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AN INVERSE PROBLEM IN THE CALCULUS OF VARIATIONS AND THE CHARACTERISTIC CURVES OF CONNECTIONS ON SO(3)-BUNDLES

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ABSTRACT. This paper concerns an inverse problem in the calculus of variations, namely, when a two-dimensional symmetric connection is globally a Riemannian or pseudo-Riemannian connection. Two new local characterizations of such connections in terms of the Ricci tensor and the Riemann curvature tensor respectively are given, together with a solution to the global problem. As an application, the problem of whether the characteristic curves of a connection on an SO(3)-bundle on a surface are the geodesics of a Riemannian metric on the surface is studied. Some applications to non-holonomic dynamics are discussed.

Introduction. In this paper we study an inverse problem in the calculus of variations, namely, for a pair of second order equations, when are they the geodesic equations of a (Riemannian or pseudo-Riemannian) metric? Or, more specifically, when is a two-dimensional symmetric connection a metric connection? We provide two new local solutions, one involving the Ricci tensor and the other in terms of the curvature tensor, and we give a global solution to this problem.

A local solution, known to some experts but unavailable in the literature, may be obtained as follows. If the connection D is the Levi-Civita connection of some (pseudo-Riemannian) metric on the surface Σ, then there exists a parallel volume form Ξ on Σ: $D\Xi = 0$ (assuming Σ is orientable). It is easy to see that one may obtain a parallel volume form by scaling an arbitrary volume form by an appropriate factor. Thus we may assume that we have obtained such a volume form Ξ. Ξ will then be the volume form of a metric g. If D is the connection of a Riemannian metric then g is uniquely determined by Ξ, but if D is defined by a pseudo-Riemannian metric then g is determined up to a negative sign. Choose a coframe $\omega = (\omega^1, \omega^2)$, such that $\Xi = \omega^1 \wedge \omega^2$. If $g = g_{ij}\omega^i \otimes \omega^j$ then $|\det(g_{ij})| = 1$. The relation between the Gaussian curvature K and the Ricci curvature S of the metric g on the surface is given by $S = Kg$. It follows that $K = \pm|\det S|^{1/2}$. Thus if S is non-zero then g may be written $g = fS$, where $f = \pm|\det S|^{-1/2}$. Therefore to determine whether a connection D with non-zero, symmetric Ricci curvature S is locally a metric connection one need only check that $D(fS) = 0$.

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The drawback of this approach is that it doesn't extend naturally to global results (for example, we have not been able to obtain a global characterization in the case of a pseudo-Riemannian metric using this approach), as it starts from a volume form.

In this paper, we take a more direct approach, which naturally leads to global results. We show that a symmetric connection D with non-zero, symmetric Ricci curvature S is locally a metric connection if and only if S is *recurrent*: there exists a 1-form ϕ such that $DS = S \otimes \phi$ (Theorem 1). Moreover, such a ϕ, if it exists, is unique and is a closed form (Lemma 5). The utility of using invariant formalism is now immediate: a non-zero, recurrent, symmetric Ricci tensor defines a cohomology class on the surface. This will enable us to obtain a global solution to the problem of the existence of a metric (pseudo-Riemannian or Riemannian, Remark 2).

The requirement that a symmetric Ricci curvature on a surface be recurrent is easily seen to amount to a system of *six* algebraic equations in two unknown variables. A slightly more satisfactory solution to our problem should provide a classification directly in terms of *four* independent conditions. As such, we give a second characterization of local metric connections on a surface in terms of covariant differentiation of the eigenspaces of the Riemann curvature tensor (Theorem 2).

There are many other solutions to the inverse problem. For example, Schmidt [13] derived a necessary condition to this inverse problem in terms of the holonomy group, which, however, is not differential-geometric and not explicit. A slightly different problem, in which any form of Lagrangian is allowed, has been studied by Douglas, Anderson-Duchamp, Morandi, and others (see [1] for a list of references). But it is not explicit in their work when the Lagrangian is actually a Riemannian metric. In the one-dimensional case, a complete solution can be found in [11].

As an application, we study the characteristic curves of a connection on an SO(3)-bundle over a surface, which is the motivation for this work.

