d/c, written $b/a \searrow d/c$ and $d/c \nearrow b/a$ respectively if $b = a \vee d$ and $c = a \wedge d$. Two quotients b/a and d/c are called *projective*, written $b/a \sim d/c$, if there exist quotients $b/a = y_0/x_0$, $y_1/x_1, \dots, y_r/x_r = d/c$ such that for each $i = 1, \dots, r$, $y_{i-1} \searrow y_i/x_i$ or $y_{i-1}/x_{i-1} \nearrow y_i/x_i$. Standard calculations now produce

PROPOSITION 6.5. *Let S be a weakly distributive semilattice and suppose $b/a \sim d/c$. Then $T(a, b) = T(c, d)$. Conversely, if $T(a, b) = T(c, d)$ then $b/a \searrow b \wedge d/a \wedge c \nearrow d/c$.*

Let S be a weakly distributive semilattice, $a, b \in S$ with $a \leqslant b$. Inductively define for $d \geqslant 1$, *a is connected b with complexity d* if there exist in S, $a = z_0 \leqslant z_1 \leqslant \cdots \leqslant z_r = b$, and x_i, y_i for $i = 1, \ldots, r$ with $x_i \leqslant y_i$ such that $z_i/z_{i-1} \sim y_i/x_i$ for each $i = 1, \ldots, r$ and x_i is connected to y_i with complexity $d - 1$.

We say that a is connected to b if a is connected to b with complexity d for some integer d, and the semilattice S is called *connected* if a is connected to b for each quotient b/a in S.

THEOREM 6.6. *Let S be a connected weakly distributive semilattice. Then $\mathcal{C}^{\omega}(S)$ is distributive.*

PROOF. It suffices to show that for each quotient y/x, $T(x, y) \subseteq \Gamma_1 \vee \Gamma_2$ implies $T(x, y) = (T(x, y) \cap \Gamma_1) \vee (T(x, y) \cap \Gamma_2)$. The case when x is connected to y with complexity 0 is done in Proposition 6.3 (it actually appeared in the proof of the proposition rather than the statement). Assume inductively that x is connected to y with complexity $d - 1$ and $T(x, y) \subseteq \Gamma_1 \vee \Gamma_2$ implies $T(x, y) = (T(x, y) \cap \Gamma_1) \vee (T(x, y) \cap \Gamma_2)$, and suppose a is connected to b with complexity d and $T(a, b) \subseteq \Gamma_1 \vee \Gamma_2$. Then there exists $a = z_0 \leqslant z_1 \leqslant \cdots \leqslant z_r = b$ and y_i/x_i such that $z_i/z_{i-1} \sim y_i/x_i$ and x_i is connected to y_i with complexity $d - 1$ for each $i = 1, \ldots, r$. Then

$$T(x_i, y_i) = T(z_{i-1}, z_i) \subseteq T(a, b) \subseteq \Gamma_1 \vee \Gamma_2$$

so that $T(z_{i-1}, z_i) = (T(z_{i-1}, z_i) \cap \Gamma_1) \vee (T(z_{i-1}, z_i) \cap \Gamma_2)$ for each $i = 1, \ldots, r$. Since $T(a, b) = T(z_0, z_1) \vee \cdots \vee T(z_{r-1}, z_r)$,

$$\begin{aligned} T(a, b) &= \bigvee_{i=1}^{r} (T(z_{i-1}, z_i) \cap \Gamma_1) \vee (T(z_{i-1}, z_i) \cap \Gamma_2) \\ &\subseteq (T(a, b) \cap \Gamma_1) \vee (T(a, b) \cap \Gamma_2) \subseteq T(a, b), \end{aligned}$$

completing the proof.

7. Congruence distributivity

The aim of this section is to prove that a congruence distributive finitely derivable weakly distributive semilattice is connected. Let E be the largest element in $\mathcal{C}^{\omega}(S)$.

LEMMA 7.1 (Hickman [9, Lemma 2.3]). *Let S be a weakly distributive semilattice, $a_1, \ldots, a_n \in S$ and suppose $\{a_1, \ldots, a_n\}$ has an upper bound. Then $\Theta(\langle a_1, \ldots, a_n \rangle_{\omega}) \vee \Psi([a_1) \cap \cdots \cap [a_n)) = E$ if and only if $a_1 \vee \cdots \vee a_n$ exists in S.*

Let I be a non-principal finitely generated ideal and $p \in S$. We say that I *shelters* p if $J \subseteq I$ for all non-principal finitely generated ideals $J \subseteq (p]$.

PROPOSITION 7.2. *Let S be a finitely derivable weakly distributive semilattice. For each maximal element I of $\mathfrak{N}(S)$, if $I \neq S$ then there exists a $p \in S$ which is sheltered by I. If S is mildly distributive, then for each $I \in \mathfrak{N}(S)$, $I \neq S$ there exists a $p \in S$ which is sheltered by I.*

PROOF. For each $J \in \mathfrak{N}(S)$, $J \not\subseteq I$, $I + J$ is principal, since is maximal, and so equals $(p_J]$ for some $p_J \in S$. Choose p such that $I \subset (p]$ and $p < \bigvee\{p_J\colon J \in \mathfrak{N}(S), J \not\subseteq I\}$. This works provided I is not the largest member of $\mathfrak{N}(S)$. If I is the largest non-principal finitely generated ideal, there exists, by our assumption, $x \in S$ such that $x \notin I$. Then $(x] + I$ is principal and I shelters its generator.

