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TIME-ISOLATED SINGULARITIES OF TEMPERATURES

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Abstract

We study singularities of solutions of the heat equation, that are not necessarily isolated but occur only in a single characteristic hyperplane. We prove a decomposition theorem for certain solutions on $D_+ = D \cap (\mathbf{R}^n \times]0, \infty[)$, for a suitable open set D, with singularities at a compact subset K of $\mathbf{R}^n \times \{0\}$, in terms of Gauss-Weierstrass integrals. We use this to prove a representation theorem for certain solutions on D_+, with singularities at K, as the sums of potentials and Dirichlet solutions. We also give conditions under which K is removable for solutions on $D \backslash K$.

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1. Introduction

Let D be an open subset of $\mathbf{R}^{n+1} = \{(x, t) : x \in \mathbf{R}^n, t \in \mathbf{R}\}$, let $D(0) = \{x \in \mathbf{R}^n : (x, 0) \in D\} \neq \emptyset$, let $D_+ = D \cap (\mathbf{R}^n \times \,]0, \infty[)$ and let $H^{\Delta}(D)$ be the family of all temperatures on D that can be written as a difference of nonnegative temperatures. The central result of this paper, Theorem 2, gives conditions under which an element u of $H^{\Delta}(D_+)$ can be written in the form $u = W\mu + W\psi + w$, where μ is a signed measure supported in a compact subset C of $D(0)$, ψ is a locally integrable function on $D(0)$ such that $W|\psi| < \infty$ on D_+, and w is a temperature on D_+ that can be extended by zero to a temperature on D. Here

$$W\mu(x, t) = \int_{\operatorname{supp}\mu} W(x - y, t)\, d\mu(y)$$

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and

$$W\psi(x,t) = \int_{\text{supp}\,\psi} W(x-y,t)\psi(y)dy,$$

with $W(x,t) = (4\pi t)^{-n/2}\exp(-\|x\|^2/4t)$ for all $(x,t) \in \mathbf{R}^{n+1}_+$. Because w tends to zero at $D(0)\times\{0\}$, this decomposition enables us to use theorems on Gauss-Weierstrass integrals to prove results about temperatures in any $H^{\Delta}(D_+)$.

In Section 3, we use the decomposition theorem to prove a representation theorem, which extends one established by Aronson [2] for solutions of a wide class of parabolic partial differential equations on $B(0,\rho)\times\,]0,T[$ with singularities at $(0,0)$. Working only with temperatures, we are able to considerably weaken the constraints on the solutions, replace the circular cylinder by an arbitrary D_+, and replace the point of singularity by an arbitrary compact subset K of $D(0)\times\{0\}$. A representation of the form $u = G_D\mu + h$ is obtained, where $G_D\mu$ is the potential on D of a signed measure supported in K, and h is a Dirichlet solution on D_+.

In Section 4, we consider temperatures u on $D\backslash(C\times\{0\})$ for an arbitrary compact subset C of $D(0)$, and give a mild constraint which ensures that they can be written as the sum of a temperature on D and the potential of a signed measure supported in $C\times\{0\}$. The idea here is that, because the restriction of u to $D\backslash\overline{D}_+$ has a continuous extension to u^* (say) on $D\backslash D_+$, we can take $\psi = u^*(\cdot,0)$ in the decomposition theorem, so that $W\psi + w$ can be extended to a temperature on D. Given this result, known conditions which imply that μ is null can be converted into conditions for $C\times\{0\}$ to be removable.

Many other papers have been written about removable singularities, including [1, 5–7]. Isolated singularities of nonnegative temperatures have been characterized by Widder [16, p. 119], and those of arbitrary temperatures by Chung and Kim [3].

2. The decomposition theorem

If μ and ψ are, respectively, a measure and a function defined on a subset of $\mathbf{R}^N$, they are assumed to be extended by zero to the whole space. Their restrictions to a set A are denoted by μ_A and ψ_A.

A temperature u on D_- is called *initially zero* if $u(x,t)\to 0$ as $(x,t)\to(y,0+)$ for all $y\in D(0)$.

A family $\mathscr{F}$ of closed balls is called an *abundant Vitali covering* of $\mathbf{R}^n$ if, given $x\in\mathbf{R}^n$ and $\epsilon>0$, $\mathscr{F}$ contains uncountably many balls with centre x and radius less than ϵ. See [13] for a discussion.