Let $\pi: F \longrightarrow \Sigma$ be a principal SO(3)-fiber bundle over a surface Σ, with a connection

$$TF = H \oplus K, \tag{1}$$

where K is the distribution tangent to the fibers. We assume that the *holonomy algebra* of the connection is so(3), *i.e.* the horizontal distribution H is of the type studied by E. Cartan [5]). A *characteristic curve* γ on Σ is the projection of an integral curve of the characteristics of the sub-bundle $H^{\perp} \subset T^*F$ (*cf.* [3]). The characteristic curves correspond to singularities in the space of horizontal curves on F with fixed end points (considered as a sub-space in the space of C^1 curves with fixed end points with the induced topology), and play an important role in the study of exterior differential systems and control theory. Recently, Bryant-Hsu studied their local rigidity ([4]).

The characteristic curves satisfy a system of second order differential equations on Σ (see Section 3.1). A natural question to ask is whether these equations come from some Lagrangian variational principle, where the Lagrangian has the form of kinetical energy plus potential energy. More precisely, we shall study the following problem: For which

SO(3)-connection is there a Riemannian metric on Σ such that all the characteristic curves coincide with the geodesics of this metric? (We will call such a connection *metrizable*.)

Applying the solution to the inverse problem, we obtain that if an SO(3)-connection satisfies a certain set of five equations, then it is metrizable. Moreover, this metric is unique up to a constant multiple. So all the metrizable connections form a 'subvariety' of codimension less than or equal to five in the space of all SO(3)-connections.

1. The inverse problem.

1.1. *Statement of results.* For the local problem we have the following two characterizations of those connections which are metric connections.

THEOREM 1. *Let D be a connection on a surface Σ with Ricci curvature $S \neq 0$ everywhere. D is locally a metric connection if and only if*

(1) D and S are symmetric,

(2) $\det S \neq 0$, and

(3) S is recurrent.

This (local) metric is defined up to multiplication by a constant.

REMARK 1. Anderson and Thompson have shown (*cf.* [1] Proposition 7.5) that under the above conditions, the geodesic equations of the given connection is the Euler-Lagrange equation for many independent Lagrangians. Here we have shown that one of those Lagrangians is in fact a (pseudo) Riemannian metric.

REMARK 2. Theorem 1 is easy to generalize to the global setting. Recall that S is recurrent if there exists a unique 1-form ϕ such that $DS = S \otimes \phi$. Theorem 1 implies that ϕ is always closed and so defines a cohomology class $[\phi] \in H^1(\Sigma)$. Then D is a global metric connection if and only if $[\phi] = 0$.

A more intrinsic solution is the following:

THEOREM 2. *Let D be a connection on a surface Σ with Ricci curvature $S \neq 0$. D is locally a metric connection if and only if*

(1) D and S are symmetric,

(2) $\det S \neq 0$, and

(3) covariant differentiation preserves the eigenspaces of the curvature.

This (local) metric is defined up to multiplication by a constant.

REMARK 3. The eigenspaces of the map $R(x,y)\colon T_m\Sigma \longrightarrow T_m\Sigma$ are independent of the choice of $x, y \in T_m\Sigma$. The eigenspaces of the curvature are therefore well-defined.

REMARK 4. If $\det S > 0$, then the metric is Riemannian, otherwise it is pseudo-Riemannian.

1.2. *Proofs of Theorems 1 and 2.* Let D be a symmetric connection on Σ with Christoffel symbols Γ^i_{jk} with respect to a local coordinate system $x = (x^1, x^2)$ on $U \subseteq \Sigma$. D is a metric connection on U if and only if there is a non-degenerate, symmetric solution $g = g_{ij}(x)$ to the system of partial differential equations

$$2g_{hi}\Gamma^h_{jk} = \partial_k g_{ij} + \partial_j g_{ki} - \partial_i g_{jk}, \tag{2}$$

where $1 \leq i, j, k, h \leq 2$.

Let $e_i := \partial / \partial x^i$ for $i \in \{1, 2\}$. The Ricci curvature is given in these components by

$$S = \begin{pmatrix} -R^2_{112} & R^1_{112} \\ -R^2_{212} & R^1_{212} \end{pmatrix},$$

where R is the Riemann curvature tensor.

If K denotes the Gaussian curvature of the metric $g = g_{ij}dx^i \otimes dx^j$ then a necessary condition for D to be the connection associated with g is

$$S = Kg. \tag{3}$$

This equation is equivalent to the existence of integral elements of the Pfaffian system representing (2). Under the assumption that S is non-zero, K is also non-zero and we observe that S must be symmetric and $\det S \neq 0$. Moreover we may write

$$g = hS$$

and this value for g must satisfy (2).

For notational convenience, we shall write $a = R^1_{112}$, $b = -R^2_{212}$ and $c = -R^1_{212}$, and we let f_i denote $e_i(f)$ for a function $f \colon U \longrightarrow R$.

