EXAMPLES OF MALFORMED SUBSETS OF A RIEMANN SURFACE

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Let R be a hyperbolic Riemann surface and W an open subset of R with ∂W piecewise analytic. Denote by $\tilde{M}(R)$ the space of Dirichlet finite Tonelli functions on R and by π the harmonic projection of $\tilde{M}(R)$. Consider the relative HD-class on W, $HD(W;\partial W)=\{u \in \tilde{M}(R) \mid u \mid W \in HD(W)$ and $u \mid R\setminus W=0\}$. The extremization operation μ_D^W is the linear mapping of $HD(W;\partial W)$ into $HD(R)$ defined by $\mu_D^W = \pi \mid HD(W;\partial W)$. Since π preserves values of functions at the Royden harmonic boundary, the maximum principle implies that μ_D^W is an order preserving injection and that $\mu_{BD}^W = \mu_D^W \mid HBD(W;\partial W)$ is an isometry with respect to the supremum norms.

The term *malformed* was introduced by Nakai [**2**] for subsets W of R with the propery that μ_{BD}^W is surjective yet μ_D^W is not. Royden's theorem [**4**] on the CD-denseness of $HBD(R)$ in $HD(R)$ (and that of $HBD(W;\partial W)$ in $HD(W;\partial W)$) leads one to believe that there are no malformed subsets. However, Nakai [**1, 2**] showed that on an arbitrary hyperbolic Riemann surface R corresponding to each unbounded function $h \in HD(R)$ there is a malformed region W with $h \notin \mu_D^W(HD(W;\partial W))$. The proof of this important result on the classification of Riemann surfaces is elaborate and difficult. In this paper we show that on a suitably chosen R the task of exhibiting malformed subregions is considerably simpler. From this point of view the result presented here is weaker. However, due to its simplicity one can observe a new phenomenon: the existence of malformed subregions such that the *deficiency* of μ_D^W, i.e. the dimension of $HD(R)/\mu_D^W(HD(W;\partial W))$, is infinite.

1. We use the Royden ideal boundary extensively and refer to [**5**] for the notations and results. Let T be the Tôki surface, a Riemann surface in the class $O_{HD}\setminus O_G$. The harmonic boundary of T consists of a single point p^*. Thus, if U is any subregion of T with $T\setminus U$ compact, then $\dim HD(U;\partial U)=\dim HBD(U;\partial U)=1$. Let g_T be the Green's function on T with pole at a fixed point $q_0 \in T$. For any $\alpha \in (0,+\infty)$ let $G_\alpha = \{p \in T \mid g_T(p)<\alpha\}$ and set $\omega_\alpha = (1-\alpha^{-1}g_T)\cup 0$. Note that $\omega_\alpha \in HBD(G_\alpha;\partial G_\alpha)$ and $\omega_\alpha(p^*)=1$.

LEMMA. *For $\alpha \in (0,+\infty)$ the Dirichlet integral of ω_α over T is given by*

$$D_T(\omega_\alpha)=2\pi/\alpha. \tag{1}$$

For the proof consider the polar coordinate differentials $dr, d\theta$ on T defined by $dr/r=-dg_T$, $d\theta=-{}^*dg_T$. Whereas $r=e^{-g_T}$ is a global function on T, $\theta=-g_T^*$ is defined only locally. There is a neighborhood of each point of T except the countably many

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critical points of g_T in which $re^{i\theta}$ may be used as the local parameter. Thus we have

$$D_{G_\alpha}(f)=\int_{e^{-\alpha}}^{1}\left(\int_{C_\rho}\left\{\left(\frac{\partial f}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial f}{\partial\theta}\right)^2\right\}d\theta\right)\rho\,d\rho, \tag{2}$$

where C_ρ is the positively oriented boundary of $\{p\in T\mid r(p)<\rho\}$ and f is a suitable function on G_α. For all but countably many $\rho\in(0,1)$ we know that

$$\int_{C_\rho} d\theta=2\pi \tag{3}$$

(cf. [**5**, p. 199]). Setting $f=g_T$ in (2) and using (3) to evaluate the right hand side gives $D_{G_\alpha}(g_T)=2\pi\alpha$. Since $D_T(\omega_\alpha)=D_{G_\alpha}(\alpha^{-1}g_T)$, we conclude that (1) holds.