The proof of the decomposition theorem requires a preliminary theorem.

THEOREM 1. *Suppose that* $u = W\mu + v$ *on* D_+, *where* v *is an initially zero*

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temperature, and μ *is a signed measure concentrated on* $D(0)$ *such that* $W|\mu| < \infty$ *on* D_+*. Let* $\mathscr{F}$ *be an abundant Vitali covering of* $\mathbf{R}^n$*. If there is a signed measure* ν *concentrated on* $D(0)$ *such that*

$$\lim_{t\to 0+} \int_{A\cap V} u(x,t)dx = \nu(A\cap V) \tag{1}$$

whenever $A, V \in \mathscr{F}$, $V \subseteq D(0)$, *and* $A \cap V \neq \emptyset$, *then* $\mu = \nu$.

PROOF. By [13, Theorem 7.3(i)], there is an abundant Vitali covering $\mathscr{F}_0 \subseteq \mathscr{F}$ such that $|\mu|(\partial A) = 0$ for all $A \in \mathscr{F}_0$. Given $V \in \mathscr{F}_0$ such that $V \subseteq D(0)$, put $w_V = W\mu_V$ and $w_{D\setminus V} = W\mu_{D(0)\setminus V}$. Then $w_V = u - v - w_{D\setminus V}$ on D_+.

If $A \in \mathscr{F}_0$ and $A \cap V \neq \emptyset$, then $A \cap V$ is a compact subset of $D(0)$, so that

$$\lim_{t\to 0+} \int_{A\cap V} v(x,t)dx = 0. \tag{2}$$

Furthermore, because the boundaries of $A \cap V$ and $A\setminus V$ are both μ-null, it follows from [13, Theorem 7.2(i)] that

$$\lim_{t\to 0+} \int_{A\cap V} w_{D\setminus V}(x,t)dx = 0 = \lim_{t\to 0+} \int_{A\setminus V} w_V(x,t)dx. \tag{3}$$

Combining (1), (2) and (3), we obtain

$$\lim_{t\to 0+} \int_A w_V(x,t)dx = \lim_{t\to 0+} \int_{A\cap V} w_V(x,t) = \nu_V(A).$$

On the other hand, if $A \in \mathscr{F}_0$ and $A \cap V = \emptyset$, then it follows from [13, Theorem 7.2(i)] that

$$\lim_{t\to 0+} \int_A w_V(x,t)dx = 0 = \nu_V(A).$$

Therefore $\mu_V = \nu_V$, by [13, Theorem 7.3(ii)].

Given any open subset U of $D(0)$, choose a sequence of sets $\{V_k\}$ in $\mathscr{F}_0$ with union U, and put $X_1 = V_1$, $X_j = V_j \setminus \cup_{k=1}^{j-1} V_k$ for all $j \geq 2$. Then, by the above,

$$\mu(U) = \sum_{j=1}^{\infty} \mu(X_j) = \sum_{j=1}^{\infty} \mu_{V_j}(X_j) = \sum_{j=1}^{\infty} \nu_{V_j}(X_j) = \nu(U).$$

The result now follows from the regularity of Radon measures. $\square$

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NOTE. If, in Theorem 1, $u(x, 0+)$ is finite whenever $x \in D(0)$ and the limit exists, then the same is true of $W\mu(x, 0+)$, and the two are equal. Therefore $d\mu(x) = u(x, 0+)dx$, by [12, Theorem 1], so that $u(\cdot, 0+)$ is locally integrable, and

$$\lim_{t\to 0+} \int_B u(x,t)dx = \int_B u(x, 0+)dx$$

for each bounded Borel subset B of $D(0)$ such that $m_n(\partial B) = 0$, by [13, Theorem 7.2].

THEOREM 2. *Suppose that $u \in H^\Delta(D_+)$, that C is a compact subset of $D(0)$, that $\mathscr{F}$ is an abundant Vitali covering of $\mathbf{R}^n$, and that ψ is a locally m_n-integrable function on $D(0)$ such that $W|\psi| < \infty$ on D_+. If*

$$\lim_{t\to 0+} \int_{A\cap V} u(x,t)dx = \int_{A\cap V} \psi(x)dx$$

whenever $A, V \in \mathscr{F}$, $V \subseteq D(0)\backslash C$, and $A \cap V \neq \emptyset$, then there exist a unique signed measure μ, supported in C and with finite total variation, and a unique initially zero temperature w on D_+, such that $u = W\mu + W\psi + w$ on D_+.

