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QUASIGRAPHS

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1. Introduction. In the study of direct differential geometry, families of oriented arcs and curves have been employed extensively to define the differentiability of an arc at a point in various kinds of planes; cf. [**2**]. In [**6**], P. Scherk used lines in the projective plane; in [**3**] and [**4**], N. D. Lane and P. Scherk used circles in the conformal plane; conic-sections in the projective plane were employed in [**5**] and [**7**] by N. D. Lane and K. D. Singh; in [**1**], M. Gupta and N. D. Lane used the graphs of polynomials of degree at most n in the affine plane. For non-linear differentiability, the families of curves which were employed sometimes contained degenerate curves such as isolated points, pairs of lines, rays and even lines and rays counted with a multiplicity greater than one. These different investigations on direct differentiability, order and characteristic followed surprisingly similar patterns and led naturally to a search for a general theory of differentiability which would include, as particular cases, the linear, circular, conic-sectional and polynomial theories. In the present paper, the authors introduce structures called quasigraphs which appear to form a suitable basis for such a general theory.

A *quasigraph* in the unit disk G consists, roughly speaking, of a finite graph $[K]$ in G, together with a decomposition of $G\backslash[K]$ into two distinct open sets K^1 and K^{-1}. By means of an isotopy of G, we then obtain a family $\mathfrak{A}$ of quasigraphs. If $Q \in [K_1] \cap [K_2]$ for two distinct quasigraphs K_1 and K_2 in $\mathfrak{A}$, we require $Q \in [K]$ for all K in $\mathfrak{A}$. If $Q \in [K_1] \cap [K_2]$, then K_1 and K_2 can intersect at Q, or support at Q, or do neither, depending on the number 4, 3, or $\leqq 2$ of non-void sets $K_1^{\pm 1} \cap K_2^{\pm 1} \cap N$, where N is a small neighbourhood of Q.

Our first theorem asserts that if there are two distinct quasigraphs in $\mathfrak{A}$ which support (intersect) at Q, then any two quasigraphs in $\mathfrak{A}$ will support (intersect) at Q. This property of the families $\mathfrak{A}$ will be needed for the definition of differentiability and the introduction of the characteristic of a point of an arc.

Suppose any two quasigraphs of $\mathfrak{A}$ support (intersect) at Q. Let N be a small neighbourhood of Q. Consider h distinct quasigraphs $K_1, \ldots, K_h$ in $\mathfrak{A}$. Then exactly $h + 1$ (exactly $2h$) of the 2^h sets $K_1^{\pm 1} \cap \ldots \cap K_h^{\pm 1} \cap N$ are non-void; $h \geqq 2$ (Theorem 2 (Theorem 3)).

Our final Theorem 4 asserts that our construction of quasigraphs is equivalent to their definition by means of certain equivalence classes of sets of oriented

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Jordan curves and arcs. It is these classes of sets which constitute the immediate generalization of the examples mentioned at the beginning of this introduction.

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2. Basic definitions.

2.1. Our domain is the closed unit disk G in the Euclidean plane. A *Jordan arc* (*curve*) is the homeomorphic image in G of a closed interval (of the circle).

2.1.1. We consider a finite set of points called *vertices* and of "edges" in G. An *edge* is either a Jordan curve, possibly with one point removed, or the relative interior of a Jordan arc. Any two edges shall be disjoint and no edge shall meet bd$G$ or contain a vertex. Every endpoint of an edge shall be a vertex.

A *loop* is an edge whose closure contains at most one vertex. No vertex in int G shall be the endpoint of precisely two edges (loops counted twice). One or both of the sets of vertices and edges may be void.

2.1.2. Given any such set of vertices and edges in G, we shall denote by $[K]$ the set of all those points that either are vertices or lie on edges.

Let K^1 and K^{-1} be any open sets which partition $G\backslash[K]$. Thus every connected component of $G\backslash[K]$ lies entirely in K^1 or K^{-1}. Then we call the ordered triple $([K], K^1, K^{-1})$ a *quasigraph* and denote it by K. In particular, we call

$$(\emptyset, G, \emptyset) \quad \text{and} \quad (\emptyset, \emptyset, G)$$

the *void quasigraphs*.

