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THE BRUSS–ROBERTSON–STEELE INEQUALITY

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Abstract

The Bruss–Robertson–Steele (BRS) inequality bounds the expected number of items of random size which can be packed into a given suitcase. Remarkably, no independence assumptions are needed on the random sizes, which points to a simple explanation; the inequality is the integrated form of an ω-by-ω inequality, as this note proves.

Keywords: Bruss–Robertson–Steele inequality; linear program; pathwise

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1. The basic problem

The Bruss–Robertson–Steele (BRS) inequality was first proved in [2], and later generalized in [3]. The recent survey in [1] gives a fine review.

You have N objects which you would like to take in your suitcase on a flight. The weight of object j is Z_j, but the total weight you are allowed to take on the flight must not exceed $s > 0$. You want to maximise the number of objects that you can take, subject to this constraint. This can be posed as a linear programme:

$$\max_{x\geq 0} \sum_i x_i \quad \text{subject to} \quad x_i \leq 1 \text{ for all } i, \ \sum_i x_i Z_i \leq s.$$

Strictly, we have to have that each x_i is in $\{0, 1\}$, but this additional constraint will only reduce the value; since we are looking for upper bounds, the gap here will help us. We can write this in canonical matrix form,

$$\max_{x\geq 0} c^\top x \quad \text{subject to} \quad Ax \leq b,$$

where $c^\top = (1, \ldots, 1)$, $b^\top = (1, \ldots, 1, s)$, and

$$A = \begin{pmatrix} I \\ Z^\top \end{pmatrix}.$$

The dual linear programme is

$$\min_{y\geq 0} \ b^\top y \quad \text{subject to} \quad c \leq A^\top y.$$

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Written out more fully, this is

$$\min_{y\geq 0} \sum_{j=1}^{N} y_j + s y_{N+1} \quad \text{subject to} \quad 1 \leq y_j + Z_j y_{N+1} \text{ for all } j. \tag{1}$$

The value of the dual problem is the value of the primal problem (e.g. [4, Section 4.2]), and for any dual-feasible y the value $b^\top y$ is an upper bound for the value of the problem. If we write $y_{N+1} = \eta$ for short, the problem in (1) requires

$$\min_{y\geq 0} \sum_{j=1}^{N} y_j + \eta s \quad \text{subject to} \quad 1 \leq y_j + \eta Z_j \text{ for all } j.$$

Obviously, once $\eta > 0$ has been chosen, the best dual-feasible choice of $y_1, \ldots, y_N$ will be $y_j = (1 - \eta Z_j)^+$. Thus, for any $\eta > 0$, the value Φ^* of the problem is bounded above by

$$\Phi(\eta) \equiv \sum_{j=1}^{N} (1 - \eta Z_j)^+ + \eta s,$$

which is clearly a convex piecewise-linear function of η.

2. The BRS inequality

In the problem studied by Bruss and Robertson, and later in greater generality by Steele, $Z_1, \ldots, Z_N$ are positive random variables, and the distribution function of Z_j is F_j, assumed for convenience to be continuous. In this situation, the value Φ^* of the suitcase packing problem will of course be random, and the BRS inequality gives an upper bound for $E\Phi^*$. Let us see how the BRS inequality follows easily from the linear-programming story of the previous section.

Clearly, for any $\eta > 0$ we have

$$\begin{aligned} \mathbb{E}\Phi^* \leq \mathbb{E}\Phi(\eta) &= \mathbb{E} \sum_{j=1}^{N} (1 - \eta Z_j)^+ + \eta s \\ &= \sum_{j=1}^{N} \int_0^{1/\eta} (1 - \eta z)\, F_j(\mathrm{d}z) + \eta s. \end{aligned} \tag{2}$$

Now we optimize the bound in (2) by differentiating:

$$0 = -\sum_{j=1}^{N} \int_0^{1/\eta} z\, F_j(\mathrm{d}z) + s,$$

which will be satisfied when $\eta = \eta^*$, the root of

$$\sum_{j=1}^{N} \int_0^{1/\eta} z\, F_j(\mathrm{d}z) = s. \tag{3}$$

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Taking $\eta = \eta^*$ and using (3), the bound in (2) for $\mathbb{E}\Phi^*$ is easily seen to be

$$\mathbb{E}\Phi^* \le \sum_{j=1}^{N} \int_0^{1/\eta^*} F_j(\mathrm{d}x),$$

which is the BRS inequality.

Remark 1. Even if the implicit equation in (3) cannot be solved explicitly, the bound in (2) can still be applied for any choice of η.

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