PROOF. By [15, Theorem 1], there is a unique signed measure ν on $D(0)$ with the following property. Given any bounded open set E such that $\overline{E} \subseteq D$ and $E(0) \neq \emptyset$, there is a unique initially zero temperature v on E_+ such that $u = W\nu_E + v$ on E_+.

Choose E such that $C \subseteq E(0)$. Applying Theorem 1 on $E\backslash(C \times \{0\})$, we obtain $d\nu_{E\backslash C}(y) = \psi_{E\backslash C}(y)dy$. Since C is compact $|\nu|(C) < \infty$, so that $W|\nu_E| \leq W|\nu_C| + W|\psi| < \infty$ on D_+. It follows that $W|\nu| < \infty$ on D_+, so that there is a unique initially zero temperature w on D_+ such that $u = W\nu + w$ on D_+, by [15, Theorem 1]. Putting $d\mu(y) = d\nu_C(y) - \psi_C(y)dy$, we obtain

$$u = W\nu + w = W\nu_C + W\psi_{D\backslash C} + w = W\mu + W\psi + w$$

on D_+, as asserted. □

REMARK. The measure ν, associated with $u \in H^\Delta(D_+)$ by [15, Theorem 1] and described in the first paragraph of the above proof, is called the *initial measure* of u.

If the initial measure of u is absolutely continuous with respect to m_n, then the following corollary may be easier to use than the theorem. Note that, for any $u \in H^\Delta(D_+)$, the limit $u(x, 0+)$ exists and is finite for m_n-almost every $x \in D(0)$, by [15, Theorem 2].

COROLLARY. *Suppose that $u \in H^{\Delta}(D_+)$, that C is a compact subset of $D(0)$, that $u(x, 0+)$ is finite whenever $x \in D(0)\backslash C$ and the limit exists, and that $W|u(\cdot, 0+)| < \infty$ on D_+. Then there exist a unique signed measure μ, supported in C and with finite total variation, and a unique initially zero temperature w on D_+, such that $u = W\mu + Wu(\cdot, 0+) + w$ on D_+.*

PROOF. Let ν be the initial measure of u, and let $\mathscr{F}$ be an abundant Vitali covering of $\mathbf{R}^n$. Given $A, V \in \mathscr{F}$ such that $V \subseteq D(0)\backslash C$ and $A \cap V \neq \emptyset$, choose a bounded open set E such that $\overline{E} \subseteq D$ and $A \cap V \subseteq E(0)$. If $F = E\backslash(C \times \{0\})$, then $u = W\nu_F + v$ on F_+ for some initially zero temperature v, and $u(\cdot, 0+)$ is finite whenever $x \in F(0)$ and the limit exists. Therefore, by the note following Theorem 1,

$$\lim_{t\to 0+} \int_{A\cap V} u(x, t)dx = \int_{A\cap V} u(x, 0+)dx.$$

Since $W|u(\cdot, 0+)| < \infty$ on D_+, $u(\cdot, 0+)$ is locally integrable on $D(0)$, and the result follows from Theorem 2. □

REMARK. If $C = \bigcup_{j=1}^{k}\{x_k\}$ in Theorem 2, then

$$u(x, t) = \sum_{j=1}^{k} \alpha_j W(x - x_j, t) + W\psi(x, t) + w(x, t),$$

where $\alpha_j = \mu(\{x_j\})$ for all j. We can show that

$$\alpha_j = \lim_{t\to 0+} \int_{B(x_j, r)} (u(x, t) - \psi(x))dx$$

for any r such that $\overline{B}(x_j, r) \subseteq D(0)$ and $\overline{B}(x_j, r) \cap C = \{x_j\}$. Given such an r, we have $|\mu|(\partial B(x_j, r)) = 0$, so that [13, Theorem 7.2(i)] implies that

$$\begin{aligned}\lim_{t\to 0+} \int_{B(x_j, r)} u(x, t)dx &= \lim_{t\to 0+} \int_{B(x_j, r)} (W\mu(x, t) + W\psi(x, t))dx \\ &= \mu(B(x_j, r)) + \int_{B(x_j, r)} \psi(x)dx \\ &= \alpha_j + \int_{B(x_j, r)} \psi(x)dx,\end{aligned}$$

as asserted. Compare [2, Theorem 3].