2. We follow the procedure of [**3**] for forming a Riemann surface by welding subsurfaces of T to a region in the complex plane $\mathbb{C}$. Fix $\nu\in(0,+\infty)$ such that $T\setminus G_\nu$ is homeomorphic to a closed disc and does not contain critical points of g_T. Set $c=2\pi/\nu$ and fix a number $a\in(1,2)$. Define two sequences

$$\alpha_k=a^k\nu,\qquad \beta_k=2^k\nu,\qquad k=1,2,\ldots.$$

Note that $G_\nu\subset G_{\alpha_k}\subset G_{\beta_k}$. By the lemma in Section 1 we have

$$D_T(\omega_{\alpha_k})=c/a^k,\qquad k=1,2,\ldots. \tag{4}$$

We also remark that $1-\omega_{\beta_k}\mid\partial G_\nu=\beta_k^{-1}(g_T\mid\partial G_\nu)\cap 1=\beta_k^{-1}\nu\cap 1=2^{-k}$ and in particular,

$$1-\omega_{\beta_k}\mid\partial G_\nu\le\tfrac{1}{2},\qquad k=1,2,\ldots. \tag{5}$$

We prepare copies T_n, $n=1,2,\ldots$, of T and for any $\alpha\in(0,+\infty)$ we let G_α^n be a copy of G_α in T_n. For $n=1,2,\ldots$ define $X_n=G_\nu^n$, $W_n=G_{\alpha_n}^n$, $V_n=G_{\beta_n}^n$. We view ω_α as being defined on T_n and set $w_n=\omega_{\alpha_n}\mid W_n$, $v_n=\omega_{\beta_n}\mid V_n$. We weld the surfaces V_n to the region $D=\mathbb{C}\setminus\bigcup_1^\infty\{|z-3n|\le 1\}$ by identifying ∂V_n with $\{|z-3n|=1\}$, $n=1,2,\ldots$. Denote the resulting Riemann surface by R. The functions w_n, v_n can be viewed as being defined on the subsurfaces W_n, V_n of R. Then according to (4) and (5) we have

$$D_{W_n}(w_n)=c/a^n \tag{6}$$

and

$$1-v_n\mid\partial X_n\le\tfrac{1}{2},\qquad n=1,2,\ldots. \tag{7}$$

3. Denote the harmonic boundary of R by Δ. Since $\dim HD(X_n;\partial X_n)=1$ we see that $\bar{X}_n\cap\Delta$ consists of a single point which we denote by p_n^*. Set $\Delta_1=\{p_1^*,p_2^*,\ldots\}$.

LEMMA. *Δ_1 is dense in Δ.*

Assume that there is a point $q^*\in\Delta\setminus\bar{\Delta}_1$. Then we can find a function $u\in HD(R)$ such that $0<u<1$, $u\mid\bar{\Delta}_1=0$, $u(q^*)=1$. For each n, $u(p_n^*)=1-v_n(p_n^*)=0$ and $u\mid\partial V_n<1=1-v_n\mid\partial V_n$. Thus $u\mid V_n<1-v_n\mid V_n$ and in view of (7) we have $u\mid\partial X_n\le\frac{1}{2}$. Consider

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the region $R_0 = R \setminus \bigcup_1^\infty \bar{X}_n$. Clearly, $u \mid \partial R_0 \leq \frac{1}{2}$. Since R_0 is planar we conclude that $u \mid R_0 \leq \frac{1}{2}$ and consequently $u \mid R \leq \frac{1}{2}$. This contradicts $u(q^*) = 1$ and the assertion follows.

4. Set $W = \bigcup_1^\infty W_n$ and define a function w on R by setting $w \mid R \setminus W = 0$, $w \mid W_n = w_n$, $n = 1, 2, \ldots$. Clearly $w \in HB(W; \partial W)$. By (6) we see that $D_R(w) = \sum_1^\infty D_{W_n}(w_n) < +\infty$ and hence $w \in HBD(W; \partial W)$. Since $w \mid \Delta_1 = 1$, the lemma in Section 3 implies that $w \mid \Delta = 1$. Thus $\Delta \subset \{p^* \in R^* \mid w(p^*) > 0\} \subset \bar{W}$; i.e. $\bar{W}$ is a neighborhood of Δ in R^*.