Substituting hS for g into (2) we obtain the following necessary and sufficient conditions for the existence of a local integral manifold of I with independence condition $dx^1 \wedge dx^2 \neq 0$:

$$\begin{aligned} 2(-b\Gamma^1_{11} + a\Gamma^2_{11}) &= -b_1 - bh_1h^{-1} \\ 2(-b\Gamma^1_{12} + a\Gamma^2_{12}) &= -b_2 - bh_2h^{-1} \\ \left(-b\Gamma^1_{12} + a(\Gamma^1_{11} + \Gamma^2_{12}) + c\Gamma^2_{11}\right) &= a_1 + ah_1h^{-1} \\ \left(-b\Gamma^1_{22} + a(\Gamma^1_{12} + \Gamma^2_{22}) + c\Gamma^2_{12}\right) &= a_2 + ah_2h^{-1} \\ 2(a\Gamma^1_{12} + c\Gamma^2_{12}) &= c_1 + ch_1h^{-1} \\ 2(a\Gamma^1_{22} + c\Gamma^2_{22}) &= c_2 + ch_2h^{-1}. \end{aligned} \tag{3}$$

These relations must be algebraically consistent for h_1h^{-1}, and h_2h^{-1} whereby we obtain

$$\begin{aligned} 2(-b\Gamma^1_{11} + a\Gamma^2_{11}) &= -b_1 - bh_1h^{-1} \\ 2(-b\Gamma^1_{12} + a\Gamma^2_{12}) &= -b_2 - bh_2h^{-1} \\ ab_1 - a_1b - ab\Gamma^1_{11} - b^2\Gamma^1_{12} + (2a^2 + bc)\Gamma^2_{11} + ab\Gamma^2_{12} &= 0 \\ ab_2 - a_2b - ab\Gamma^1_{12} - b^2\Gamma^1_{22} + (2a^2 + bc)\Gamma^2_{12} + ab\Gamma^2_{22} &= 0 \\ ac_1 - a_1c + ac\Gamma^1_{11} - (2a^2 + bc)\Gamma^1_{12} + c^2\Gamma^2_{11} - ac\Gamma^2_{12} &= 0 \\ ac_2 - a_2c + ac\Gamma^1_{12} - (2a^2 + bc)\Gamma^1_{22} + c^2\Gamma^2_{12} - ac\Gamma^2_{22} &= 0. \end{aligned}$$

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The integrability condition for the first two of these six equations is

$$(b_1a - ba_1)\Gamma^2_{12} + b(b\partial_1\Gamma^1_{12} - a\partial_1\Gamma^2_{12}) + (ba_2 - b_2a)\Gamma^2_{11} + b(a\partial_2\Gamma^2_{11} - b\partial_2\Gamma^1_{11}) = 0,$$

which turns out to be an algebraic consequence of the remaining four equations. Thus we have shown

LEMMA 3. *$(I, dx^1 \wedge dx^2)$ has the following integrability conditions:*

$$\begin{aligned}
ab_1 - a_1b - ab\Gamma^1_{11} - b^2\Gamma^1_{12} + (2a^2 + bc)\Gamma^2_{11} + ab\Gamma^2_{12} &= 0\\
ab_2 - a_2b - ab\Gamma^1_{12} - b^2\Gamma^1_{22} + (2a^2 + bc)\Gamma^2_{12} + ab\Gamma^2_{22} &= 0\\
ac_1 - a_1c + ac\Gamma^1_{11} - (2a^2 + bc)\Gamma^1_{12} + c^2\Gamma^2_{11} - ac\Gamma^2_{12} &= 0\\
ac_2 - a_2c + ac\Gamma^1_{12} - (2a^2 + bc)\Gamma^1_{22} + c^2\Gamma^2_{12} - ac\Gamma^2_{22} &= 0.
\end{aligned}$$

Moreover, when these conditions are satisfied, the solutions $g = g(x^1, x^2)$ are determined up to an arbitrary non-zero constant multiple.