3. The representation theorem

In this section, we prove an extension of Aronson's representation theorem [2, Theorem 3]. This requires a hypothesis of Dirichlet regularity, which is to be understood in the sense of [10], as that allows us a much wider class of open sets than does the usual potential-theoretic sense in [4]. We therefore recall the necessary definitions.

Let $(y, s) \in \partial D$. We call (y, s) an *abnormal boundary point*, and write $(y, s) \in ab(\partial D)$, if there is an open ball B centred at (y, s) such that $B \cap (\mathbf{R}^n \times]-\infty, s[) \subseteq D$. If B can be found such that $B \cap (\mathbf{R}^n \times]-\infty, s[) = B \cap D$, then (y, s) is of the *first kind*, and so belongs to $ab_1(\partial D)$; otherwise, it is of the *second kind*, and belongs to $ab_2(\partial D)$. The *essential boundary* $\text{ess}(\partial D)$ consists of all boundary points that are not in $ab_1(\partial D)$. If $(y, s) \in \text{ess}(\partial D)$, we put $D(y, s) = D \cap (\mathbf{R}^n \times]s, \infty[)$ if $(y, s) \in ab_2(\partial D)$, and $D(y, s) = D$ otherwise.

If D is bounded, then every continuous function $f : \text{ess}(\partial D) \to \mathbf{R}$ is resolutive. A point $(y, s) \in \text{ess}(\partial D)$ is called *regular* if

$$\lim_{\substack{(x,t)\to(y,s) \\ (x,t)\in D(y,s)}} S_f^D(x, t) = f(y, s)$$

for every continuous $f : \text{ess}(\partial D) \to \mathbf{R}$, where S_f^D denotes the generalized solution to the Dirichlet problem for f on D. The set D is called *regular* if every $(y, s) \in \text{ess}(\partial D)$ is regular.

We now give conditions which ensure that a temperature u on D_+ belongs to $H^\Delta(D_+)$. These will be used in the representation theorem. We write $D(t)$ for $\{x : (x, t) \in D\}$.

THEOREM 3. *Suppose that D_+ is bounded and Dirichlet regular, that u is a temperature on D_+, and that there is a continuous function $\psi : \text{ess}(\partial D_+) \to \mathbf{R}$ such that*

$$\lim_{\substack{(x,t)\to(y,s) \\ (x,t)\in D_+(y,s)}} u(x, t) = \psi(y, s) \tag{4}$$

whenever $(y, s) \in \text{ess}(\partial D_+)\backslash D$. Then u can be extended to $\widetilde{D}_+ = D_+ \cup ab(\partial D_+)$ by putting

$$u(y, s) = \lim_{(x,t)\to(y,s-)} u(x, t), \tag{5}$$

and if

$$\liminf_{t\to 0+} \int_{\widetilde{D}_+(t)} u^+(x, t)\,dx < \infty, \tag{6}$$

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then $u \in H^{\Delta}(D_+)$ *and the function*

$$t \mapsto \int_{\widetilde{D}_+(t)} u^+(x,t)\,dx$$

is bounded.

PROOF. Let $a = \sup\{t : D_+(t) \neq \emptyset\}$. Whenever $0 < c < a$, u is bounded on the set $E_c = D_+ \cap (\mathbf{R}^n \times]c, a[)$. For suppose that $\{x_j\}$ is a sequence in $D(c)$ such that $(x_j, c) \to (y_0, c) \in \partial D_+$. Then $(y_0, c) \in \operatorname{ess}(\partial D_+)$, so that $u(x_j, c) \to \psi(y_0, c)$ by (4). Therefore $u(\cdot, c)$ is bounded, so that u is bounded on E_c, by (4) and the maximum principle [10, Theorem 2]. For any $(y, s) \in ab(\partial D_+)$, the boundedness of u on $E_{\frac{1}{2}s}$ implies that the limit in (5) exists and is finite [4, p. 274].