We have

$$G = [K] \cup K^1 \cup K^{-1} \tag{2.1.1}$$

and

$$[K] = \mathscr{C}K^1 \cap \mathscr{C}K^{-1}.$$

We say K *decomposes* G if both K^1 and K^{-1} are non-void. It *decomposes G at a point Q* if

$$K^1 \cap N \neq \emptyset \quad \text{and} \quad K^{-1} \cap N \neq \emptyset$$

for every neighbourhood N of Q.

If $K = ([K], K^1, K^{-1})$ is a quasigraph, so is $L = ([K], K^{-1}, K^1)$. We call K and L *opposite* quasigraphs.

Obviously if K decomposes G at one point of an edge E, then it will do so at every point of E. We then call E *odd*. Any non-odd edge is called *even*. Then K decomposes G at a vertex P if and only if P is the endpoint of at least one odd edge. A vertex is the endpoint of an even number of odd edges, counting odd loops twice.

2.2. The open sets K^1 and K^{-1} being disjoint, we have

$$(2.2.1) \qquad \overline{K^\alpha} \cap K^{-\alpha} = \emptyset$$

and

$$(2.2.2) \qquad \overline{K^\alpha} \cap \operatorname{int} \overline{K^{-\alpha}} = \emptyset; \quad \alpha = \pm 1.$$

By (2.2.1) and (2.1.1),

$$\overline{K^\alpha} \subset K^\alpha \cup [K].$$

Finally, since $[K] \subset \overline{K^1} \cup \overline{K^{-1}}$, (2.1.1) implies

$$G = \overline{K^1} \cup \overline{K^{-1}},$$

i.e.

$$\mathscr{C}\overline{K^\alpha} \subset \overline{K^{-\alpha}}.$$

Hence

$$\mathscr{C}\overline{K^\alpha} = \operatorname{int} \mathscr{C}\overline{K^\alpha} \subset \operatorname{int} \overline{K^{-\alpha}},$$

i.e.

$$G = \overline{K^\alpha} \cup \operatorname{int} \overline{K^{-\alpha}}; \quad \alpha = \pm 1.$$

2.3. Define

$$(2.3.1) \qquad [\tilde{K}] = \overline{K^1} \cap \overline{K^{-1}} = \operatorname{bd} K^1 \cap \operatorname{bd} K^{-1}$$

and

$$(2.3.2) \qquad \tilde{K}^\alpha = \operatorname{int} \overline{K^\alpha}; \quad \alpha = \pm 1.$$

Then $[\tilde{K}]$ is the union of the closures of the odd edges of K. In particular, $[\tilde{K}] \subset [K]$. Also we have

$$(2.3.3) \qquad K^\alpha \subset \tilde{K}^\alpha \subset (K^\alpha \cup [K]).$$

2.4. By (2.3.1) and (2.3.2), $[\tilde{K}]$ is closed while $\tilde{K}^1$ and $\tilde{K}^{-1}$ are open. Since every point of G belongs to one and only one of the three sets $[\tilde{K}]$, $\tilde{K}^1$, $\tilde{K}^{-1}$, the triple $\tilde{K} = ([\tilde{K}], \tilde{K}^1, \tilde{K}^{-1})$ is a quasigraph and our notation is justified. We shall $\tilde{K}$ the *reduced quasigraph* of K.

K decomposes G at Q if and only if $\tilde{K}$ does.

Let $\tilde{K}$ and $\tilde{L}$ be reduced quasigraphs, $[\tilde{K}] = [\tilde{L}]$. Then $\tilde{K}$ and $\tilde{L}$ are equal or opposite.

2.5. Every edge of $\tilde{K}$ is odd and is the union of vertices and odd edges of K. Conversely, every odd edge of K is contained in some edge of $\tilde{K}$.

The set of vertices of $\tilde{K}$ is a (possibly improper) subset of the set of vertices of K. More precisely, a vertex of K in int G is a vertex of $\tilde{K}$ if and only if it is an endpoint of an even number greater than two of odd edges, odd loops being counted twice.

Starting with $\tilde{K}$ instead of K, we can construct $\tilde{\tilde{K}}$. Obviously, $\tilde{\tilde{K}} = \tilde{K}$.

2.6. $\tilde{K}^{-1} = \mathscr{C}\overline{K^1}$ *or, equivalently*, $\overline{K^1} = \mathscr{C}\tilde{K}^{-1} = [\tilde{K}] \cup \tilde{K}^1$.