The equation

$$DS = S \otimes \phi,$$

is equivalent to the six equations

$$\begin{aligned}
2(-b\Gamma^1_{11} + a\Gamma^2_{11}) &= -b_1 + b\phi_1\\
2(-b\Gamma^1_{12} + a\Gamma^2_{12}) &= -b_2 + b\phi_2\\
\left(-b\Gamma^1_{12} + a(\Gamma^1_{11} + \Gamma^2_{12}) + c\Gamma^2_{11}\right) &= a_1 - a\phi_1\\
\left(-b\Gamma^1_{22} + a(\Gamma^1_{12} + \Gamma^2_{22}) + c\Gamma^2_{12}\right) &= a_2 - a\phi_2\\
2(a\Gamma^1_{12} + c\Gamma^2_{12}) &= c_1 - c\phi_1\\
2(a\Gamma^1_{22} + c\Gamma^2_{22}) &= c_2 - c\phi_2,
\end{aligned}$$

where $\phi = \phi_1 dx^1 + \phi_2 dx^2$. Thus we see that if $DS = S \otimes \phi$ and $S \neq 0$ then ϕ is uniquely determined. Moreover the existence of such a ϕ is equivalent to the integrability equations for (2). This completes the proof of Theorem 1.

To prove the statement in Remark 2 we need the following two lemmas.

LEMMA 4. *Let D be a connection on a surface Σ with Ricci curvature $S \neq 0$ everywhere. D is a metric connection if and only if*

(1) D and S are symmetric,

(2) $\det S \neq 0$, and

(3) $DS = S \otimes dh$ for some function $h: M \longrightarrow R$.

PROOF. Follows by taking the covariant differential of the equation (3). ∎

LEMMA 5. *If S is a non-zero, symmetric Ricci tensor and $DS = S \otimes \phi$ then ϕ is closed.*

PROOF. This follows from Theorem 1, Lemma 4 and from the uniqueness of ϕ in the equation $DS = S \otimes \phi$. ∎

The statement in Remark 2 now follows.

In the proof of Theorem 2, it remains to be shown that the integrability conditions for (2) are equivalent to the condition that the eigenspaces of Ω are preserved by the connection. To this end, note that the eigenvalues of Ω are $\pm i\lambda$ where

$$\lambda = |a^2 + bc|^{1/2}.$$

The corresponding eigenspaces are

$$\mathcal{E}_1 = \left\{\big((a-\lambda)e_1 + be_2\big)l : l \in R\right\} = \left\{\big(ce_1 + (-a-\lambda)e_2\big)l : l \in R\right\}$$

and

$$\mathcal{E}_2 = \left\{\big((a+\lambda)e_1 + be_2\big)l : l \in R\right\} = \left\{(ce_1 + \big(-a+\lambda)e_2\big)l : l \in R\right\}.$$

Let W_1 and W_2 denote the sections of $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively. The following two lemmas describe when $DW_i \in \mathcal{E}_i$ for $i = 1, 2$. Let $k_i \in W_i$, for $i = 1, 2$. That is,

$$k_1 = \big((a-\lambda)e_1 + be_2\big)s_1 \quad \text{and} \quad k_2 = \big((a+\lambda)e_1 + be_2\big)s_2$$

for some $s_i \colon U \longrightarrow R - \{0\}$.

We now need the following lemma, which is proved by a direct computation.

LEMMA 6. *The equations*

$$\begin{aligned}
\big(-bdx^1 + (a-\lambda)dx^2\big)D_{e_1}k_1 &= 0\\
\big(-bdx^1 + (a-\lambda)dx^2\big)D_{e_2}k_1 &= 0\\
\big(-bdx^1 + (a+\lambda)dx^2\big)D_{e_1}k_2 &= 0\\
\big(-bdx^1 + (a+\lambda)dx^2\big)D_{e_2}k_2 &= 0
\end{aligned}$$

are equivalent to the four equalities

$$\begin{aligned}
ab_1 - a_1b - ab\Gamma^1_{11} - b^2\Gamma^1_{12} + (2a^2+bc)\Gamma^2_{11} + ab\Gamma^2_{12} &= 0\\
ab_2 - a_2b - ab\Gamma^1_{12} - b^2\Gamma^1_{22} + (2a^2+bc)\Gamma^2_{12} + ab\Gamma^2_{22} &= 0\\
\lambda(b\Gamma^1_{11} - 2a\Gamma^2_{11} - b\Gamma^2_{12}) + \lambda_1 b - \lambda b_1 &= 0\\
\lambda(b\Gamma^1_{12} - 2a\Gamma^2_{12} - b\Gamma^2_{22}) + \lambda_2 b - \lambda b_2 &= 0.
\end{aligned}$$

We may also write

$$k_1 = \big((ce_1 + (-a-\lambda)e_2\big)t_1 \quad \text{and} \quad k_2 = \big((ce_1 + (-a+\lambda)e_2\big)t_2$$

for some $t_i \colon U \longrightarrow R - \{0\}$.