If S is mildly distributive, then for each $J \not\subseteq I$ there exists a p_J in S with $I \subseteq (p_j]$ and $J \not\subseteq (p_J]$. The proof now proceeds as above.

PROPOSITION 7.3. *Let S be a weakly distributive semilattice.*

(i) *If I is a maximal element of $\mathfrak{N}(S)$ and $I \neq S$ then I is prime.*

(ii) *If $\mathfrak{N}(S)$ has a largest element $\langle a_1,\ldots,a_n\rangle_\omega$ and $\langle a_1,\ldots,a_n\rangle_\omega \neq S$, and $S = \langle a_1,\ldots,a_n\rangle_\omega \cup ([a_n))$, then $\mathcal{C}^\omega(S)$ is not distributive.*

PROOF. (i) If $a \wedge b \in I$ then $I = I + ((a] \cap (b]) = (I + (a]) \cap (I + (b])$. If neither a nor b is in I, then we have expressed I as the intersection of two ideals, both of which are principal by the maximality of I. This would imply that I is principal, which is a contradiction.

(ii) Since $\langle a_1,\ldots,a_n\rangle_\omega \neq S$, $[a_1) \cap \cdots \cap [a_n) \neq \varnothing$ as $S = \langle a_1,\ldots,a_n\rangle_\omega \cup ([a_1) \cap \cdots \cap [a_n))$. We show that $\Psi(a_1) \cap \cdots \cap \Psi(a_n) = \mathcal{J}([a_1) \cap \cdots \cap [a_n))$ where $\Psi(a_i)$ abbreviates $\Psi([a_i))$. Suppose $x \wedge a_i = y \wedge a_i$ for all $i = 1,\ldots,n$. If $y \in [a_1) \cap \cdots \cap [a_n)$ then $y \wedge a_i = a_i = x \wedge a_i$ so $x \in [a_1) \cap \cdots \cap [a_n)$ and $x \equiv y(\Psi([a_1) \cap \cdots \cap [a_n)))$. If $y \notin [a_1) \cap \cdots \cap [a_n)$ then $y \in \langle a_1,\ldots,a_n\rangle_\omega$ and so $y = (y \wedge a_1) \vee \cdots \vee (y \wedge a_n) = (x \wedge a_1) \vee \cdots \vee (x \wedge a_n)$. Also, since $y \notin [a_1) \cap \cdots \cap [a_n)$, $x \notin [a_1) \cap \cdots \cap [a_n)$, whence $x \in \langle a_1,\ldots,a_n\rangle_\omega$ and $x = (x \wedge a_1) \vee \cdots \vee (x \wedge a_n) = y$. Once again, $x \equiv y$ $(\Psi([a_1) \cap \cdots \cap [a_n)))$. This shows that $\Psi(a_1) \cap \cdots \cap \Psi(a_n) \subseteq \Psi([a_1) \cap \cdots \cap [a_n))$ and the reverse inequality comes straight from the definition of $\Psi([a_1) \cap \cdots \cap [a_n))$.

If $\mathcal{C}^\omega(S)$ were distributive, we would have

$$\begin{aligned}\mathbf{Q}(\langle a_1,\ldots,a_n\rangle_\omega) &\vee \Psi([a_1) \cap \cdots \cap [a_n)) \\ &= \big(\Theta(\langle a_1,\ldots,a_n\rangle_\omega) \vee \Psi(a_1)\big) \cap \cdots \cap \big(\mathbf{Q}(\langle a_1,\ldots,a_n\rangle_\omega) \vee \Psi(a_r)\big).\end{aligned}$$

It is easy to see that $\Theta(\langle a_1,\ldots,a_n\rangle_\omega) \vee \Psi(a_i) = E$ for each i, and hence $\Theta(\langle a_1,\ldots,a_n\rangle_\omega) \vee \Psi([a_1) \cap \cdots \cap [a_n)) = E$ which implies by Lemma 7.1 that $a_1 \vee \cdots \vee a_n$ exists in S, which is a contradiction. Thus $\mathcal{C}^\omega(S)$ is not distributive.

The idea of the proof of the main theorem is to examine the congruences on certain principal ideals. The next three lemmas prepare for this.

LEMMA 7.4. *Let S be a weakly distributive semilattice. Then for each $b \in S$ the restriction map ρ: $\mathcal{C}^\omega(S) \to \mathcal{C}^\omega((b])$ defined by $\rho(\Phi) = \Phi \cap (b]^2$ for each $\Phi \in$ $\mathbf{C}^\omega(S)$ is a lattice homomorphism of $\mathcal{C}^\omega(S)$ onto $\mathcal{C}^\omega((b])$.*

PROOF. Define ψ: $\mathcal{C}^\omega((b]) \to \mathcal{C}^\omega(S)$ by $x \equiv y(\psi(\Gamma))$ if and only if $x \wedge b \equiv y \wedge b(\Gamma)$ in $(b]$. It is easy to check that $\psi(\Gamma)$ is a join partial congruence for each Γ in $\mathcal{C}^\omega((b])$, and furthermore that $\Phi \subseteq \psi(\rho(\Phi))$ and $\rho(\psi(\Gamma)) \subseteq \Gamma$. Thus ρ is a residuated mapping with residual ψ. Since residuated mappings preserve all joins —see for example T. S. Blyth and M. F. Janowitz [2, Theorem 5.2, page 37]—ρ is a lattice homomorphism. Finally, if x, $y \in (b]$ then $\rho(T_S(x, y)) = T_{(b]}(x, y)$ and this is enough to show that ρ is onto.

The author would like to thank Dr. Brian Davey for providing this great improvement on the original proof.

