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AN ANALOGUE OF EULER'S IDENTITY AND SPLIT PERFECT PARTITIONS

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Abstract

We give the generating function of split $(n + t)$-colour partitions and obtain an analogue of Euler's identity for split n-colour partitions. We derive a combinatorial relation between the number of restricted split n-colour partitions and the function $\sigma_k(\mu) = \sum_{d|\mu} d^k$. We introduce a new class of split perfect partitions with $d(a)$ copies of each part a and extend the work of Agarwal and Subbarao ['Some properties of perfect partitions', *Indian J. Pure Appl. Math* **22**(9) (1991), 737–743].

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1. Introduction and definitions

For a natural number λ, the rising q-factorial of a with base q is defined by $(a; q)_0 = 1$ and $(a; q)_\lambda = (1 - a)(1 - aq) \cdots (1 - aq^{\lambda-1})$, where $|q| < 1$. Any series involving this rising q-factorial is called a q-series (or basic series or Eulerian series).

Definition 1.1. The partition function $p(n)$ represents the number of distinct ways of representing n as a sum of natural numbers (with order irrelevant). The generating function of $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1 - q^n}.$$

The first result in the history of partitions is Euler's famous discovery for ordinary partitions.

Theorem 1.2 (Euler's identity). *The number of partitions of a positive integer n into distinct parts equals the number of partitions of n into odd parts.*

Let us recall the celebrated Rogers–Ramanujan identities

$$\sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2}}{(q;q)_\lambda} = \prod_{\lambda=1}^{\infty} \frac{1}{(1-q^{5\lambda-1})(1-q^{5\lambda-4})},$$
$$\sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2+\lambda}}{(q;q)_\lambda} = \prod_{\lambda=1}^{\infty} \frac{1}{(1-q^{5\lambda-2})(1-q^{5\lambda-3})}.$$

MacMahon [18] interpreted the Rogers–Ramanujan identities combinatorially by using the ordinary partition function. Several identities of Rogers–Ramanujan type have been interpreted combinatorially by means of ordinary partitions (see, for example, [2, 9, 12, 13]). In 1987, Agarwal and Andrews [1] introduced and studied the following generalised partition function $P_{n+t}(\mu)$.

Definition 1.3. For $t \geq 0$, the partition function $P_{n+t}(\mu)$ represents the number of distinct ways of representing μ as a sum of natural numbers (with order irrelevant) using $(n+t)$ copies of each part n. The partitions are called $(n+t)$-colour partitions. The generating function for $(n+t)$-colour partitions is given by

$$\sum_{\mu=0}^{\infty} P_{n+t}(\mu) q^{\mu} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{n+t}}.$$

The $(n+t)$-colour partitions are used as a combinatorial tool to interpret many more q-series identities combinatorially in [1, 16, 17]. An n-colour analogue of Euler's identity can be obtained by the use of generating functions [4].

Theorem 1.4. *The number of n-colour partitions of μ into distinct parts equals the number of n-colour partitions of μ such that the parts are either odd or even with even subscripts only.*

Recently, Agarwal and Sood [6] further generalised the $(n+t)$-colour partition function $P_{n+t}(\mu)$ to the split $(n+t)$-colour partition function $SP_{n+t}(\mu)$.

Definition 1.5. The partition function $SP_{n+t}(\mu)$ represents the number of distinct ways of representing μ as a sum of natural numbers (with order irrelevant) using $(n+t)$ copies of each part n such that the subscript p of each part a_p is further split into two parts (with order relevant): green colour (g) and red colour (r), such that $1 \leq g \leq p$, $0 \leq r \leq p-1$ and $p = g + r$. These partitions are called split $(n+t)$-colour partitions.

As shown in [6], this new set of partitions is very helpful in interpreting q-series identities combinatorially when they cannot be interpreted combinatorially using ordinary partitions or $(n+t)$-colour partitions (see, for instance, [5, 15, 20]). Using split $(n+t)$-colour partitions, Agarwal and Sood [6] interpreted two basic identities of Gordon and MacIntosh [14] combinatorially.