Proof. By (2.2.2), $\overline{K^1} \cap \tilde{K}^{-1} = \emptyset$. Hence $\tilde{K}^{-1} \subset \mathscr{C}\overline{K^1}$.

Conversely, by (2.3.1) and (2.3.2), $[\tilde{K}] \cup \tilde{K}^1 \subset \overline{K^1}$. Taking the complements, we obtain $\mathscr{C}\overline{K^1} \subset \tilde{K}^{-1}$.

2.7. *Let K be any quasigraph and S be any open set in G. If $[\tilde{K}] \cap S \neq \emptyset$, then $S \cap K^\alpha \neq \emptyset$, $\alpha = \pm 1$.*

Proof. Let $P \in [\tilde{K}] \cap S$. Let E denote an odd edge of K through P or with the endpoint P. Since S is open, $E \cap S$ contains an interior point Q of E. Thus $Q \in S \cap E \subset S \cap [\tilde{K}]$.

Choose any small neighbourhood $N \subset S$ of Q. Since E is odd, $N \cap K^\alpha \neq \emptyset$ for $\alpha = \pm 1$. This proves our assertion.

2.8. *Let K and L be reduced quasigraphs such that $[K]$ and $[L]$ are homeomorphic and $[K] \subset [L]$. Then $[K] = [L]$.*

Proof. Every vertex of K is one of L. Since K and L have the same finite number of vertices, every vertex of L is also one of K. Let $Q_1, \ldots, Q_n$ denote these vertices. Let $f_{ij}[g_{ij}]$ be the number of edges of K[of L] connecting O_i and Q_j, $i, j = 1, 2, \ldots, n$. Every edge of K connecting Q_i and Q_j is one of L. Hence $f_{ij} \leqq g_{ij}$ for all i, j. Since K and L are homeomorphic, $\Sigma_{i,j} f_{ij} = \Sigma_{i,j} g_{ij}$. Hence $f_{ij} = g_{ij}$ for all i, j. Thus every edge of L connecting Q_i and Q_j is also an edge of K. A similar argument shows that K and L have the same loops without vertices. Thus $[K] = [L]$.

Since K and L are reduced, they can have only the same or opposite orientations. Thus K and L are either identical or opposite quasigraphs.

3. The metric space of the quasigraphs.

3.1. Let K be a quasigraph. Then $[K]$, $\mathscr{C}K^1$ and $\mathscr{C}K^{-1}$ are compact sets. We provide the collection of all the non-void compact subsets of G with its Hausdorff metric δ and define the distance d between two non-void quasigraphs K and K' by

$$d(K, K') = \delta(\mathscr{C}K^1, \mathscr{C}K'^1) + \delta(\mathscr{C}K^{-1}, \mathscr{C}K'^{-1}).$$

Thus $d(K, K') = 0$ if and only if $K = K'$. This defines a metric in the space of the non-void quasigraphs.

We complete this metric by postulating that each of the two void quasigraphs has the distance 4 from every other quasigraph.

3.2. *If K and K' are two non-void quasigraphs, then*

$$\delta([K], [K']) \leqq d(K, K').$$

Proof. Let ρ denote the ordinary Euclidean distance between two points of G. If P is any point of G and A is any non-empty compact subset of G, we write

$$\sigma(P, A) = \min_{Q \in A} \rho(P, Q).$$

Let $P' \in [K']$. If $P' \in [K]$, then

$$\sigma(P', [K]) = 0 \leqq d(K, K').$$

Let $P' \in K^\alpha$. Since $[K] \subset \mathscr{C}K^\alpha$, we readily verify

$$\sigma(P', [K]) = \sigma(P', \mathscr{C}K^\alpha).$$

The right hand term is not greater than

$$\delta(\mathscr{C}K'^\alpha, \mathscr{C}K^\alpha) \leqq d(K, K').$$

Thus

$$\sigma(P', [K]) \leqq d(K, K') \quad \text{for all } P' \in [K'].$$

Symmetrically, $\sigma(P, [K']) \leqq d(K, K')$, for all $P \in [K]$. Hence

$$\delta([K], [K']) \leqq d(K, K').$$

3.3. We study a family of quasigraphs

$$\{K_s | s \in I = (0, 1)\}$$

where K_s depends continuously on s. Note that $\mathscr{C}K_s^\alpha$ must then also be continuous in the sense of the δ-metric; $\alpha = \pm 1$. By 3.2, so is $[K_s]$.