PROPOSITION. *The extremization $\mu_{BD}^W: HBD(W; \partial W) \to HBD(R)$ is surjective.*

This follows from Theorem 2 of [**1**] or can be established directly. Let h be any function in $HBD(R)$. Then wh is in the Royden algebra since each factor is and we can set $u = \pi_{R \setminus W}(wh)$, the harmonic projection of wh on W (cf. [**5**, p. 165]). Since $u \in HBD(W; \partial W)$ and $\mu_{BD}^W u \mid \Delta = u \mid \Delta = wh \mid \Delta = h \mid \Delta$, it follows that $\mu_{BD}^W u = h$.

5. Define a measure σ on Δ as follows:

$$\sigma(\Delta \setminus \Delta_1) = 0, \qquad \sigma(\{p_n^*\}) = D_{W_n}(w_n), \qquad n = 1, 2, \ldots. \tag{8}$$

The space $HD(W; \partial W)$ is completely determined by σ.

PROPOSITION. $HD(W; \partial W) \mid \Delta = L^2(\Delta, \sigma)$.

Assume that $u \in HD(W; \partial W)$. Then $u \mid W_n = u(p_n^*) w_n$ and hence by (8) we see that $+\infty > D_W(u) = \sum_1^\infty D_{W_n}(u) = \sum_1^\infty (u(p_n^*))^2 \sigma(\{p_n^*\})$. Conversely, let $\xi \in L^2(\Delta, \sigma)$ and define $\xi_n = \xi(p_n^*)$. The function $u = \sum_1^\infty \xi_n w_n$ is in $HD(W; \partial W)$ and $u(p_n^*) = \xi_n$, $n = 1, 2, \ldots$. Thus $u \mid \Delta = \xi$, σ-a.e.

6. Recall that the definitions of W and σ depend on an arbitrary constant $a \in (1, 2)$. Set $W^a = W$ and $\sigma^a = \sigma$ to express this dependence.

PROPOSITION. *The subset W^a of R has the property that the deficiency of $\mu_D^{W^a}$ is infinite.*

For the proof set $Z^a = \mu_D^{W^a}(HD(W^a; \partial W^a))$. By (8) and (6) it is easily seen that for a', a'' with $a \leq a' < a'' < 2$ the space $L^2(\Delta, \sigma^{a'})$ is properly contained in $L^2(\Delta, \sigma^{a''})$. Since extremization leaves values on Δ fixed, the proposition in Section 5 gives $Z^a \mid \Delta = L^2(\Delta, \sigma^a)$ and similarly for a', a''. We conclude that Z^a is contained in $Z^{a'}$ and that $Z^{a'}$, $a < a' < 2$, form a family of properly increasing subspaces of $HD(R)$.

7. We revert to the notations W and σ. In the remainder of this paper we show that the open set W is contained in a region Ω such that

$$HD(\Omega; \partial\Omega) \mid \Delta = HD(W; \partial W) \mid \Delta. \tag{9}$$

Actually, in order to make $\partial\Omega$ piecewise analytic it will be necessary to shift our

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considerations to a Riemann surface R' obtained from R by removing a point s_0 from R. Since the harmonic boundary of R' can be identified with Δ and s_0 will not be in $\bar{W}$, the propositions in Sections 4 and 6 will not be affected. The result in (9) implies that the image of μ_D^Ω coincides with the image of μ_D^W in $HD(R')$ and in particular that the image of μ_{BD}^Ω coincides with the image of μ_{BD}^W. Thus our main result will follow from (9) and the two propositions just mentioned.

THEOREM. *There exists a Riemann surface R' and a subregion Ω with $\partial\Omega$ piecewise analytic such that μ_{BD}^Ω is surjective yet the deficiency of μ_D^Ω is infinite.*

The region Ω will consist of the union of W, a fixed parametric disc and a collection of thin strips joining the components of W to this disc. Fix a point p_0 in the subset D of R (cf. Section 2) and let g_R be the Green's function for R with pole at p_0. Choose $\gamma \in (0, +\infty)$ such that the set $B_\gamma = \{p \in R \mid g_R(p) > \gamma\}$ is homeomorphic to a disc and $\bar{B}_\gamma$ contains no critical points of g_R. We now fix the arbitrary constant $a \in (1, 2)$ that determines W so that none of the critical points of g_R lie on ∂W.