The purpose of this paper is to study analytical aspects of split $(n+t)$-colour partitions. In Section 2, we obtain the generating functions of split $(n+t)$-colour partitions and several restricted split n-colour partition functions. Then we give an analytic proof of an analogue of Euler's identity for split n-colour partitions. Further, we establish a combinatorial connection between a certain class of restricted split n-colour partitions and the number theoretic function $\sigma_k(\mu) = \sum_{d|\mu} d^k$. In Section 3, we define new sets of partitions, split partitions and split perfect partitions with $d(a)$ copies of a and we extend the work of Agarwal and Subbarao [7]. Before we state our main results, we first recall some definitions.

Definition 1.6 [19]. A *perfect partition* of a nonnegative integer n is a partition that contains exactly one partition of every $m < n$ when repeated parts are regarded as indistinguishable.

Example 1.7. There are four perfect partitions of 7:

$$4+1+1+1, \quad 4+2+1, \quad 2+2+2+1 \quad \text{and} \quad 1+1+1+1+1+1+1.$$

Definition 1.8 [1]. A *partition with $(n+t)$ copies of n*, $t \geq 0$, is a partition in which a part of size n, $n \geq 0$, can come in $(n+t)$ different colours that are denoted by subscripts: $n_1, n_2, \ldots, n_{n+t}$. Note that zeros are permitted if and only if t is greater than or equal to one. Also, zeros are not permitted to repeat in any partition.

Remark 1.9. If we take $t = 0$, then these partitions are just the n-colour partitions.

Definition 1.10. The *weighted difference* of two parts a_p, b_q $(a \geq b)$ is defined by $a - b - p - q$ and is denoted by $((a_p - b_q))$.

Definition 1.11 [6]. Let a_p be a part in an $(n+t)$-colour partition of a nonnegative integer μ. We split the colour 'p' into two parts, 'the green part' and 'the red part', and denote them by 'g' and 'r', respectively, such that $1 \leq g \leq p$, $0 \leq r \leq p-1$ and $p = g + r$. An $(n+t)$-colour partition in which each part is split in this manner is called a split $(n+t)$-colour partition.

Example 1.12. In 5_{2+1}, the green part is 2 and the red part is 1.

Remark 1.13. If $r = 0$, then we will not write it. Thus, we will write 5_3 for 5_{3+0}.

Definition 1.14 [3]. A *partition with $d(a)$ copies of a*, where $d(a)$ is the number of positive divisors of a, is a partition of μ in which a part of size a can come in $d(a)$ different colours that are denoted by subscripts $a_1, a_2, \ldots, a_{d(a)}$.

Example 1.15. There are five partitions of 3 with $d(a)$ copies of a:

$$3_1, \quad 3_2, \quad 2_1 + 1_1, \quad 2_2 + 1_1 \quad \text{and} \quad 1_1 + 1_1 + 1_1.$$

Definition 1.16 [7]. A *partition of μ with $d(a)$ copies of a* is perfect if it contains exactly one partition with $d(a)$ copies of a of each $\nu < \mu$ when repeated parts are regarded as indistinguishable.

Example 1.17. There are only three perfect partitions of 3 with $d(a)$ copies of a:

$$2_1 + 1_1, \quad 2_2 + 1_1, \quad \text{and} \quad 1_1 + 1_1 + 1_1.$$

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2. Generating function and split *n*-colour analogue of Euler's identity

In this section, we derive the generating function of split $(n+t)$-colour partitions. Using this generating function and some straightforward partition techniques, we give generating functions for several restricted split n-colour partition functions and we deduce a split n-colour analogue of Euler's identity using generating functions.