The continuity of our family implies that either no K_s is void or every K_s is void.

3.4. *If $P \in K_s^\alpha$, then $P \in K_t^\alpha$ for all t near s.*

Proof. Let $P \in K_s^\alpha$. Since $P \notin \mathscr{C}K_s^\alpha$, the distance $\sigma(P, \mathscr{C}K_s^\alpha)$ from P to the compact set $\mathscr{C}K_s^\alpha$ is positive. By 3.3, $\sigma(P, \mathscr{C}K_t^\alpha)$ varies continuously with t and this distance remains positive for every t close to s. Hence $P \in K_t^\alpha$ for all such t.

3.5. *Let J be an open subsegment of $I = (0, 1)$. If $P \notin \bigcup_{s \in J}[K_s]$, then there is an $\alpha = \pm 1$ such that $P \in K_s^\alpha$ for all $s \in J$.*

Proof. Let $J_\alpha = \{s \in J | P \in K_s^\alpha\}$; $\alpha = \pm 1$. Then J_1 and J_{-1} are disjoint, $J = J_1 \cup J_{-1}$ and, by 3.4, J_1 and J_{-1} are open. Since J is connected, one of J_1 and J_{-1} is void.

3.5.1. Corollary. *Let $s_1 < s_2$. Then*

$$(K_{s_1}{}^1 \cap K_{s_2}{}^{-1}) \cup (K_{s_1}{}^{-1} \cup K_{s_2}{}^1) \subset \bigcup_{s_1 < s < s_2} [K_s]. \tag{3.5.1}$$

Proof. Let $P \in K_{s_1}{}^\alpha \cap K_{s_2}{}^{-\alpha}$. Suppose $P \notin \bigcup_{s_1 < s < s_2}[K_s]$. Then, by 3.5, $P \in K_s^\alpha \subset \mathscr{C}K_s^{-\alpha}$ for all $s \in (s_1, s_2)$. Hence $P \in \mathscr{C}K_{s_2}{}^{-\alpha} = K_{s_2}{}^\alpha \cup [K_{s_2}]$, a contradiction.

3.5.2. Obviously, (3.5.1) can be improved to

$$(K_{s_1}{}^1 \cap K_{s_2}{}^{-1}) \cup (K_{s_1}{}^{-1} \cap K_{s_2}{}^1) \subset \bigcup_{s_1 < s < s_2} [K_s] \setminus \bigcap_{s \in I} [K_s];$$

cf. 4.4.

4. Certain families of quasigraphs.

4.1. Let $I = (0, 1)$. In the following, we study families $\mathfrak{A} = \{K_s | s \in I\}$ of quasigraphs with the following property: there exists a quasigraph K and a continuous map $F : G \times I \to G$ such that, for each s, $F|_{G \times s}$ is a homeomorphism satisfying $F([K] \times s) = [K_s]$ and $F(K^1 \times s) = K_s{}^1$ (hence $F(K^{-1} \times s) = K_s{}^{-1}$, and $\mathfrak{A}$ is generated by an isotopy). Thus F is an open mapping.

If $J = [s, t]$ is a closed subinterval of I, and R is an interior point of G, then $[F|_{G \times J}]^{-1}(R)$ is readily seen to be a Jordan arc whose endpoints lie in int $(G \times \{s\})$ and int $(G \times \{t\})$ and which does not meet the boundary of $G \times J$ elsewhere (cf. 4.7 ff).

More conditions on $\mathfrak{A}$ will be added in 4.4 and 6.3.

4.1.1. $F|_{G \times s}$ maps each edge of K onto an edge of K_s and each vertex of K onto one of K_s. Loops are mapped onto loops. The parity of an edge is preserved (cf. 2.1).

4.1.2. If E is an edge of K and $Q \in G$, put $E_s = F(E, s)$, $Q_s = F(Q, s)$, etc.

4.2. *With* $\mathfrak{A}$, *the reduced family*

$$\tilde{\mathfrak{A}} = \{\tilde{K}_s | s \in I, K_s \in \mathfrak{A}\}$$

satisfies 4.1.