For each positive integer n let p_n be a point in the compact set ∂W_n such that $g_R(p_n) = \max_{\partial W_n} g_R$. Then grad g_R at p_n is perpendicular to the tangent to ∂W_n at p_n. Here we are using the notations dr, $d\theta$ for the polar coordinate differentials of R obtained from g_R. In terms of the local parameter $re^{i\theta}$ we can represent ∂W_n in a neighborhood of p_n in the form $r = f_n(\theta)$, $\theta \in I_n$, an open interval. The function θ has a harmonic continuation along any integral curve of grad g_R with θ being constant on the curve. Thus we may choose $\theta_n \in I_n$ such that the maximal integral curve of grad g_R starting at the point $q_n \in \partial W_n$ with $\theta(q_n) = \theta_n$, $r(q_n) = f_n(\theta_n)$ does not terminate at any of the critical points of g_R. Therefore, this curve intersects $\partial B_{3\gamma}$ and we denote by l_n the portion connecting q_n with $\partial B_{3\gamma}$.

Let S_n be a region on which θ is well-defined and which contains l_n but does not contain any critical point of g_R. Then $re^{i\theta}$ may be used as the parameter on S_n. Denote by ε_0 the minimum of the positive quantities $\min_{t \in \partial I_n} |t - \theta_n|$ and $\inf_{p \in \partial S_n} |\theta(p) - \theta_n|$. For any $\varepsilon \in (0, \varepsilon_0]$ define

$$U_n^\varepsilon = W_n \cup \{p \in R \mid |\theta(p) - \theta_n| < \varepsilon \text{ and } e^{-3\gamma} < r(p) \le f_n(\theta)\}.$$

Clearly U_n^ε is a region containing W_n with ∂U_n^ε piecewise analytic and $\bigcap_{0 < \varepsilon \le \varepsilon_0} U_n^\varepsilon = W_n \cup \tilde{l}_n$, where $\tilde{l}_n = l_n \setminus \partial B_{3\gamma}$. Moreover, if $n \neq n'$, then the U_n^ε, $U_{n'}^{\varepsilon'}$ constructed here are disjoint.

8. Consider a fixed positive integer n. For each integer $m > \varepsilon_0^{-1}$ there is a function $u_m \in HD(U_n^{1/m}; \partial U_n^{1/m})$ such that $u_m(p_n^*) = 1$. Extend the definition of w_n to R by setting $w_n \mid R \setminus W_n = 0$. It is easily seen that for $m' > m > \varepsilon_0^{-1}$, $w_n \le u_{m'} \le u_m < 1$ on R. Thus $u = \lim_m u_m$ exists on R,

$$w_n \le u \le u_m \text{ on } R, \tag{10}$$

$u \mid W_n$ is harmonic and $u \mid R \setminus (W_n \cap \tilde{l}_n) = 0$. By (10) we see that if we set $u(p_n^*) = 1$, then u is continuous on $W_n \cap \{p_n^*\}$. Since each point in $\partial W_n \setminus \{q_n\}$ is eventually contained in $R \setminus U_n^{1/m}$, (10) implies that u vanishes continuously on $\partial W_n \setminus \{q_n\}$. Let $g_R(\cdot, q_n)$ be the

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Green's function on R with pole at q_n. We apply the HB-maximum principle (cf. [**5**, p. 160]) to $w_n+\eta g_R(\cdot, q_n)-u$ for any $\eta>0$ and let $\eta\to 0$ to obtain $w_n\geq u$ on W_n. Consequently, $u=w_n$ on $R\setminus l_n$. Since the sequence $\{u_m\}$ is eventually harmonic at each point of $R\setminus(\partial W_n\cup l_n)$, we conclude that

$$\lim_m \operatorname{grad} u_m=\operatorname{grad} w_n \quad \text{on} \quad R\setminus(\partial W_n\cup l_n). \tag{11}$$