2.1. Generating function of split *n*-colour partitions. Let $SP_{n+t}(\mu)$ denote the partition function for split $(n+t)$-colour partitions. From Definition 1.11, it is clear that in a split $(n+t)$-colour partition of μ, the subscript p of each part a_p of an $(n+t)$-colour partition of μ is further partitioned into at most two parts (with order relevant). Thus in a split $(n+t)$-colour partition of μ, we use $\binom{p-1}{0}+\binom{p-1}{1}=p$ copies of each part a_p of an $(n+t)$-colour partition of μ. Thus a part of size 'a' can come in $1+2+\cdots+a+t=\frac{1}{2}(a+t)(a+t+1)$ different colours. Hence we have the generating function

$$\sum_{\mu=0}^{\infty} SP_{n+t}(\mu)q^{\mu}=\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^{(n+t)(n+t+1)/2}}\quad\text{for all } t\geq 0.$$

When $t=0$, we get the generating function for split n-colour partitions given by

$$\sum_{\mu=0}^{\infty} SP_{n}(\mu)q^{\mu}=\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^{n(n+1)/2}}=1+q+4q^2+10q^3+26q^4+\cdots.$$

EXAMPLE 2.1. The ten split n-colour partitions of 3 are

$$3_1,\ 3_2,\ 3_3,\ 3_{1+1},\ 3_{2+1},\ 3_{1+2}, 2_2+1_1,\ 2_1+1_1,\ 2_{1+1}+1_1, 1_1+1_1+1_1.$$

2.2. Generating functions for several restricted split *n*-colour partitions. Let $SP_n(T,U,V,W,\mu)$ be a restricted split n-colour partition function that counts the split n-colour partitions of μ of the form $a_{p=g+r}$ such that $a\in T$, $p\in U$, $g\in V$ and $r\in W$. We denote the set of all positive integers, the set of all odd positive integers, the set of all even positive integers, the set of whole numbers and the set of all distinct positive integers by N, O, E, W and D, respectively. Now using the standard techniques of partition theory [8], we obtain the generating functions shown in Table 1.

2.3. A split *n*-colour analogue of Euler's identity.

THEOREM 2.2. *Let $A(\mu)$ denote the number of split n-colour partitions of μ with distinct parts and let $B(\mu)$ denote the number of split n-colour partitions of μ in which the even parts that have both green and red colours odd together are not allowed. Then $A(\mu)=B(\mu)$.*

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TABLE 1. Generating functions of restricted split n-colour partitions.

Partition function	Generating function	Partition function	Generating function
$1+\sum_{\mu=1}^{\infty} SP_n(E,N,N,W,\mu)q^{\mu}$	$\prod_{n=1}^{\infty} 1/(1-q^{2n})^{n(2n+1)}$	$1+\sum_{\mu=1}^{\infty} SP_n(O,E,E,E\cup\{0\},\mu)q^{\mu}$	$\prod_{n=1}^{\infty} 1/(1-q^{2n-1})^{n(n-1)/2}$
$1+\sum_{\mu=1}^{\infty} SP_n(O,N,N,W,\mu)q^{\mu}$	$\prod_{n=1}^{\infty} 1/(1-q^{2n-1})^{n(2n-1)}$	$1+\sum_{\mu=1}^{\infty} SP_n(E,E,O,O,\mu)q^{\mu}$	$\prod_{n=1}^{\infty} 1/(1-q^{2n})^{n(n+1)/2}$
$1+\sum_{\mu=1}^{\infty} SP_n(E,E,N,W,\mu)q^{\mu}$	$\prod_{n=1}^{\infty} 1/(1-q^{2n})^{n(n+1)}$	$1+\sum_{\mu=1}^{\infty} SP_n(O,E,O,O,\mu)q^{\mu}$	$\prod_{n=1}^{\infty} 1/(1-q^{2n-1})^{n(n+1)/2}$
$1+\sum_{\mu=1}^{\infty} SP_n(O,O,N,W,\mu)q^{\mu}$	$\prod_{n=1}^{\infty} 1/(1-q^{2n-1})^{n^2}$	$1+\sum_{\mu=1}^{\infty} SP_n(D,N,N,W,\mu)q^{\mu}$	$\prod_{n=1}^{\infty} (1+q^{n})^{n(n+1)/2}$
$1+\sum_{\mu=1}^{\infty} SP_n(E,O,N,W,\mu)q^{\mu}$	$\prod_{n=1}^{\infty} 1/(1-q^{2n})^{n^2}$	$1+\sum_{\mu=1}^{\infty} SP_n(D\cap E,N,N,W,\mu)q^{\mu}$	$\prod_{n=1}^{\infty} (1+q^{2n})^{n(2n+1)}$
$1+\sum_{\mu=1}^{\infty} SP_n(O,E,N,W,\mu)q^{\mu}$	$\prod_{n=1}^{\infty} 1/(1-q^{2n-1})^{n(n-1)}$	$1+\sum_{\mu=1}^{\infty} SP_n(D\cap O,N,N,W,\mu)q^{\mu}$	$\prod_{n=1}^{\infty} (1+q^{2n-1})^{n(2n-1)}$
$1+\sum_{\mu=1}^{\infty} SP_n(E,E,E,E\cup\{0\},\mu)q^{\mu}$	$\prod_{n=1}^{\infty} 1/(1-q^{2n})^{n(n+1)/2}$	$1+\sum_{\mu=1}^{\infty} SP_n(D\cap E,E,E,E\cup\{0\},\mu)q^{\mu}$	$\prod_{n=1}^{\infty} (1+q^{2n})^{n(n+1)/2}$