Proof. Let $\tilde{K}$ be the reduced quasigraph of K. Since $F|_{G \times s}$ is a homeomorphism, the definitions of 2.3 – 2.6 yield

$$\begin{aligned}[\tilde{K}_s] &= \overline{K_s{}^1} \cap \overline{K_s{}^{-1}} \\ &= \overline{F(K^1 \times s)} \cap \overline{F(K^{-1} \times s)} \\ &= F(\overline{K^1} \times s) \cap F(\overline{K^{-1}} \times s) \\ &= F((\overline{K^1} \cap \overline{K^{-1}}) \times s) \\ &= F([\tilde{K}] \times s)\end{aligned}$$

and

$$\begin{aligned}\tilde{K}_s{}^\alpha &= \operatorname{int} \overline{K_s{}^\alpha} \\ &= \operatorname{int} \overline{F(K^\alpha \times s)} \\ &= \operatorname{int} F(\overline{K^\alpha} \times s) \\ &= F(\operatorname{int} \overline{K^\alpha} \times s) \\ &= F(\tilde{K}^\alpha \times s).\end{aligned}$$

4.3. K_s *is continuous in the topology of the metric* 3.1.

Proof. Let $s \in I$. Choose a closed subinterval J of I which contains s. In the compact set $\mathscr{C}K^\alpha \times J$, F is uniformly continuous; $\alpha = \pm 1$. Let $\epsilon > 0$. Then there exists an $\eta > 0$ such that, in particular,

(4.3.1) $\quad \rho(F(x, s_1), F(x, s_2)) < \epsilon/2$

for all (x, s_1), (x, s_2) in $\mathscr{C}K^\alpha \times J$ such that $|s_1 - s_2| < \eta$.

Let $|s_1 - s_2| < \eta$ and $y_1 \in \mathscr{C}K_{s_1}{}^{\alpha}$. Thus $y_1 = F(x, s_1)$ for some $x \in \mathscr{C}K^{\alpha}$. Put $y_2 = F(x, s_2)$. Thus $y_2 \in \mathscr{C}K_{s_2}{}^{\alpha}$ and, by (4.3.1), $\rho(y_1, y_2) < \epsilon/2$. Thus y_1 lies in the $\epsilon/2$-neighbourhood of $\mathscr{C}K_{s_2}{}^{\alpha}$. As this applies to any $y_1 \in \mathscr{C}K_{s_1}{}^{\alpha}$, we obtain that $\mathscr{C}K_{s_1}{}^{\alpha}$ lies in the $\epsilon/2$-neighbourhood of $\mathscr{C}K_{s_2}{}^{\alpha}$. Symmetrically, $\mathscr{C}K_{s_2}{}^{\alpha}$ lies in the $\epsilon/2$-neighbourhood of $\mathscr{C}K_{s_1}{}^{\alpha}$. Hence

$$\delta(\mathscr{C}K_{s_1}{}^{\alpha}, \mathscr{C}K_{s_2}{}^{\alpha}) < \epsilon/2, \quad \alpha = \pm 1.$$

Therefore $d(K_{s_1}, K_{s_2}) < \epsilon$. In particular, $d(K_s, K_t) < \epsilon$ for all t close to s.

4.3.1. By 4.2, $\tilde{K}_s$ is also continuous in the topology of 3.1.

4.3.2. Let E be an edge of K and let E_s denote the corresponding edge of K_s. Then, $F|_{\bar{E}\times I}$ being continuous, our argument shows that the closure $\bar{E}_s$ of E_s depends continuously on s.

4.4. Let $M = \bigcap_{s \in I} [K_s]$ and $\tilde{M} = \bigcap_{s \in I} [\tilde{K}_s]$. We assume:

4.4.1. If $s \neq t$, then $K_s \neq K_t$ and

$$[K_s] \cap [K_t] = M.$$

Thus $\mathfrak{A}$ is a simple arc in the space of the quasigraphs.

4.4.2. Either $[\tilde{K}_s] = \tilde{M}$ for all $s \in I$ or, if $s \neq t$, then $\tilde{K}_s \neq \tilde{K}_t$ and

$$[\tilde{K}_s] \cap [\tilde{K}_t] = \tilde{M}.$$

The following example shows that 4.4.1 does not imply 4.4.2. Let

$$E_1 = \{(x, 0) | -1 < x < 0\}, E_2 = \{(x, 0) | 0 < x < 1\},$$

$E_3 = \{(0, y) | 0 < y < 1\}$; $V = (0, 0)$. K shall have the edges E_1, E_2, E_3 and vertices V, $(-1, 0)$, $(0, 1)$ and $(1, 0)$. $K^1 = \{(x, y) \in G | x < 0 \text{ or } y < 0\}$; $K^{-1} = \{(x, y) \in G | x > 0, y > 0\}$. Define K_s by sliding V on the x-axis from $(-\frac{1}{2}, 0)$ to $(\frac{1}{2}, 0)$, moving $E_{3,s}$ parallel to itself, expanding $E_{1,s}$ and shrinking $E_{2,s}$.