By [10, Theorem 32], ψ is resolutive for D_+. Let $h = S_\psi^{D_+}$, and let $g = u - h$. Then (4) and the regularity of D_+ imply that

$$\lim_{\substack{(x,t)\to(y,s)\\(x,t)\in D_+(y,s)}} g(x,t) = 0 \tag{7}$$

whenever $(y, s) \in \operatorname{ess}(\partial D_+)\backslash D$. Since h is bounded, g is bounded on E_c whenever $0 < c < a$, and therefore g can be extended to $\widetilde{D}_+$ as u was. Since D_+ is bounded, it now follows from (6) that

$$\liminf_{t\to 0+} \int_{\widetilde{D}_+(t)} g^+(x,t)\,dx < \infty.$$

Put $w = g^+$ on $\widetilde{D}_+$, and $w = 0$ elsewhere on $\mathbf{R}^{n+1}_+$. Then w is continuous on $\mathbf{R}^{n+1}_+\backslash\partial D_+$, and also on $n(\partial D_+)$ because of (7). Furthermore, w is upper semicontinuous on $ab_1(\partial D_+)$, and also on $ab_2(\partial D_+)$ in view of (7). On $\mathbf{R}^{n+1}_+\backslash ab(\partial D_+)$, w satisfies locally the mean value inequality characteristic of subtemperatures. An application of Fatou's lemma shows that w also satisfies locally the mean value inequality at points of $ab(\partial D_+)$. Hence w is a subtemperature, by [8, Theorem 15].

Since g is bounded on each E_c, and D_+ is bounded, given $c \in]0, a[$ there is $\kappa_c < \infty$ such that

$$\int_{\widetilde{D}_+(t)} g^+(x,t)\,dx \le \kappa_c$$

whenever $c < t < a$. Therefore, if $0 < c < t < d < b$, then

$$\int_{\mathbf{R}^n} W(x, b-t)\,w(x,t)\,dx \le (4\pi(b-d))^{-\frac{1}{2}n}\kappa_c.$$

Thus the function $t \mapsto \int_{\mathbf{R}^n} W(x, b-t)w(x,t)dx$ is locally bounded on $]0, b[$, for any $b > 0$. Furthermore,

$$\liminf_{t \to 0+} \int_{\mathbf{R}^n} W(x, b-t)w(x,t)dx \le (2\pi b)^{-\frac{1}{2}n} \liminf_{t \to 0+} \int_{\widetilde{D}_+(t)} g^+(x,t)dx < \infty.$$

Hence, in the notation of [9, Theorem 19], $w \in \Phi_b$ whenever $0 < b < \infty$, so that there is a temperature v which majorizes w on $\mathbf{R}^{n+1}_+$. Now $u - h \le g^+ \le v$ on D_+, so that

$$u - h = v - (v - u + h) \in H^\Delta(D_+).$$

Since h is the generalized Dirichlet solution for ψ, we have $h \in H^\Delta(D_+)$, so that $u \in H^\Delta(D_+)$ as asserted.

For the last part, choose r such that $D_+(t) \subseteq B(0, r)$ for all $t \in]0, a[$, and choose $b \in]a, \infty[$. Then, whenever $0 < t < a$,

$$\begin{aligned} \int_{\widetilde{D}_-(t)} u^-(x,t)dx &\le \int_{\widetilde{D}_+(t)} (g^+(x,t) + h^+(x,t))dx \\ &\le \int_{\widetilde{D}_+(t)} g^+(x,t)dx + \sup |h| v_n r^n, \end{aligned}$$

where v_n is the volume of the unit ball in $\mathbf{R}^n$. Furthermore,

$$\int_{\widetilde{D}_-(t)} g^+(x,t)dx \le (4\pi b)^{\frac{1}{2}n} \exp\left(\frac{r^2}{4(b-a)}\right) \int_{\mathbf{R}^n} W(x, b-t)w(x,t)dx,$$

and the integral on the right is bounded as a consequence of [9, Theorem 16]. □

We can now prove our extension of Aronson's result. Here G_D denotes the Green function for D in the sense of [10], and

$$G_D\mu(x,t) = \int_D G_D(x, t; y, s)d\mu(y, s)$$

for a signed measure μ of finite total variation. If G is G_D with $D = \mathbf{R}^{n+1}$, and $\nu = \lambda \times \delta_0$ with δ_0 the unit mass at 0, then

$$G\nu(x,t) = \int_{\mathbf{R}^n \times \{0\}} W(x-y, t)d\nu(y, 0) = W\lambda(x,t)$$

whenever $t > 0$.