LEMMA 7.5. *Let S be a weakly distributive semilattice and $b \in S$. Then $\mathfrak{N}((b]) =$ $\{I \in \mathfrak{N}(S)\colon I \subseteq (b]\}$.*

LEMMA 7.6. *Let S be a weakly distributive semilattice, a, $b \in S$ with $a \leqslant b$. If a is connected to b in some principal ideal then a is connected to b in S.*

PROOF. Suppose $a \leqslant b \leqslant x$ and a is connected to b with complexity d in $(x]$. We show by induction on d that a is connected to b in S. The case $d = 0$ is obvious. Suppose a is connected to b with complexity d in $(x]$. Then there exist $a = z_0 \leqslant z_1 \leqslant \cdots \leqslant z_r = b$ and y_i/x_i in $(x]$ such that $z_i/z_{i-1} \sim y_i/x_i$ and x_i is connected to y_i with complexity $d-1$ in $(x]$. Then $z_i/z_{i-1} \sim y_i/x_i$ in S and y_i is connected to x_i in S, so a is connected to b in S.

THEOREM 7.7. *Let S be a finitely derivable weakly distributive semilattice and suppose $\mathcal{C}^\omega(S)$ is distributive. Then S is connected.*

PROOF. The proof is by induction on $|\mathfrak{N}(S)|$. Assume that $|\mathfrak{N}(S')| < k$ and $\mathcal{C}^\omega(S')$ distributive implies S' is connected and suppose $|\mathfrak{N}(S)| = k$ and $\mathcal{C}^\omega(S)$ is distributive. By Lemma 7.6 it suffices to show that each principal ideal of S is connected, and so without loss of generality, assume that S has a largest element. Also observe that if $a \in S$ and a is not contained within a maximal element of $\mathfrak{N}(S)$ then $a \vee x$ exists in S for all $x \in S$.

Suppose $a, b \in S$ with $a < b$. If a is not contained in a maximal non-principal finitely generated ideal, then a is strong below b, so that a is connected to b. Suppose $a \in I$ where I is a maximal element of $\mathfrak{N}(S)$. If $I \not\subseteq (b]$ then $|\mathfrak{N}((b])| < k$ by Lemma 7.5, and this would imply by the inductive hypothesis and Lemma 7.4 that a is connected to b in $(b]$ and hence also in S. Thus assume $I \subseteq (b]$. By Proposition 7.2 choose a $p \in S$ which is sheltered by I. Then I also shelters $b \wedge p$ and $b \wedge p$ is strong below b. It remains to show that a is connected to $b \wedge p$. Since $C^{\omega}((b \wedge p])$ is distributive there exists, by Proposition 7.3(ii), $c \in (b \wedge p]$ such that $c \notin I$ and $I \subseteq (c]$. Then $a \vee c$ exists and is strong under $b \wedge p$. Finally, $a \vee c/a \searrow c/a \wedge c$ and $a \wedge c$ is connected to c in $(c]$ since $|\mathfrak{N}((c])| < k$ and consequently a is connected to $a \vee c$ in S.

It is interesting to note that the conclusion of this theorem is no longer true if S is not finitely derivable. Example 4 of Hickman [9] is congruence distributive but not connected.

8. Connected finitely derivable mildly distributive semilattices

The results of the last two sections have special significance for mildly distributive semilattice, because, as we will show in this section, connected finitely derivable mildly distributive semilattices can be constructed in a systematic way.

LEMMA 8.1. *Let S be a weakly distributive semilattice and suppose a is connected to b with complexity d. Then for all $x \in S$, $a \wedge x$ is connected to $b \wedge x$ with complexity d.*

PROOF. The proof is by induction on d. To begin with, suppose a is strong below b. Then for all $y \leqslant b \wedge x$, $(a \wedge x) \vee y = x \wedge (a \vee y)$, which exists since a is strong below b. Hence $a \wedge x$ is strong below $b \wedge x$. If a is connected to b with complexity 0, then there exist $a = z_0 \leqslant z_1 \leqslant \cdots \leqslant z_r = b$ such that z_{i-1} is strong under z_i for each $i = 1, \ldots, r$. Then

$$a \wedge x = z_0 \wedge x \leqslant z_1 \leqslant x \leqslant \cdots \leqslant z_r \wedge x_r \wedge x = b \wedge x$$

and so $a \wedge x$ is connected to $b \wedge x$ with complexity 0.

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If a is connected to b with complexity d then there exist $a = z_0 \leqslant z_1 \leqslant \cdots \leqslant z_r = b$ and y_i/x_i such that $z_i/z_{i-1} \sim y_i/x_i$ and x_i is connected to y_i with complexity $d-1$. It remains to show that $z_i \wedge x/z_{i-1} \wedge x \sim y_i \wedge x_i \wedge x$ and this is done by showing that $T(z_{i-1} \wedge x, z_i \wedge x) = T(x_i \wedge x, y_i \wedge x)$. Indeed $(x_i \wedge x) \wedge (z_{i-1} \wedge x) = (y_i \wedge x) \wedge (z_{i-}\wedge x)$ since $x_i \wedge z_{i-1} = y_i \wedge z_{i-1}$. Also $\langle x_i \wedge x, z_i \wedge x \rangle_\omega = (x] \cap \langle x_i, z_i \rangle_\omega = (x] \cap \langle y_i, z_i \rangle_\omega = \langle y_i \wedge x, z_i \wedge x \rangle_\omega$ and so $T(x_i \wedge x, y_i \wedge x) \subseteq T(z_{i-1} \wedge x, z_i \wedge x)$. Since the reverse inequality is obtained similarly, the proof is complete.