PROOF. We use the generating functions for $SP_n(E,E,E,E\cup\{0\},\mu)$, $SP_n(E,O,N,W,\mu)$ and $SP_n(O,N,N,W,\mu)$ from Table 1.

$$\begin{aligned}
1+\sum_{\mu=1}^{\infty} A(\mu)q^{\mu} &= \prod_{n=1}^{\infty}(1+q^n)^{n(n+1)/2}\\
&= \prod_{n=1}^{\infty}\left(\frac{1-q^{2n}}{1-q^n}\right)^{n(n+1)/2}\\
&= \prod_{n=1}^{\infty}\frac{(1-q^{2n})^{n(n+1)/2}}{(1-q^{2n})^{n(2n+1)}(1-q^{2n-1})^{n(2n-1)}}\\
&= \prod_{n=1}^{\infty}\frac{1}{(1-q^{2n})^{n(n+1)/2}(1-q^{2n})^{n^2}(1-q^{2n-1})^{n(2n-1)}}\\
&= 1+\sum_{\mu=1}^{\infty} B(\mu)q^{\mu}.
\end{aligned}$$
□

EXAMPLE 2.3. For example, $A(3) = B(3) = 9$ and the corresponding partitions are

$$\begin{aligned}
A(3):&\quad 3_1, 3_2, 3_3, 3_{1+1}, 3_{1+2}, 3_{2+1}, 2_1+1_1, 2_2+1_1, 2_{1+1}+1_1,\\
B(3):&\quad 3_1, 3_2, 3_3, 3_{1+1}, 3_{1+2}, 3_{2+1}, 2_1+1_1, 2_2+1_1, 1_1+1_1+1_1.
\end{aligned}$$

REMARK 2.4. From Table 1, the generating function of $SP_n(E,E,E,E\cup\{0\},\mu)$ is the same as the generating function of $SP(E,E,O,O,\mu)$. So we can restate Theorem 2.2 in the following way.

THEOREM 2.5. *Let $A(\mu)$ denote the number of split n-colour partitions of μ with distinct parts and let $C(\mu)$ denote the number of split n-colour partitions of μ in which the even parts that have both green and red colours even together are not allowed. Then $A(\mu) = C(\mu)$.*

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Example 2.6. Here $C(3) = 9$ and the corresponding partitions are

$$3_1, 3_2, 3_3, 3_{1+1}, 3_{1+2}, 3_{2+1}, 2_1 + 1_1, 2_{1+1} + 1_1, 1_1 + 1_1 + 1_1.$$