To complete the proof, we need the following result, LEMMA 7. *The equations*

$$\begin{aligned}
\big((a+\lambda)dx^1 + cdx^2\big)D_{e_1}k_1 &= 0\\
\big((a+\lambda)dx^1 + cdx^2\big)D_{e_2}k_1 &= 0\\
\big((a-\lambda)dx^1 + cdx^2\big)D_{e_1}k_2 &= 0\\
\big((a-\lambda)dx^1 + cdx^2\big)D_{e_2}k_2 &= 0
\end{aligned}$$

are equivalent to the four equalities

$$\begin{aligned}
ac_1 - a_1c + ac\Gamma^1_{11} - (2a^2+bc)\Gamma^1_{12} + c^2\Gamma^2_{11} - ac\Gamma^2_{12} &= 0\\
ac_2 - a_2c + ac\Gamma^1_{12} - (2a^2+bc)\Gamma^1_{22} + c^2\Gamma^2_{12} - ac\Gamma^2_{22} &= 0\\
\lambda(c\Gamma^1_{11} - 2a\Gamma^1_{12} - c\Gamma^2_{12}) + \lambda c_1 - \lambda_1 c &= 0\\
\lambda(c\Gamma^1_{12} - 2a\Gamma^1_{22} - c\Gamma^2_{22}) + \lambda c_2 - \lambda_2 c &= 0.
\end{aligned}$$

It follows from these two lemmas that the condition that the connection preserves the eigenspaces of the curvature form is equivalent to the four equalities

$$\begin{aligned}
ab_1 - a_1b - ab\Gamma^1_{11} - b^2\Gamma^1_{12} + (2a^2+bc)\Gamma^2_{11} + ab\Gamma^2_{12} &= 0\\
ab_2 - a_2b - ab\Gamma^1_{12} - b^2\Gamma^1_{22} + (2a^2+bc)\Gamma^2_{12} + ab\Gamma^2_{22} &= 0\\
ac_1 - a_1c + ac\Gamma^1_{11} - (2a^2+bc)\Gamma^1_{12} + c^2\Gamma^2_{11} - ac\Gamma^2_{12} &= 0\\
ac_2 - a_2c + ac\Gamma^1_{12} - (2a^2+bc)\Gamma^1_{22} + c^2\Gamma^2_{12} - ac\Gamma^2_{22} &= 0,
\end{aligned}$$

which we recognize to be the integrability conditions for the system (2). This completes the proof of Theorem 2.

It should be pointed out, however, that in general a connection which is everywhere locally a metric connection need not be determined by a global metric as the following example shows.

EXAMPLE. Let g be a Riemannian metric

$$a_1(dx^1)^2 + a_2(dx^2)^2$$

where $a_1, a_2\colon R^2 \longrightarrow R$ are positive functions satisfying

$$a_i(x^1+1, x^2) = 2a_i(x^1, x^2),\ a_i(x^1, x^2+1) = a_i(x^1, x^2), \quad i = 1, 2$$

and Γ^i_{jk} the corresponding Christoffel symbols. It is easy to see that all Γ^i_{jk} are periodic in x^1, x^2, but the metric g is not itself. This means that Γ^i_{jk} projects to a connection on the torus T^2, but the metric g does not project down. In other words, the connection Γ^i_{jk} on T^2 is not a global metric connection. ∎

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2. The characteristics of connections on SO(3) bundles.

2.1. *Characteristic curves.* Let (x^1, x^2) be a local coordinates on Σ, and $\omega_4 = \pi^*(dx^1)$, $\omega_5 = \pi^*(dx^2)$. Take 1-forms $\omega_1, \omega_2, \omega_3$ on F, such that H is defined by $\omega_1 = \omega_2 = \omega_3 = 0$ and

$$\begin{pmatrix} d\omega_1 \\ d\omega_2 \end{pmatrix} = \begin{pmatrix} a_{11}\omega_4 + a_{21}\omega_5 & b_{11}\omega_4 + b_{21}\omega_5 \\ a_{21}\omega_4 + a_{22}\omega_5 & b_{21}\omega_4 + b_{22}\omega_5 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} + \begin{pmatrix} \omega_3 \wedge \omega_5 \\ -\omega_3 \wedge \omega_4 \end{pmatrix} + A_1; \tag{4}$$
$$d\omega_3 = \omega_4 \wedge \omega_5;$$

where $A_1 = c_1\omega_1 \wedge \omega_2 + c_2\omega_1 \wedge \omega_3 + c_3\omega_2 \wedge \omega_3$ for some vector-valued functions c_1, c_2, c_3. Of course, these ω_i's are not uniquely defined, but are determined up to a linear transformation of the form

$$q = \begin{pmatrix} JAJ^{-1}\det A & 0 & 0 \\ B & \det A & 0 \\ C & D & A \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5}$$

where A is a 2×2 matrix.