4.4.3. By 2.8, either $[\tilde{K}_s] = \tilde{M}$ for all $s \in I$ or $[\tilde{K}_s] \neq \tilde{M}$ for all $s \in I$.

4.5. *Suppose a vertex of K_s lies in* int G *and is the endpoint of three edges or more. Then it lies in M. In particular, every vertex of $\tilde{K}_s$ in* int G *belongs to $\tilde{M}$. Every vertex of K_s on* bd G *which is the endpoint of two edges or more is fixed.* (In these statements, loops are counted twice.)

Proof. Let Q be a vertex of K which is in int G and the endpoint of at least three edges of K. Suppose $Q_s \notin M$. Choose a neighbourhood N of Q_s so small that (i) its closure does not meet M or any edge of K_s which has not Q_s as an endpoint, (ii) this closure does not contain any other vertex of K_s, and (iii) $\mathscr{C}N$ meets every loop of K_s with the vertex Q_s.

Let $t > s$. Thus $Q_t \neq Q_s$. Choose t so close to s that $Q_u \in N$ for all u with $s < u \leqq t$. Let t_0 denote the smallest parameter value $> s$ for which Q_{t_0} lies on the circle C_t about Q_s through Q_t.

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For each edge of K_s with the endpoint Q_s, we consider the first point in which this edge meets C_t, (in the case of a loop, we consider the two points with this property closest to Q_s). These points divide C_t into a finite number of open arcs. As $Q_{t_0} \notin M$, one of them, say the arc A_t, contains Q_{t_0}. Let $P^{1,t}$ and $P^{2,t}$ denote the endpoints of A_t and let $E_s^{1,t}$ and $E_s^{2,t}$ be the edges (or the loop) of K_s through $P^{1,t}$ and $P^{2,t}$, respectively. Since there are only finitely many edges ending in Q_s, we may choose t such that

$$E_s^{1,u} = E_s^{1,t} = E_s^1 \quad \text{and} \quad E_s^{2,u} = E_s^{2,t} = E_s^2$$

for infinitely many $u > s$ and converging to s.

If $E_s^1 = E_s^2$, then this edge is a loop.

The arc A_t and the subarcs of E_s^1 and E_s^2 with the endpoints Q_s and $P^{1,t}$ and $P^{2,t}$ respectively, constitute the boundary of a region $R \subset N$. The arc $\{Q_u | s < u < t_0\}$ lies in C_t and connects Q_s with $Q_{t_0} \in \text{bd } R$ without meeting E_s^1 or E_s^2; hence it lies in R.

Choose an edge E_s^3 with the endpoint Q_s and distinct from E_s^1 and E_s^2. Choose $P_s \in E_s^3$ close to Q_s. If $u > s$ is sufficiently close to s, E_u^3 will be close to E_s^3. The point $P_u \in E_u^3$ will be close to P_s and hence outside R, while $Q_u \in R$. Hence the subarc of E_u^3 with the endpoints P_u and Q_u must meet bd R. As $\bar{A}_t$ has a positive distance from E_s^3, we have $E_u^3 \cap A_t = \emptyset$. Hence E_u^3 would have to meet either E_s^1 or E_s^2; a contradiction.

The proof of the last assertion follows similar lines.

4.6. *Let E_s be an edge of K_s and $Q_s \in E_s \backslash M$. Then there exists a neighbourhood N' of Q_s and an interval $[s_1, s_2]$ containing s in its interior such that*

$$N' \subset \bigcup_{t \in [s_1, s_2]} E_t.$$

Proof. Since M is compact, there is a neighbourhood N_s of Q_s such that $N_s \cap M = \emptyset$. Thus each point Q' of N_s lies on not more than one $[K_t]$. In particular, every $Q' \in N_s$ lies on not more than one E_t.