2.4. Combinatorial relation between restricted split n-colour partitions and $\sigma_k(\mu)$.

Definition 2.7. The number theoretic function $\sigma_k(\mu)$ is defined by

$$\sigma_k(\mu) = \sum_{d|\mu} d^k \quad \text{for all } \mu \geq 1.$$

Theorem 2.8. *Let $E(\mu)$ denote the number of split n-colour partitions of μ into equal parts. Then*

$$E(\mu) = \tfrac{1}{2}(\sigma(\mu) + \sigma_2(\mu)).$$

Proof. For any positive integer μ, let $\Pi = \underbrace{\lambda_\nu + \lambda_\nu + \cdots + \lambda_\nu}_{k}$ be the split n-colour partition of μ into equal parts, each of size λ_ν. Then, obviously, we must have $k \times \lambda_\nu = \mu$ or $\lambda_\nu \mid \mu$. Thus, for each divisor d of μ, we have $\frac{1}{2}d(d+1)$ split n-colour partitions of μ with equal parts of size d. Hence the total number of split n-colour partitions of μ into equal parts is

$$E(\mu) = \sum_{d|\mu} \frac{d(d+1)}{2} = \sum_{d|\mu} \frac{1}{2}(d + d^2) = \frac{1}{2}(\sigma(\mu) + \sigma_2(\mu)).$$

□

3. Split perfect partitions with $d(a)$ copies of a

Analogous to the definitions of partitions with $d(a)$ copies of a and perfect partitions with $d(a)$ copies of a, we define split partitions with $d(a)$ copies of a and split perfect partitions with $d(a)$ copies of a.

Definition 3.1. Let a_q be a part in a partition with $d(a)$ copies of a of an integer μ. We split the colour 'q' into two parts, 'the green part' and 'the red part', and we denote them by 'g' and 'r', respectively, such that $1 \leq g \leq d(a)$, $0 \leq r \leq d(a) - 1$ and $q = g + r$. A partition with $d(a)$ copies of a in which each part is split in this manner is called a split partition with $d(a)$ copies of a.

Example 3.2. There are seven split partitions of 3 with $d(a)$ copies of a:

$$3_1,\ 3_2, 3_{1+1},\ 2_1 + 1_1,\ 2_2 + 1_1, 2_{1+1} + 1_1, 1_1 + 1_1 + 1_1.$$

Let $SP_{d(a)}(\mu)$ denote the partition function of split partitions with $d(a)$ copies of a. Then the generating function of $SP_{d(a)}(\mu)$ is given by

$$1 + \sum_{\mu=1}^{\infty} SP_{d(a)}(\mu) q^\mu = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{d(n)(d(n)+1)/2}}.$$

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Definition 3.3. A *split partition of μ with $d(a)$ copies of a* is perfect if it contains exactly one split partition with $d(a)$ copies of a of each $\nu < \mu$ when repeated parts are regarded as indistinguishable.

Example 3.4. There are only four split perfect partitions of 3 with $d(a)$ copies of a:

$$2_1 + 1_1,\ 2_2 + 1_1, 2_{1+1} + 1_1, 1_1 + 1_1 + 1_1.$$

The following theorem is a direct consequence of Definition 3.3.

Theorem 3.5. *Let $SPP_{d(a)}(\mu)$ denote the number of split perfect partitions of μ with $d(a)$ copies of a. Let $\Pi = {p_1}^{q_1}{p_2}^{q_2}\cdots {p_l}^{q_l}$ be an ordinary perfect partition of μ in which the part p_i repeats q_i times. Then*

$$SPP_{d(a)}(\mu) = \sum_{\Pi}\Big(\frac{d(p_1)(d(p_1)+1)}{2}\Big)\Big(\frac{d(p_2)(d(p_2)+1)}{2}\Big)\cdots\Big(\frac{d(p_l)(d(p_l)+1)}{2}\Big).$$

Example 3.6. For $\mu = 5$, there are three perfect ordinary partitions: $3 + 1^2$, $2^2 + 1$ and 1^5. This implies that

$$\begin{aligned} SPP_{d(a)}(5) &= \Big(\frac{d(3)(d(3)+1)}{2}\Big)\Big(\frac{d(1)(d(1)+1)}{2}\Big) + \Big(\frac{d(2)(d(2)+1)}{2}\Big)\Big(\frac{d(1)(d(1)+1)}{2}\Big) \\ &\quad + \Big(\frac{d(1)(d(1)+1)}{2}\Big) \\ &= \Big(\frac{2.3}{2}\Big)\Big(\frac{1.2}{2}\Big) + \Big(\frac{2.3}{2}\Big)\Big(\frac{1.2}{2}\Big)\Big(\frac{1.2}{2}\Big) = 3 + 3 + 1 = 7. \end{aligned}$$