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THEOREM 4. *Suppose that D_+ is bounded and Dirichlet regular, that C is a compact subset of $D(0)$, that $\mathscr{F}$ is an abundant Vitali covering of $\mathbf{R}^n$, and that ψ is a continuous real-valued function on* $\operatorname{ess}(\partial D_+)$. *If u is a temperature on D_+ such that*

$$\lim_{\substack{(x,t)\to(y,s)\\(x,t)\in D_+(y,s)}} u(x,t) = \psi(y,s) \tag{8}$$

for every $(y,s) \in \operatorname{ess}(\partial D_+)\backslash D$,

$$\liminf_{t\to 0+} \int_{\widetilde{D}_+(t)} u^+(x,t)dx < \infty,$$

and

$$\lim_{t\to 0+} \int_{A\cap V} u(x,t)dx = \int_{A\cap V} \psi(x,0)dx$$

whenever $A, V \in \mathscr{F}$, $V \subseteq D(0)\backslash C$, and $A \cap V \neq \emptyset$, then

$$u = G_D\nu + S_\psi^{D_+}$$

on D_+ for some signed measure ν of finite total variation supported in $C \times \{0\}$.

PROOF. By Theorem 3, $u \in H^\Delta(D_+)$. Therefore, by Theorem 2, there exist a signed measure μ, supported in C and with finite total variation, and an initially zero temperature w on D_+, such that $u = W\mu + W\psi(\cdot,0) + w$ on D_+. Let $\nu = \mu \times \delta_0$, so that $G\nu = W\mu$ on $\mathbf{R}^{n+1}_+$ and $G\nu = 0$ elsewhere. By the Riesz decomposition theorem, $G\nu^+ = G_D\nu^+ + h_1$ and $G\nu^- = G_D\nu^- + h_2$ on D, where h_1 and h_2 are the greatest thermic minorants of $G\nu^+$ and $G\nu^-$ on D, so that each h_i is initially zero on D_+. Thus, if $h = w + h_1 - h_2$ then $u - G_D\nu = W\psi(\cdot,0) + h$ on D_+. It follows from (8) and [11, Theorem 2] that (8) holds with u replaced by $u - G_D\nu$. Furthermore, if $v = W\psi(\cdot,0) + h$ then $v(x,t) \to \psi(y,0)$ as $(x,t) \to (y,0+)$ whenever $(y,0) \in \operatorname{ess}(\partial D_+) \cap D$. The result now follows from [10, Theorem 31]. □

4. Removable singularities

We now consider the situation where u is a temperature on $D\backslash K$, for some compact set $K = C \times \{0\}$ with $C \subseteq D(0)$. In Theorem 5, under a mild constraint on u, we show that u is the sum of a temperature v on D and the potential $G\nu$ of a signed measure supported in K. Thus, if $\nu = \lambda \times \delta_0$ then $u = W\lambda + v$ on D_+, and conditions which ensure that $W\lambda = 0$ become conditions for the removability of K. The proofs of Theorems 6, 7 and 8 all use this idea.

A temperature in $H^\Delta(D_+)$ is called *initially nonnegative* if its initial measure is nonnegative. Conditions that imply initial nonnegativity can be found in [15].

THEOREM 5. *Let C be a compact subset of $D(0)$, let $K = C \times \{0\}$, and let u be a temperature on $D\backslash K$ such that*

$$\liminf_{t\to 0+} \int_U u^+(x,t)dx < \infty \tag{9}$$

for some open superset U of C in $\mathbf{R}^n$. Then u can be written uniquely as the sum of a temperature on D and the potential $G\nu$ of a signed measure supported in K.

PROOF. Let $\{V_k\}$ be an exhaustion of $U \cap D(0)$ by bounded open subsets of $\mathbf{R}^n$ which are Dirichlet regular for Laplace's equation. Choose j such that $C \subseteq V_j$, and put $V = V_j$. Then $\overline{V}$ is a compact subset of $D(0)$, so that we can find $a > 0$ such that the set $E = V \times \,]-a, a[$ has its closure in D. Note that E_+ is Dirichlet regular for the heat equation.