LEMMA 8.2. *Let S be a weakly distributive semilattice and suppose a is connected to b with complexity d. Then there exist $a = z_0 \leqslant z_1 \leqslant \cdots \leqslant z_r = b$ and v_i/u_i for $i = 1,\ldots,r$ such that $z_i/z_{i-1} \searrow v_i/u_i$ for $i = 1,\ldots,r$ and u_i is connected to v_i with complexity $d-1$.*

PROOF. Since a is connected to b with complexity d, there exist $a = z_0 \leqslant z_1 \leqslant \cdots \leqslant z_r = b$ and y_i/x_i such that $z_i/z_{i-1} \sim y_i/x_i$ and x_i is connected to y_i with complexity $d-1$. By Proposition 6.5, this implies $z_i/z_{i-1} \searrow z_i \wedge y_i/z_{i-1} \wedge x_i$. Since $z_{i-1} \equiv z_i(T(x_i, y_i))$, $z_{i-1} \wedge x_i = z_i \wedge x_i$, so that $z_i/z_{i-1} \searrow z_i \wedge y_i/z_i \wedge x_i$ and now the result follows from the previous lemma by setting $v_i = z_i \wedge y_i$ and $u_i = z_i \wedge x_i$.

THEOREM 8.3. *Let S be a connected finitely derivable mildly distributive semilattice with deriving sequence $S_0,\ldots,S_r$. Then S_i is connected for each $i = 1,\ldots,r$.*

PROOF. By induction, it suffices to show that S_{r-1} is connected, and by definition, $S = S_{r-1} \backslash \{x\}$ where x is completely removable in S_{r-1}. Suppose $a, b \in S_{r-1}$ and $a \leqslant b$.

Case 1. $a, b \neq x$. We show by induction on d that a connected to b with complexity d in S implies a is connected to b in S_{r-1}, but first it is necessary to do it when a is strong below b in S. Note that $a \vee y$ exists in S_{r-1} for all $y \leqslant b$, $y \neq x$. Since r is completely removable in S_{r-1}, $x = c_1 \vee \cdots \vee c_n$ for some $c_i < x$. If $\langle a, c_1,\ldots,c_n \rangle_\omega$ is principal in S then $a \vee x$ exists in S_{r-1} and a is strong below b. So suppose $\langle c_1,\ldots,c_n \rangle_\omega \subseteq (b]$ and $\langle a, c_1,\ldots,c_n \rangle_\omega$ is non-principal in S. Choose $p' \in S$ which is sheltered by $\langle c_1,\ldots,c_n \rangle_\omega$, and this gives rise to ip $= p' \wedge b$ which is also sheltered by $\langle c_1,\ldots,c_n \rangle_\omega$. Since a is strong below b in S, $a \vee p$ exists and clearly $a \vee p/a \searrow p/a \wedge p$. We claim that $a \wedge p$ is strong below p in S_{r-1} and consequently a is connected to $a \vee p$ in S_{r-1}. Indeed if $y \leqslant p$ and $y \neq x$ then $a \vee y$ exists in S_{r-1} and hence so does $(a \wedge p) \vee y = (a \vee y) \wedge p$. If $y = x$ then $(x] \subseteq \langle a, y \rangle_\varepsilon \subseteq (p]$ and so $a \vee y$ exists in S_{r-1} and hence so does $(a \wedge p) \vee y$. The next step is to show that p is connected to b in S_{r-1}. This is

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done by induction on $|\mathfrak{N}((b])\backslash\mathfrak{N}((p])|$, or to be more precise, we show by induction on $|\mathfrak{N}((b])\backslash\mathfrak{N}((q])|$ that for all q and b, $x < q \leqslant b$ implies q is connected to b in S_{r-1}. If $|\mathfrak{N}((b])\backslash\mathfrak{N}((q])| = 0$ then q is strong below b in S_{r-1}. Suppose $|\mathfrak{N}((b])|\backslash\mathfrak{N}((q])| = k$. If q is not strong below b then there exists $y \in S_{r-1}$ with $y \leqslant b$ such that $q \vee y$ does not exist in S_{r-1}. Consequently, $\langle q, y\rangle_\omega$ is non-principal in S, and without loss of generality, we may assume that $\langle q, y\rangle_\omega$ is a minmal such ideal. That is, if $(q] \subseteq \langle q, y'\rangle_\omega \underset{\neq}{\subset} \langle q, y\rangle_\omega$ then $\langle q, y'\rangle_\omega$ is principal. Since S is mildly distributive there is $c \in S$ which is sheltered by $\langle q, y\rangle_\omega$, and without loss of generality, $c \leqslant b$. Since $\mathcal{C}^\omega((c])$ is distributive, there exists