The seven split perfect partitions of 5 with $d(a)$ copies of a are: $3_1 + 1_1 + 1_1$, $3_2 + 1_1 + 1_1$, $3_{1+1} + 1_1 + 1_1$, $2_1 + 2_1 + 1_1$, $2_2 + 2_2 + 1_1$, $2_{1+1} + 2_{1+1} + 1_1$, $1_1 + 1_1 + 1_1 + 1_1 + 1_1$.

Remark 3.7. It can be easily seen that the split partition $\underbrace{1_1 + 1_1 + \cdots 1_1}_{\mu}$ of any positive integer μ and the split partitions $\underbrace{2_1 + 2_1 + \cdots 2_1}_{\lambda} + 1_1$, $\underbrace{2_2 + 2_2 + \cdots 2_2}_{\lambda} + 1_1$ and $\underbrace{2_{1+1} + 2_{1+1} + \cdots 2_{1+1}}_{\lambda} + 1_1$ of any odd positive integer $\mu = 2\lambda + 1$ are always perfect. These four partitions are called trivial split perfect partitions of μ with $d(a)$ copies of a.

From a result of Efang [11, Corollary 2, page 268], one can easily get the following theorem.

Theorem 3.8. *A positive integer μ has a nontrivial split perfect partition with $d(a)$ copies of a into distinct parts if and only if*

$$\mu = 2^{\lambda+1} - 1 \quad \textit{where } \lambda = 2, 3, 4, \ldots.$$

Finally, we establish a combinatorial connection between split perfect partitions with $d(a)$ copies of a and the factorial function by giving an explicit formula for the number of nontrivial split perfect partitions of μ with $d(a)$ copies of a into distinct parts.

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THEOREM 3.9. *Let $D(\mu)$, for $\mu = 2^{\lambda+1} - 1$ and $\lambda = 2, 3, 4, \ldots$, be the number of nontrivial split perfect partitions of μ with $d(a)$ copies of a into distinct parts. Then*

$$D(\mu) = \frac{(\lambda+1)!\,(\lambda+2)!}{2^{\lambda+1}}.$$

PROOF. For any integer $\mu = 2^{\lambda+1} - 1$, $\lambda = 2, 3, 4, \ldots$, the only nontrivial perfect partition into distinct parts is $1 + 2 + 2^2 + 2^3 + \cdots + 2^\lambda$. Thus the number of corresponding nontrivial split perfect partitions of μ with $d(a)$ copies of a into distinct parts is

$$\frac{d(1)(d(1)+1)}{2} \times \frac{d(2)(d(2)+1)}{2} \times \frac{d(2^2)(d(2^2)+1)}{2} \times \cdots \times \frac{d(2^\lambda)(d(2^\lambda)+1)}{2}$$
$$= \frac{1.2}{2} \times \frac{2.3}{2} \times \frac{3.4}{2} \times \cdots \times \frac{(\lambda+1)(\lambda+2)}{2} = \frac{(\lambda+1)!\,(\lambda+2)!}{2^{\lambda+1}}. \qquad \square$$

4. Conclusion

Agarwal and Mullen [3] proved that the number of partitions of n with $d(a)$ copies of a is the same as the number of factorisation patterns of n and they studied the graphical representation of these partitions. It will be interesting to explore the existence of similar results for split partitions of μ with $d(a)$ copies of a and to study the graphical aspects of split perfect partitions. Dai *et al.* [10], gave bounds on the number of odd k-perfect numbers using the multiplicative partition function $f(m)$, that is, the number of ways of factorising m into a product of integers greater than one. It will be interesting to see whether such bounds can be established for split perfect partitions.

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