Let $H^\perp \subset T^*F$ be the sub-bundle spanned by $\omega_1, \omega_2, \omega_3$, and $i: H^\perp \longrightarrow T^*F$ the inclusion. Recall that a characteristic $X \in TH^\perp$ is such that $i^*\omega(X, \cdot) = 0$, where ω is the symplectic two-form on T^*F (*cf.* [3]). The projection of an integral curve of characteristic elements to Σ is to be called a *characteristic curve*.

The following result is due to E. Cartan [5].

LEMMA 8. *The characteristic curves are given by the* 2*-nd order differential equations*

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0, \quad i = 1, 2, \tag{6}$$

where $\Gamma^1_{11} = a_{11}$, $\Gamma^1_{12} = \Gamma^1_{21} = (a_{12} + a_{21})/2$, $\Gamma^1_{22} = a_{22}$, $\Gamma^2_{11} = b_{11}$, $\Gamma^2_{12} = \Gamma^2_{21} = (b_{12} + b_{21})/2$, $\Gamma^2_{22} = b_{22}$.

PROOF. By Pontryagin's maximal principle. Let X_4, X_5 be vector fields on M spanning H and satisfying $p_*(X_4) = \partial/\partial x^1$, $p_*(X_5) = \partial/\partial x^2$, and $X_3 = [X_4, X_5] \bmod(H)$, $X_2 = -[X_5, X_3] \bmod(X_3, X_4, X_5)$, $X_1 = [X_4, X_3] \bmod(X_3, X_4, X_5)$. Now $X_1, \ldots, X_5$ is dual to $\omega_1, \ldots, \omega_5$. Introduce functions on T^*F

$$f_i : T^*F \longrightarrow R, \quad f_i(x, p) = \langle X_i(x), p \rangle, \quad i = 1, \ldots, 5.$$

Then an abnormal extremal is the solution of the Hamiltonian system $af_4 + bf_5$ subject to the constraint $f_1 = f_2 = f_3 = 0$ for some functions a, b on F. An abnormal extremal projects to a characteristic on Σ.

Now differentiating f_3 along the abnormal extremal, one obtains

$$0 = \frac{d}{dt}f_3 = a\{f_3, f_4\} + b\{f_3, f_5\}$$

where $\{\ ,\ \}$ denotes the Poisson bracket on T^*F. Using the fact that $\{f_3, f_4\} = f_2$, $\{f_3, f_5\} = -f_1$ along the abnormal extremal, one has $af_2 - bf_1 = 0$. So there is a function c such that $a = cf_1, b = cf_2$. Hence the Hamiltonian function can be written as $c(f_1f_4 + f_2f_5)$. Reparametrizing the time variable, we can take $c = 1$. Writing out the Hamiltonian system of $f_1f_4 + f_2f_5$ on T^*F, and eliminating the variable p, we obtain the second order equations (6). ∎

2.2. *The normalization.* The group of linear transformations as in (5) is too large. For example, if we make a change of variables $(x^1, x^2) \longrightarrow (\bar{x}^1, \bar{x}^2)$, then the equation (6) changes from one into another under the transformation $(x^1, x^2, t) \longrightarrow (\bar{x}^1, \bar{x}^2, \bar{t})$ where

$$\bar{t} = \left(\det \frac{\partial(\bar{x}^1, \bar{x}^2)}{\partial(x^1, x^2)}\right)^{-1} t,$$

so the time-variable is also changed. We need to reduce the group of transformations as in (5) to a smaller group. To this end, we will explore the action of SO(3) on F.

First note that on the distribution K there is a metric, induced from the Killing form on so(3), which is unique up to a multiplication by a constant. Fix such a metric. Then the $1-D$ sub-distribution $H_1 = K \bigcap [H, H]$ inherits a metric g. Suppose that there is a global section e of H_1, *i.e.*, the Euler characteristic class of H_1 is zero. Normalize e so that it has norm 1 in g, *i.e.* $g(e, e) = 1$. Since e is dual to ω_3, we can normalize ω_3 by

$$\langle \omega_3, e\rangle = 1. \tag{7}$$

Henceforth we will always assume that ω_3 is normalized, *i.e.* fixed.