$$e \in (c]\backslash(\langle q, y\rangle_\omega \cup ([q) \cap [y))).$$

If $q < e$ then $|\mathfrak{N}((e])\backslash\mathfrak{N}((q])| < k$ and $|\mathfrak{N}((b])\backslash\mathfrak{N}((e])| < k$, implying that q is connected to e in S_{r-1}, e is connected to b in S_{r-1} and hence q is connected to b in S_{r-1}. If $q \nleqslant e$ then $q \vee e$ exists in S and if $\langle q, y\rangle_\omega \not\subseteq (q \vee e]$ then repeat the last calculation with e replaced by $q \vee e$. If $\langle q, y\rangle_\omega \subseteq (q \vee e]$ then $q \vee e/q \searrow e/q \wedge e$. We claim that $q \wedge e$ is strong in e. Indeed, if $z \leqslant e$ then $q \vee z$ exists in S_{r-1} (since $\langle q, y\rangle_\omega$ has minimal) and hence $(q \wedge e) \vee z = (q \vee z) \wedge e$ exists in S_{r-1}. Thus, $q \wedge e$ is connected to e in S_{r-1}, q is connected to $q \vee e$ in S_{r-1} and also $|\mathfrak{N}((b])\backslash\mathfrak{N}((q \vee e])| < k$ so $q \vee e$ is connected to b in S_{r-1}. Thus q is connected to b in S_{r-1}, which completes this stage of the proof of case 1. The remainder of case 1 is now quite easy. If a is connected to b with complexity 0 in S then there exist $a = z_0 \leqslant z_1 \leqslant \cdots \leqslant z_m = b$ in S with z_{i-1} strong in z_i. By the first stage z_{i-1} is connected to z_i in S_{r-1} for each i and so a is connected to b in S_{r-1}. If a is connected to b with complexity d in S, then there exist $a = z_0 \leqslant \cdots \leqslant z_m = b$ and y_i/x_i in S such that $z_i/z_{i-1} \searrow y_i/x_i$ and x_i is connected to y_i with complexity $d-1$. By induction this implies x_i is connected to y_i in S_{r-1} and so a is connected to b.

Case 2. $a = x$. Choose $p \in S$, $p \leqslant b$ such that p shelters $\langle c_1, \ldots, c_n\rangle_\omega$ in S. Then a is strong below p in S_{r-1} and p is connected to b by case 1.

Case 3. $b = x$. We show by induction on $|\mathfrak{N}((b])\backslash((y])|$ $y \leqslant b = x$ implies y is connected to b in S_{r-1}. If $|\mathfrak{N}((b])\backslash\mathfrak{N}((y])| = 0$ then y is strong below b. Suppose $|\mathfrak{N}((b])\backslash\mathfrak{N}((y])| = k$. If y is not strong below b then there exists $z \leqslant b$ such that $y \vee z$ does not exist in S_{r-1}. Hence there exists $q < b$ such that $\langle y, z\rangle_\omega \subseteq (q]$. Then y is connected to q by case 1 and q is connected to b since $|\mathfrak{N}((b])\backslash\mathfrak{N}((q])| < k$.

We are now in a position to describe how to form congruence distributive mildly distributive semilattices.

THEOREM 8.4. *Let H be a connected finitely derivable mildly distributive, semilattice, x a completely removable element in H and suppose $(x]$ is a lattice and $S = H\backslash\{x\}$ is mildly distributive. Then S is congruence distributive if and only if, whenever p is sheltered by $\langle c_1,\ldots,c_n\rangle_\omega$ in S, where $x = c_1 \vee \cdots \vee c_n$ in H, then $(p]$ is mildly distributive but not distributive.*

PROOF. If $(p]$ is distributive, then $\mathcal{C}^\omega((p])$ is not distributive since $(p]$ is not a lattice, and consequently $\mathcal{C}^\omega(S)$ is not distributive. Suppose $(p]$ is mildly distributive but not distributive. We show that $(p]$ is connected. Indeed suppose $a < b \leqslant p$ and a is not strong below b. Then there exists $y \leqslant b$ such that $a \vee y$ does not exist. Since $|\mathfrak{N}((p])| = 1$ and $(p]$ is not distributive there exists $e \leqslant p$ with $e \notin \langle a, y\rangle_\omega \cup ([a) \cap [y))$, and clearly $a \vee e$ exists in $(p]$. Then $a \vee e/a \searrow e/a \wedge e$ and $a \wedge e$ is strong below e, and $a \vee e$ is strong below p, so that a is connected to p. By Lemma 8.1, this implies a is connected to b.

Now suppose $a \nleqslant b$ and $b \nleqslant x$ in H. If $b \ngtr x$ then $(b]$ is S equals $(b]$ in H, and so is congruence distributive and connected in S. So assume $b > x$ and first consider the case when $a > x$. Then, using the same argument detailed in the last theorem, it can be shown that a is connected to b. Finally, suppose $a \ngtr x$. Then, for some d, a is connected to b with complexity d in H. We show by induction on d, that a is connected to b in S. If a is connected to b with complexity 0 in H, then there exist $a = z_0 \leqslant z_1 \leqslant \cdots \leqslant z_r = b$ such that z_{i-1} is strong below z_i. Let j be the smallest integer such that $z_j > x$, whence $z_{j-1} \ngtr x$. If $z_{j-1} = x$ then choose $p \in S$ such that p is sheltered by $\langle c_1,\ldots,c_n\rangle_\omega$ and $p \leqslant z_j$. Then a is connected to p and p is connected to b, so a is connected to b. If $z_{j-1} \neq x$ then a is connected to z_{j-1} in S and z_j is connected to b in S by the previous cases. If z_{j-1} is strong below z_j in S then we are finished, so suppose $z_{j-1} \vee y$ does not exist in S. Clearly $\langle z_{j-1}, y\rangle_\varepsilon = \langle c_1,\ldots,c_n\rangle_\omega$ since z_{j-1} is strong below z_j in H. Choose $p \in S$ which is sheltered by $\langle z_{j-1}, y\rangle_\omega$ and which is below z_j. Then z_{j-1} is connected to p, p is connected to z_j, whence z_{j-1} is connected to z_j.