The essential boundary of $V \times \,]-a, 0[$ is a compact subset of $D\backslash K$, so that u is bounded there and hence also on $V \times \,]-a, 0[$. Therefore we can define a continuous, real-valued function ψ on $\operatorname{ess}(\partial E_+)$ by putting

$$\psi(y,0) = \lim_{(x,t)\to(y,0-)} u(x,t)$$

for all $y \in V$, and

$$\psi(y,s) = u(y,s)$$

for all $(y,s) \in \partial V \times [0,a]$. Note that $\psi = u$ on $(V\backslash C) \times \{0\}$, and that $u(x,t) \to \psi(y,s)$ as $(x,t) \to (y,s)$ with $(x,t) \in E_+$, whenever $(y,s) \in \operatorname{ess}(\partial E_+)\backslash E$. It follows from (9) and Theorem 3 that $u \in H^\Delta(E_+)$. Therefore, by the Corollary to Theorem 2, there exist a unique signed measure μ, supported in C and with finite total variation, and a unique initially zero temperature w on E_+, such that $u = W\mu + W\psi(\cdot,0) + w$ on E_+. If

$$v = \begin{cases} W\psi(\cdot,0) + w & \text{on } E_+, \\ u & \text{on } E\backslash E_+, \end{cases}$$

then v is continuous on E and a temperature on $E\backslash(V \times \{0\})$, so that v is a temperature on E, by [10, Theorem 5]. If $\nu = \mu \times \delta_0$, then $u = G\nu + v$ on E. Putting $v = u - G\nu$ on $D\backslash E$, we extend v to a temperature on D, and complete the proof. □

THEOREM 6. *Let C be a compact subset of $D(0)$ such that $m_n(C) = 0$, and let u be a temperature on $D\backslash(C \times \{0\})$ such that*

$$\liminf_{t\to 0+} \int_U u^+(x,t)dx < \infty$$

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for some open superset U *of* C *in* $\mathbf{R}^n$. *If there is an initially nonnegative* $h \in H^{\Delta}(D_+)$ *such that*

$$(10) \qquad \lim_{t\to 0+} \frac{u(x,t)}{h(x,t)} = 0$$

for all $x \in C$ *at which the limit exists, then* u *can be extended to a temperature on* D.

PROOF. Since $m_n(C) = 0$, there exists a positive temperature f on $\mathbf{R}^{n+1}_+$ such that $f(x, 0+) = \infty$ for all $x \in C$, by [11, Theorem 11]. The addition of f to h does not affect our hypotheses, and so we can assume that $h(x, 0+) = \infty$ for all $x \in C$.

By Theorem 5, there exist a signed measure μ supported in $C \times \{0\}$, and a temperature v on D, such that $u = G\mu + v$. Let $\mu = \lambda \times \delta_0$, and let ν be the initial measure of h. By [14, Theorem 2],

$$(11) \qquad \lim_{t\to 0+} \frac{W\lambda(x,t)}{W\nu(x,t)}$$

exists and is finite for ν-almost all $x \in \mathbf{R}^n$. For each $x \in C$ at which this limit exists, the limit in (10) exists and the two are equal, because $h(x, 0+) = \infty$ for all $x \in C$. Therefore the limit in (11) is zero for ν-almost all $x \in C$, and

$$\liminf_{t\to 0+} \frac{|W\lambda(x,t)|}{W\nu(x,t)} < \infty$$

for all $x \in C$. It now follows from [14, Theorem 6] (with $Z = C$ and $Y = \emptyset$) that λ_C is null. Hence μ is null, and $u = v$. □

For the final two theorems, we denote by m_q the q-dimensional Hausdorff measure on $\mathbf{R}^n$, where $0 \le q \le n$. We are only concerned that a given set is null, finite, or σ-finite with respect to m_q, so there is no need to distinguish the case $q = n$ from Lebesgue measure.

THEOREM 7. *Let* C *be a compact subset of* $D(0)$, *and let* u *be a temperature on* $D\backslash(C \times \{0\})$ *such that*

$$\liminf_{t\to 0+} \int_U u^+(x,t)\,dx < \infty$$

for some open superset U *of* C *in* $\mathbf{R}^n$. *If either*

(i) $q \in [0, n]$, $m_q(C) = 0$, *and*

$$(12) \qquad \limsup_{t\to 0+} t^{\frac{1}{2}(n-q)}|u(x,t)| < \infty \quad \textit{for all } x \in C,$$

or

(ii) *$q \in [0, n[$, C is σ-finite with respect to m_q, and*

$$\lim_{t \to 0+} t^{\frac{1}{2}(n-q)} u(x, t) = 0 \quad \textit{for all} \quad x \in C, \tag{13}$$

then u can be extended to a temperature on D.