Since F is continuous, $N = F^{-1}(N_s)$ is open in $G \times I$. Let $E_s = F(E, s)$. Let $Q_s = F(Q, s)$. Thus $(Q, s) \in N = F^{-1}(N_s)$.

Let A be a closed subarc of E with the endpoints P_1 and P_2 containing Q in its relative interior such that $A \times s \subset N$. Hence there are s_1, s_2 such that $s_1 < s < s_2$ and $S = A \times [s_1, s_2] \subset N$. Thus $F(S) \subset N_s$ and

$$A_t = F(A \times t) \subset E_t \cap N_s \quad \text{for } s_1 \leqq t \leqq s_2,$$

As S is compact and F is a continuous bijection of S onto $F(S)$, $F|_S : S \to F(S)$ is a homeomorphism. In particular $F(\text{ int } S)$ is a non-void open set containing Q_s. Every point of this set lies on some E_t; $s_1 \leqq t \leqq s_2$. Thus any neighbourhood $N' \subset F(\text{int } S)$ will satisfy our theorem.

4.7. Let E be an edge of $\tilde{K}$. The preceding remarks enable us to study the restriction of F to $E \times I$. We first collect some preliminary observations.

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4.7.1. *If $R \in \bar{E}_t$ for all $t \in I$, then $(F|_{G\times J})^{-1}(R)$ is a Jordan arc in $\bar{E} \times J$ for every closed subinterval J of I* (cf. 4.1).

4.7.2. Let B be a connected component of $\tilde{M}$ which contains a vertex V_s of $\tilde{K}_s$. Thus V is a vertex of $\tilde{K}$ and every V_t is a vertex of $\tilde{K}_t$. As $V_t \in \tilde{M}$ for all t, V_t moves continuously in B. In particular, $V_t \in B$ for all $t \in I$.

(If $V_s \in \operatorname{bd} G$, then $V_t \in B \cap \operatorname{bd} G \subset \tilde{M} \cap \operatorname{bd} G$ and since each component of $\tilde{M} \cap \operatorname{bd} G$ is a point, we obtain $V_t = V_s$ for all $t \in I$.)

4.7.3. Suppose the connected component B of $\tilde{M}$ contains no vertex of $\tilde{K}_s$. By 4.7.2, B contains no vertex of $\tilde{K}_t$ for any t. Hence B will lie on some edge $D(t)$ of $\tilde{K}_t$. For every edge E of $\tilde{K}$, the set of parameter values t such that $D(t) = E_t$ is open. Hence there is an edge E such that $D(t) = E_t$ for all $t \in I$; thus $B \subset E_t$ for all $t \in I$.

4.7.4. From 4.7.2 and 4.7.3, we obtain the following result. *Let B be any connected component of $\tilde{M}$; $B \cap \bar{E}_s \neq \emptyset$. Then $B \cap \bar{E}_t \neq \emptyset$ for all $t \in I$.*

4.8. Let E again denote an edge of $\tilde{K}$. Suppose $\bar{E}$ is defined by the homeomorphism $\Gamma : \bar{I} \to \bar{E}$. Then

$$\bigcup_{t\in I} \bar{E}_t = F(\bar{E} \times I)$$

is given by the continuous function

$$f(\lambda, t) = F(\Gamma(\lambda), t); \quad \lambda \in \bar{I}, t \in I.$$

Each restriction $f|_{\bar{I}\times t}$ is a homeomorphism of $\bar{I}$ onto $\bar{E}_t$.

Let $s \in I$; $E_s \backslash \tilde{M} \neq \emptyset$. Being open in E_s, the set $E_s \backslash \tilde{M}$ is the union of at most countably many disjoint open subarcs. Let $A(s)$ be one of them. Thus $A(s)$ has a parametric representation

$$A(s) = \{f(\lambda, s) | \rho(s) < \lambda < \rho'(s)\}, \tag{4.8.1}$$

where $0 \leqq \rho(s) < \rho'(s) \leqq 1$. The arc $A(s)$ has the end points

$$R(s) = f(\rho(s), s) \quad \text{and} \quad R'(s) = f(\rho'(s), s).$$

They are either end points of E_s; i.e. vertices, or interior points of E_s, belonging to $\tilde{M}$ (cf. 4.5).