Suppose a is connected to b with complexity d in H. Then there exist $a = z_0 \leqslant \cdots \leqslant z_r = b$ and y_i/x_i in H such that $z_i/z_{i-1} \searrow y_i/x_i$ and x_i is connected to y_i with complexity $d - 1$ in H. Again choose the smallest j such that $z_j > x$. If $z_{j-1} = x$, then a is connected to z_j as above, so suppose $z_{j-1} \neq x$. If $y_j, x_j \neq x$ then z_{j-1} is connected to z_j in S by the inductive hypothesis. Since x is meet irreducible we cannot have $x_j = x$, so suppose $y_j = x$, and choose $p \leqslant z_j$ which is sheltered by $\langle c_1,\ldots,c_n\rangle_\omega$ in S. Since $(p]$ is not distributive there exists $e \in (p]$ such that $e \notin \langle c_1,\ldots,c_n\rangle_\omega$ and $\langle c_1,\ldots,c_n\rangle_\omega \not\subseteq (e]$, and let $f = e \vee c_1 \vee \cdots \vee c_n$. Then if $z_{j-1} \wedge f \in \langle c_1,\ldots,c_n\rangle_\omega$ in S, then $z_{j-1} \wedge f = x_j$ in H so that $z_j/z_{j-1} \searrow f/x_j$ and f is connected to x_j by the first part of the proof. If $z_{j-1} \wedge f \notin \langle c_1,\ldots,c_n\rangle_\omega$ then $z_{j-1} \wedge f$ is strong below f and $z_j/z_{j-1} \searrow f/z_{j-1} \wedge f$. Thus z_{j-1} is connected to z_j in S and the proof is complete.

COROLLARY 8.5. *Let S be a finitely derivable mildly distributive semilattice and let $S_0, \ldots, S_r$ be a lower deriving sequence for S with $S_i = S_{i-1} \setminus \{x_i\}$. Then S is congruence distributive if and only if, for each i, if $x_i = c_1 \vee \cdots \vee c_n$ in S_{i-1}, $c_j < x_i$ and p shelters $\langle c_1, \ldots, c_n \rangle_\omega$ in S_i, then $(p]$ is mildly distributive but not distributive.*

COROLLARY 8.6. *Let S be a finitely derivable mildly distributive semilattice. Then S is congruence distributive if and only if whenever I is a non-principal finitely generated ideal and p is sheltered by I, then there exists $b \leqslant p$ such that $b \notin I$ and $I \not\subseteq (b]$.*

9. Congruences on principal filters

Lemma 7.4 showed that restriction map ρ: $\mathcal{C}^\omega(S) \to \mathcal{C}^\omega((a])$ is a lattice homomorphism for any weakly distributive semilattice S and any $a \in S$. For example consider the semilattice S in Figure 5. Clearly $\Theta((a]) \wedge \Psi([a)) = E$, the largest congruence on S. However, if ϕ: $\mathcal{C}^\omega(S) \to \mathcal{C}^\omega([c))$ is the restriction map then $\phi(\Theta((a])) \vee \phi(\Psi([a))) \neq \phi(E)$. This section looks at necessary conditions and sufficient conditions for the restriction maps ϕ_a: $\mathcal{C}^\omega(S) \to \mathcal{C}^\omega([a))$ to be surjective lattice homomorphisms. The investigation could be carried out by using the description of the join in $\mathcal{C}^\omega(S)$ given in Cornish and Hickman [4]; the proofs are then rather technical and provide little insight into the material. The path chosen, while no shorter, hopefully gives a better unerstanding of the matter.

A weakly distributive semilattice S is said to have full congruence restriction if for each $a \in S$, the restriction map ϕ_a: $\mathcal{C}^\omega(S) \to \mathcal{C}^\omega([a))$ is a surjective lattice homomorphism.

PROPOSITION 9.1. *A weakly distributive semilattice S has full congruence restriction if and only if $(b]$ has full congruence restriction for each $b \in S$.*

PROOF. Consider the diagram

$$\begin{array}{ccc} \mathcal{C}^\omega(S) & \xrightarrow{\rho} & \mathcal{C}^\omega((b]) \\ \phi\downarrow & & \downarrow\sigma \\ \mathcal{C}^\omega([a)) & \xrightarrow[\Theta]{} & \mathcal{C}^\omega([a) \cap (b]) \end{array}$$

in which $a \leqslant b$ are arbitrary elements of S and ρ, σ, ϕ and Θ are the restriction maps. Clearly $\sigma \circ \rho = \Theta \circ \phi$ and ρ and Θ are surjective homomorphisms by Lemma 7.4. If S has full congruence restriction then ϕ is a surjective lattice homomorphism, showing that $\sigma \circ \rho$ is a surjective homomorphism and hence so is σ.

Suppose $(b]$ has full congruence restriction for each $b \in S$, and suppose $a \in S$. Then ϕ: $\mathcal{C}^\omega(S) \to \mathcal{C}^\omega([a))$, being the restriction map, preserved intersections. Suppose $\Gamma_1, \Gamma_2 \in S$ and $x \equiv y(\phi(\Gamma_1 \vee \Gamma_2))$ for some $y \geqslant x \geqslant a$. Write b for y and examine the diagram. Clearly

$$x \equiv y(\sigma(\rho(\Gamma_1 \vee \Gamma_2)))$$

and since both σ and ρ are lattice homomorphisms, this implies $x \equiv y(\sigma(\rho(\Gamma_1)) \vee \sigma(\rho(\Gamma_2)))$. But $\sigma \circ \rho = \Theta \circ \phi$ and Θ is a lattice homomorphism. Hence

$$x \equiv y(\Theta(\phi(\Gamma_1) \vee \phi(\Gamma_2)))$$

which implies $x \equiv y(\phi(\Gamma_1) \vee \phi(\Gamma_2))$ and ϕ is a lattice homomorphism. To see that ϕ is onto first observe that for $y \geqslant x \geqslant a$, $\rho(T_S(x, y)) = T_{[a)}(x, y)$ and that ρ preserves arbitrary joins by compactness.