PROOF. By Theorem 5, u can be written as the sum of a temperature v on D, and the potential $G\mu$ of a signed measure supported in $C \times \{0\}$. If $\mu = \lambda \times \delta_0$, then $u = W\lambda + v$ on D_+. For any $x \in C$,

$$\limsup_{t \to 0+} t^{\frac{1}{2}(n-q)} |v(x, t)|$$

is zero if $q < n$, and is finite if $q = n$. Therefore conditions (12) and (13) imply similar ones on $W\lambda$. We can now apply [14, Theorem 6] (with $Z = Y = C$, so that the auxiliary function is superfluous) and conclude that λ_C is null. Hence μ is null, and $u = v$. □

In our final result we show that, if C has a certain structure, then for sets with finite m_q-measure we can weaken (13) without affecting the conclusion.

THEOREM 8. *Let $q \in [0, n[$, let C be a compact subset of $D(0)$ such that $m_q(C) < \infty$ and*

$$\liminf_{r \to 0} r^{-q} m_q(B(x, r) \cap C) > 0 \tag{14}$$

for all $x \in C$, and let u be a temperature on $D \backslash (C \times \{0\})$ such that

$$\liminf_{t \to 0+} \int_U u^+(x, t)\,dx < \infty$$

for some open superset U of C in $\mathbf{R}^n$. If

$$\liminf_{t \to 0+} t^{\frac{1}{2}(n-q)} u(x, t) \le 0 \le \limsup_{t \to 0+} t^{\frac{1}{2}(n-q)} u(x, t) \quad \textit{for } m_q\textit{-almost all} \quad x \in C, \tag{15}$$

and

$$\liminf_{t \to 0+} t^{\frac{1}{2}(n-q)} |u(x, t)| < \infty \quad \textit{for all} \quad x \in C, \tag{16}$$

then u can be extended to a temperature on D.

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PROOF. By Theorem 5, $u = v + G(\lambda \times \delta_0)$ for some temperature v on E and signed measure λ supported in C. Since $q < n$, we have $t^{\frac{1}{2}(n-q)}v(x,t) \to 0$ as $t \to 0+$ for all $x \in C$, so that (15) and (16) imply similar conditions on $W\lambda$.

Suppose that $q > 0$. Then $m_q(C) < \infty$, (14) holds, $W\lambda(x, 0+) = 0$ for all $x \in \mathbf{R}^n \backslash C$ and m_n-a.e. on $\mathbf{R}^n$, and (15), (16) hold with $W\lambda$ in place of u, so that [12, Theorem 10] shows that $W\lambda = 0$. Hence $u = v$.

Now suppose that $q = 0$, so that C is finite. (In this case, (14) and (16) are superfluous.) Given $x_0 \in C$ such that $\lambda(\{x_0\}) > 0$, put $\nu = \lambda_{\{x_0\}}$ and $\omega = \lambda - \nu$. Then [14, Lemma 1] shows that $W\omega(x_0, t) = o(W\nu(x_0, t))$ as $t \to 0+$, so that

$$W\lambda(x_0, t) \sim W\nu(x_0, t) = (4\pi t)^{-\frac{1}{2}n}\lambda(\{x_0\}).$$

Since (15) holds with u replaced by $W\lambda$, it follows that

$$\lambda(\{x_0\}) = \lim_{t \to 0+} (4\pi t)^{\frac{1}{2}n} W\lambda(x_0, t) = 0,$$

a contradiction. Therefore $\lambda(\{x\}) = 0$ for all $x \in C$, so that again $u = v$. □

REMARK. The conclusions of Theorems 7 (ii) and 8 both fail if $q = n$. For example, let $n = 1$, $D = [-1, 2]^2$, $C = [0, 1]$, $v(x, t) = e^{x+t} - (e-1)x - 1$ on D, and $u = v - Wv(\cdot, 0)_C$ on D_+, $u = v$ on $D \backslash (D_+ \cup (C \times \{0\}))$. Then u is a bounded temperature, and because $v(\cdot, 0)$ is continuous on C with $v(0,0) = v(1,0) = 0$, we have $u(x, 0+) = 0$ for all $x \in C$. Since $u(x, 0-) = v(x, 0) < 0$ whenever $0 < x < 1$, u cannot be extended to a temperature on D.

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