For $m'>m>\varepsilon_0^{-1}$, the function $u_m-u_{m'}$ vanishes on $(\overline{R\setminus U^m})\cup\{p_n^*\}$ and is therefore orthogonal to a harmonic projection on $U_m^{1/n}$ (cf. [**5**, p. 162]). In particular, $D_R(u_m-u_{m'}, u_m)=0$ and consequently

$$0\leq D_R(u_{m'}-u_m)=D_R(u_{m'})-D_R(u_m). \tag{12}$$

Similarly,

$$0\leq D_R(w_n-u_m)=D_R(w_n)-D_R(u_m). \tag{13}$$

Thus $\{D_R(u_m)\mid m>\varepsilon_0^{-1}\}$ is an increasing sequence bounded above by $D_R(w_n)$. Set $d=\lim_m D_R(u_m)$. Let $m'\to+\infty$ in (12). By (11) and Fatou's lemma we arrive at $0\leq D_R(w_n-u_m)\leq d-D_R(u_m)$ which, by the triangle inequality, implies that $\lim_m D_R(u_m)=D_R(w_n)$. Choose a positive integer m_n such that

$$D_R(w_n)/2\leq D_R(u_{m_n})\leq D_R(w_n). \tag{14}$$

We repeat this construction for each $n=1,2,\ldots$. Set $Y_n=U_n^{1/m_n}$, $y_n=u_{m_n}$ and $Y=\bigcup_1^\infty Y_n$. Also define a measure τ on Δ by setting $\tau(\Delta\setminus\Delta_1)=0$, $\tau(\{p_n^*\})=D_R(y_n)$. By (14) and (8) we see that $\sigma(A)/2\leq\tau(A)\leq\sigma(A)$ for any subset A of Δ. Thus $L^2(\Delta,\sigma)=L^2(\Delta,\tau)$. On the other hand, by the reasoning of Section 5 we can establish that

$$HD(Y;\partial Y)\mid\Delta=L^2(\Delta,\tau). \tag{15}$$

It follows that

$$HD(Y;\partial Y)\mid\Delta=HD(W;\partial W)\mid\Delta. \tag{16}$$

9. Each component Y_n of Y intersects the disc $B_{2\gamma}$ and $\partial Y\cap\partial B_{2\gamma}$ has precisely one accumulation point s_0 in R. We now regard $R'=R\setminus\{s_0\}$ as a Riemann surface in its own right. Let R'^* be its Royden compactification and denote by $\hat{B}$ the closure of a set B in R'^*. Let D_0 be a punctured disc centered at s_0. Then there is a homeomorphism between $R'^*\setminus\hat{D}_0$ and $R^*\setminus\bar{D}_0$ with harmonic boundary points corresponding to harmonic boundary points (cf. [**5**, p. 189]). Since the Green's function of R' (with pole at a fixed point in R') is a potential bounded away from 0 in $\hat{D}_0$, all the harmonic boundary points of R' are contained in $R'^*\setminus\hat{D}_0$. In this sense the harmonic boundary of R' can be identified with Δ. Thus (16) continues to hold when Y and W are viewed as subsets of R'.

Define $\Omega=Y\cup B_{2\gamma}$. Clearly Ω is a region with piecewise analytic $\partial\Omega$ in R'. The final step in our proof is to show that

$$HD(\Omega;\partial\Omega)\mid\Delta=HD(Y;\partial Y)\mid\Delta. \tag{17}$$

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Let $v \in HD(Y; \partial Y)$. Set $u = \pi_M v$ (where $M = \widehat{R' \setminus \Omega}$). Then $u \mid \Delta = v \mid \Delta$ and since $v \mid R \setminus \Omega = 0$ we have $u \in HD(\Omega; \partial\Omega)$. To establish the converse let $f \in M(R')$ such that $f \mid R' \setminus B_\gamma = 1$ and $f \mid B_{3\gamma/2} = 0$. The support of grad f is compact in R' and consequently, for any $u \in HD(\Omega; \partial\Omega)$, $fu \in M(R')$. Moreover, $fu = 0$ on $(R' \setminus \Omega) \cup B_{3\gamma/2}$ which contains $R' \setminus Y$. Therefore the function $v = \pi_N(fu)$ (where $N = \widehat{R' \setminus Y}$) belongs to $HD(Y; \partial Y)$ and $v \mid \Delta = (fu) \mid \Delta = u \mid \Delta$.

In view of (17) and (16) we conclude that (9) holds for R'. This establishes the theorem.

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