Now the transformations on ω_i's preserving the relations (5), (7) are of the form

$$q = \begin{pmatrix} JAJ^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ C & D & A \end{pmatrix},$$

where A is a 2×2 matrix of determinant 1. Let G be the group of all such linear transformations. This means that a G-structure is defined on the cotangent bundle T^*M. In particular, it induces an SL(2)-structure on $T^*\Sigma$. Here by a G-structure on a vector bundle we mean a reduction of the structure group of the bundle to G.

Note that if $\det(A) = 1$, then the Γ^i_{jk} transform like the Christoffel symbols of a connection under a change of local coordinates. More precisely, if one has a change of coordinates (x^1, x^2) to $(\bar{x}^1, \bar{x}^2)$, and $\det \partial(\bar{x}^1, \bar{x}^2)/\partial(x^1, x^2) = 1$, then the Γ^i_{jk} changes under the law

$$\bar{\Gamma}^\alpha_{\beta\gamma} = \sum_{i,j,k} \Gamma^i_{jk} \frac{\partial x^j}{\partial \bar{x}^\beta} \frac{\partial x^k}{\partial \bar{x}^\gamma} \frac{\partial \bar{x}^\alpha}{\partial x^i} + \sum \frac{\partial^2 x^i}{\partial \bar{x}^\beta \partial \bar{x}^\gamma} \frac{\partial x^\alpha}{\partial x^i}.$$

This suggests that Γ^i_{jk} defines a connection on some SL(2)-bundle over Σ, which will be made explicit below.

Let $\omega_i, i = 1, \ldots, 5$ be as in (5), with ω_3 normalized. Take

$$\Gamma = \begin{pmatrix} \Gamma^1_{11}dx^1 + \Gamma^2_{11}dx^2 & \Gamma^1_{12}dx^1 + \Gamma^2_{12}dx^2 \\ \Gamma^1_{21}dx^1 + \Gamma^2_{21}dx^2 & \Gamma^1_{22}dx^1 + \Gamma^2_{22}dx^2 \end{pmatrix}$$

Let Y be the principal SL(2)-bundle over Σ associated with the SL(2) structure on $T^*\Sigma$. SL(2) acts on the fiber of Y by conjugation. Define a gl(2)-valued one-form on Y by

$$\theta = dAA^{-1} + A\Gamma A^{-1}. \tag{8}$$

It is easy to see that this defines a GL(2)-connection on Y, as under the change of a frame of Y by the conjugation of an element $A \in \mathrm{SL}(2)$, θ changes under the law

$$\theta^* = A\theta A^{-1}.$$

As usual, we define the curvature of the connection to be

$$\Omega = d\theta - [\theta, \theta].$$

In a local coordinates (x^1, x^2), the above can be written as

$$\Omega = d\Gamma + \Gamma \wedge \Gamma.$$

Now applying Theorem 1, we see that if the curvature form Ω is traceless, *i.e.* $\operatorname{tr}\Omega = 0$, and the eigenspaces of Ω are preserved by the connection, then the equations (6) for the characteristic curves are the geodesic equation for some (Riemannian or pseudo-Riemannian) metric.

2.3. *Examples.* In this section we will show that our results apply to the rolling ball problem on a surface in R^3. Throughout this section we fix an orthonormal frame $\bar{e}_1, \bar{e}_2, \bar{e}_3$ for R^3.

Let Σ be a surface in R^3. Let e_1, e_2 be a local frame of orthonormal vector fields tangent to Σ, and $e_3 = e_1 \times e_2$ the normal tangent vector to Σ. Denote e^1, e^2, e^3 the dual frame of e_1, e_2, e_3 with the structure equation

$$de^3 = \omega_3^1 e^1 + \omega_3^2 e^2, \quad \omega_3^1 = ae^1 + be^2, \omega_3^2 = be^1 + ce^2. \tag{9}$$

Consider a ball rolling without slipping on Σ. To describe the motion of the ball, we need to introduce an orthonormal frame $f_1(t), f_2(t), f_3(t)$ fixed with the rolling ball. We assume that at $t = 0, f_1, f_2, f_3$ agree with $\bar{e}_1, \bar{e}_2, \bar{e}_3, f_i(0) = \bar{e}_i, i = 1, 2, 3$. Now let

$$\big(f_1(t), f_2(t), f_3(t)\big) = R(t)\big(f_1(0), f_2(0), f_3(0)\big)$$

where $R(t)$ is the rotation matrix. So $R(t)$ defines a curve in the orthogonal group SO(3). As usual, we identify R^3 with the Lie algebra so(3), so if v is a vector in R^3, we denote $\hat{v}$ the corresponding element in so(3).