A semilattice is called strongly connected if for all a, b with $a \leqslant b$, a is connected to b with complexity 0.

LEMMA 9.2. *If S is a strongly connected semilattice then $[a)$ is strongly connected for each $a \in S$.*

THEOREM 9.3. *A strongly connected weakly distributive semilattice has full congruence restriction.*

PROOF. By Proposition 9.1 we may assume that our semilattice S has a largest element b, and suppose $a \in S$. Since S is strongly connected, there exist $a = z_0 \leqslant z_1 \leqslant \cdots \leqslant z_r = b$ in S with z_{i-1} strong below z_i. The first stage of the proof is to show that the restriction map ρ: $\mathcal{C}^\omega((z_i]) \to \mathcal{C}^\omega([z_{i-1}) \cap (z_i])$ is a surjective homomorphism. By Theorem 6.6 and Lemma 9.2, both $\mathcal{C}^\omega((z_i])$ and $\mathcal{C}^\omega([z_{i-1}) \cap (z_i])$ are distributive and so, by Proposition 6.1, the restriction maps

$$\lambda: \mathcal{C}(\mathcal{I}_f^\omega((z_k])) \to \mathcal{C}^\omega((z_k])$$

and

$$\mu: \mathcal{C}(\mathcal{I}_f^\omega([z_{i-1}) \cap (z_i]) \to \mathcal{C}^\omega([z_{i-1}) \cap (z_i])$$

are both lattice isomorphisms. Now consider the map

$$\chi: \mathcal{I}_f^\omega([z_{i-1}) \cap (z_i]) \to L = \{I \in \mathcal{I}_f^\omega((z_i]) : (z_{i-1}] \subseteq I\},$$

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defined by $\chi(\langle a_1,\ldots,a_n\rangle_\omega) = \langle a_1,\ldots,a_n\rangle_\omega$ in $(z_i]$. Clearly χ is a well defined injective homomorphism. If $\langle b_1,\ldots,b_m\rangle \in L$ then $z_{i-1} \in \langle b_1,\ldots,b_m\rangle_\omega$ so that $\langle b_1,\ldots,b_m\rangle_\omega = \langle b_1 \vee z_{i-1},\ldots,b_m \vee z_{i-1}\rangle_\omega$ and consequently χ is surjective. Since χ is an isomorphism it induces an isomorphism π: $\mathcal{C}(\mathcal{J}_f^\omega([z_{i-1}) \cap (z_i])) \to \mathcal{C}(L)$, and furthermore the restriction map η: $\mathcal{C}(\mathcal{J}_f^\omega((z_i])) \to \mathcal{C}(L)$ is a surjective homomorphism. This we have the following diagram.

$$\begin{array}{ccccc}
\mathcal{C}\big(\mathcal{J}_f^\omega((z_i])\big) & \overset{\eta}{\to} & \mathcal{C}(L) & \overset{\pi^{-1}}{\to} & \mathcal{C}\big(\mathcal{J}_f([z_{i-1}) \cap (z_k])\big) \\
\lambda^{-1}\uparrow & & & & \downarrow\mu \\
\mathcal{C}^\omega((z_i]) & & \xrightarrow{\qquad\quad\rho\quad\qquad} & & \mathcal{C}^\omega([z_{i-1}) \cap (z_i]).
\end{array}$$

It is straightforward to check that this diagram commutes, and that consequently ρ is a surjective homomorphism.

We proceed by induction to show that the restriction map ψ: $C^\omega((z_i]) \to \mathcal{C}^\omega([a) \cap (z_i])$ is a surjective homomorphism. As in the proof of Proposition 9.1, it suffices to show that ψ_i is a homomorphism, and the proof so far has shown that ψ_{i-1}: $\mathcal{C}^\omega((z_{i-1}]) \to \mathcal{C}^\omega([a) \cap (z_{i-1}])$ and ρ: $\mathcal{C}^\omega((z_i]) \to \mathcal{C}^\omega([z_{i-1}) \cap (z_i])$ are surjective homomorphisms, as is ν: $\mathcal{C}^\omega((z_i]) \to \mathcal{C}^\omega((z_{i-1}])$. Observe that $x \equiv y(\psi_i(\Gamma))$ if and only if $x \wedge z_{i-1} \equiv y \wedge z_{i-1}(\psi_{i-1}(\nu(\Gamma)))$ and $x \vee z_{i-1} \equiv y \vee z_{i-1}(\rho(\Gamma))$. Consequently if $x \equiv y(\psi_i(\Gamma_1 \vee \Gamma_2))$ then $x \vee z_{i-1} \equiv y \wedge z_{i-1}(\psi_{i-1}(\nu(\Gamma_1)) \vee \psi_{i-1}(\nu(\Gamma_2)))$ and $x \vee z_{i-1} \equiv y \vee z_{i-1}(\rho(\Gamma_1) \vee \rho(\Gamma_2))$. Finally $\psi_{i-1}(\nu(\Gamma_1)) \vee \psi_{i-1}(\nu(\Gamma_2)) \subseteq \psi_i(\Gamma_1) \vee \psi_i(\Gamma_2)$ and $\rho(\Gamma_1) \vee \rho(\Gamma_2) \subseteq \psi_i(\Gamma_1) \vee \psi_i(\Gamma_2)$, and hence $x \equiv y(\psi_i(\Gamma_1) \vee \psi_i(\Gamma_2))$.