The motion of the rolling ball can be described as follows.

LEMMA 9. *Let $c(t)$ be the center of the ball, then*

$$\begin{aligned} \frac{dc(t)}{dt} &= v_1 e_1 + v_2 e_2 \\ \frac{dR(t)}{dt} &= R(t)(v_2 \hat{e}_1 - v_1 \hat{e}_2) \end{aligned} \tag{10}$$

where v_1, v_2 are the components of the velocity of the center.

Note that (10) defines a connection on the fiber bundle $\Sigma \times SO(3)$ whose parallel transport is described by (10).

Take the vector fields $X_1 = (e_1, -R\hat{e}_2), X_2 = (e_2, R\hat{e}_1), X_3 = X_1 \times X_2$. Then we compute $X_3 = (0, R\hat{e}_3) \bmod (X_1, X_2)$. In particular, with respect to the Killing form on so(3) X_3 has norm 1. This justifies the renormalization procedure in Section 3.

Next, using (9), we have

$$[X_1, X_3] = \Big(0, R\big(-\hat{e}_1 - (b\hat{e}_1 + c\hat{e}_2)\big)\Big), \quad [X_2, X_3] = \Big(0, R\big(-\hat{e}_2 + (a\hat{e}_1 + b\hat{e}_2)\big)\Big) \bmod (X_1, X_2)$$

So we see that the connection is Cartan type if and only if

$$\det \begin{pmatrix} 1+b & c \\ a & b-1 \end{pmatrix} \neq 0,$$

that is, the Gaussian curvature of Σ is not -1. Moreover, it is easy to see that the characteristics project down to the geodesics on Σ (see [10] for the case where Σ is a plane in R^3).

Before concluding, we shall mention that our result has an interesting application in control theory. Note that (1) defines a control system in the following way. Suppose H is spanned by X_1, X_2, then the control system is $\dot{y}(t) = u_1 X_1 + u_2 X_2$, where $y(t)$ is a path in F, and u_1, u_2 the controls. Then the characteristic curves correspond to the abnormal extremals of the control system (*cf.* [7], [8] for a discussion of abnormal extremals). A part of the optimal control problem is to choose a cost-functional of the form

$$\int a_{ij}(y) u_i u_j \, dt,$$

which we want to minimize over the space of curves subject to the control law $\dot{y}(t) = u_1 X_1 + u_2 X_2$. For a metrizable connection, our result provides a canonical way of selecting a cost-functional such that *all the abnormal extremals are the extremals of the cost-functional.*

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REFERENCES

1. I. Anderson and G. Thompson, *The inverse problem of the calculus of variations for ordinary differential equations*, Mem. Amer. Math. Soc. **473**(1992).
2. L. Bates and J. Sniatycki, *Non-holonomic reduction*, preprint.
3. R. Bryant, S. S. Chern, R. Gardner, H. Goldschmidt and P. Griffiths, *Exterior Differential Systems*, Springer-Verlag, 1991.
4. R. Bryant and L. Hsu, *Rigidity of integral curves of rank two distributions*, 1993, preprint.
5. E. Cartan, *Les systemes de Pfaff a cinq variables et les equations aux derivees partielles du second ordre*, Ann. Sci. École Norm. Sup. **27**(1910), 109–192.
6. J. Douglas, *Solution of the inverse problem of the calculus of variations*, Trans. Amer. Math. Soc. **50**(1941), 71–128.
7. Zhong Ge, *On a class of constrained variational problems for systems linear in the controls*, preprint.

8. ———, *Caustics in constrained variational problems and optimal control*, preprint.
9. P. A. Griffiths, *Exterior Differential Systems and the Calculus of Variations*, Progr. Math. **25**, Birkhauser, 1983.
10. V. Jurdjevic, *The geometry of the plate-ball problem*, preprint.
11. N. Kamran, K. G. Lamb and W. Shadwick, *The local equivalence problem for* $d^2y/dx^2 = F(x,y,dy/dx)$ *and the Painlevé transcendents*, J. Differential Geom. **22**(1985), 117–123.
12. G. Morandi, C. Ferrario and G. Vecchio, *The inverse problem in the calculus of variations and the geometry of the tangent bundle*, Phys. Rep. **188**(1990), 147–284.
13. B. Schmidt, *Conditions on a connection to be a metric connection*, Comm. Math. Phys. **29**(1973), 55–59.

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