Our aim is to prove the converse of this theorem under the additional assumption that S is finitely derivable.

LEMMA 9.4. *Let S be a finitely derivable weakly distributive semilattice with full congruence restriction. Suppose $x, y, a \in S$ are such that $x, y \leqslant a$ and $x \vee y$ does not exist in S. Then there exists $z \in S$ with $x \leqslant z \leqslant a$ and $z \ngeqslant y$ such that $\langle z, y\rangle_\omega \neq \langle x, y\rangle_\omega$.*

PROOF. Since S has full congruence restriction so does $(a]$ for each $a \in S$. Let x, y, a be as in the statement of the lemma and work within $(a]$.

Let $J = \{b \in (a]: b \geqslant x$ and $\langle b, y\rangle_\omega = \langle x, y\rangle_\omega\}$. Clearly J is an ideal in $[x)$. We claim that $[x)\backslash J \neq [x) \cap [y)$. For if this were not the case then J would be a prime ideal in $[x)$ and the partition $\Gamma = \{J, [x)\backslash J\}$ would be a join partial congruence on $[x)$. Now consider $\Theta((y])$, $\Psi([y))$ and the restriction map ρ: $C^\omega((a]) \to C^\omega([x) \cap (a])$. Then, $\rho(\Theta((y]) \vee \Psi([y))) = \rho(E) = E_{[x)}$. Also $b \equiv x(\rho(\Theta((y])))$ if and only if $b \in J$, and os J is a congruence class of $\rho(\Theta((y])),$

where $\rho(\Theta((y]))\subseteq\Gamma$. Since $[y)$ is a congruence class of $\Psi([y))$, $[x)\backslash J=[x)\cap[y)$ is a congruence class of $\rho(\Psi([y)))$ and hence $\rho(\Psi([y)))\subseteq\Gamma$. This shows that

$$\rho(\Theta((y])\vee\Psi([y)))\neq E_{[x)}$$

which contradicts the assumption that ρ is a lattice homomorphism. Thus our claim that $[x)\cap[y)\neq[x)\backslash J$ has been verified. Since $[x)\cap[y)\subseteq[x)\backslash J$ there exists z in $(a]$ such that $z\geqslant x$, $z\notin J$ and $z\ngeqslant y$, which is the desired result.

THEOREM 9.5. *Let S be a fintely derivable weakly distributive semilattice with full congruence restriction. Then S is strongly connected.*

PROOF. Let $x\leqslant b$ be arbitrary elements of S. We have to show that x is connected to b with complexity 0. If $x\vee y$ exists for all $y\leqslant b$ then there is nothing to prove, so suppose the contrary. Let P be the partially ordered set of all non-principal ideals of the form $\langle x,y\rangle_\omega$ where y varies over $(b]$. By assumption P is finite and has a finite number of minimal elements, any $\langle x,y_0\rangle_\omega,\dots,\langle x,y_n\rangle_\omega$ (all distinct). Set $a=b$, $y=y_0$ and choose z_0 such that $z_0\ngeqslant y$, $z_0\geqslant x$ and $\langle z_0,y_0\rangle_\omega\neq\langle x,y_0\rangle_\omega$, as in the lemma. If $\langle x,y_i\rangle_\omega\subseteq(z_0]$ for some i, then set $a=z_0$, $y=y_i$ and choose z_1 as in the lemma. Continue this process until $\langle x,y_i\rangle_\omega\nsubseteq(z_k]$ for each $i=1,\dots,n$. Set $x_1=z_k$. Then $x<x_1$ and $x\vee y$ exists for all $y\leqslant x_1$. Repeat the whole process with x_1 instead of x, and hence find an $x_1\geqslant x_1$ such that $x_1\vee y$ exists in S for all $y\leqslant x_2$. Continuing, we get a sequence $x=x_0<x_1<x_2<\cdots<b$ such that x_{i-1} is strong below x_i. It remains to show that this sequence terminates. Consider the following set of non-principal finitely generated ideals,

$$X=\left\{\left\langle x_0,y^0\right\rangle_\omega,\left\langle x_1,y^1\right\rangle_\omega,\dots,\left\langle x_n,y^n\right\rangle_\omega,\dots\right\}$$

where y^i is the final y used in the construction of x_{i+1}. We claim that $\langle x_i,y^i\rangle_\omega\neq\langle x_{i+1},y^{i+1}\rangle_\omega$ for each i. Otherwise we would have, for some i, $x_{i+1}\in\langle x_i,y^i\rangle_\omega$ and so $\langle x_{i+1},y^i\rangle_\omega=\langle x_i,y^i\rangle_\omega$, a contradiction. This also shows that $x_j\notin\langle x_i,y^i\rangle_\omega$ for each $j>i$ and so, for each i,j with $i\neq j$,

$$\left\langle x_i,y^i\right\rangle_\omega\neq\left\langle x_j,y^j\right\rangle_\omega.$$

Since S is fintely derivable, X is finite and so $x=x_0<x_1<x_2<\cdots<b$ is a sequence which connects x to b with complexity 0.

COROLLARY 9.6. *A finitely derivable weakly distributive semilattice with full congruence restriction is congruence distributive. Consequently a finitely derivable distributive semilattice with full congruence restriction has the upper bound property.*

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The final part of this corollary is actually true without the assumption of finite derivability.

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