If $R(s) \notin \tilde{M}$, it is a vertex of $\tilde{K}_s$ on $\operatorname{bd} G$; if $R(s) = R_s = F(R, s)$, then put $R(t) = R_t$ for all $t \in I$. Thus $R(t)$ depends continuously on t. In this case, define $\rho(t) = 0$ for all t. Then $R(t) = f(\rho(t), t)$ for all $t \in I$.

Let $R(s)$ and $R'(s)$ be in $\tilde{M}$. Let B and B' denote the connected components of $\tilde{M}$ containing $R(s)$ and $R'(s)$ respectively.

Let

$$V(t) = f(0, t) \quad \text{and} \quad V'(t) = f(1, t)$$

be the end points of $\bar{E}_t$.

Suppose $B \neq B'$. By 4.7.4,

$$B \cap \bar{E}_t \neq \emptyset \neq B' \cap \bar{E}_t \quad \text{for all } t \in I.$$

Define

$$(4.8.2) \qquad \begin{cases} \rho(t) = \max\{\lambda \in \bar{I} | f(\lambda, t) \in B \cap \bar{E}_t\}, \\ \rho'(t) = \min\{\lambda \in \bar{I} | f(\lambda, t) \in B' \cap \bar{E}_t\}, \\ R(t) = f(\rho(t), t), \quad R'(t) = f(\rho'(t), t). \end{cases}$$

We wish to show that $R(t)$ depends continuously on t. Let $t_0 \in I$.

(i) Let $R(t_0) \neq V(t_0)$. Let $R_0 = f(\lambda_0, t_0)$ be an accumulation point of $R(t)$ as t tends to t_0. As $R_0 \in B$, we have $0 \leqq \lambda_0 \leqq \rho(t_0)$. We may assume that $B \cap \bar{E}_{t_0}$ contains more than one point. Let Q be any interior point of this arc. As $\bar{E}_t$ depends continuously on t, we have $Q \in \bar{E}_t$ for all t sufficiently close to t_0. Hence R_0 lies in the closed subarc of $\bar{E}_{t_0}$ bounded by Q and $R(t_0)$. As this holds true for every choice of Q, we have $R_0 = R(t_0)$.

(ii) Let $R(t_0) = V(t_0)$. Define R_0 as before. As

$$R_0 \in B \cap \bar{E}_{t_0} = \{V(t_0)\},$$

we obtain again $R_0 = R(t_0)$.

Now let

$$A(t) = \{f(\lambda, t) | \rho(t) < \lambda < \rho'(t)\}$$

denote the open subarc of E_t bounded by $R(t)$ and $R'(t)$. If $A(u)$ were to contain a point R'' of $\tilde{M}$ for some u, then R'' would lie on $\bar{E}_u$ between $R(u)$ and $R'(u)$. Thus R'' would belong to a component B'' of $\tilde{M}$ distinct from B and B'. By 4.7.4, $B'' \cap \bar{E}_t \neq \emptyset$ for all $t \in I$. By the continuity of $\bar{E}_t$, the order in which $\bar{E}_t$ meets B, B'', B' remains fixed as t ranges through I. Choosing $t = s$ yields a contradiction. Thus $A(t) \cap \tilde{M} = \emptyset$ for all t. As the end points of $\overline{A(t)}$ lie in $\tilde{M}$, $A(t)$ is a connected component of $E_t \backslash \tilde{M}$. With E_t, $R(t)$, $R'(t)$, $\overline{A(t)}$ depends continuously on t.

These results remain valid if $B = B'$. In this case, $\bar{E}_s$ and thus all $\bar{E}_t$ meet only the one component B of $\tilde{M}$.

The case that $R(s) \in \tilde{M}$ but $R(s') \notin \tilde{M}$ is similarly dealt with.

4.9. *Let $P_s \in E_s \backslash \tilde{M}$. Then there exists a continuous function $\lambda : I \to I$ such that*

$$P(t) = f(\lambda(t), t) \in E_t \backslash \tilde{M}$$

for all $t \in I$ and $P(s) = P_s$.

Proof. Let $A(s)$, defined by (4.8.1), be the connected component of $E_s \backslash \tilde{M}$ which contains P_s. Then

$$P_s = f(\lambda_0, s)$$

for some $\lambda_0 \in (\rho(s), \rho'(s))$. Let ξ be defined by

$$\lambda_0 = (1 - \xi)\rho(s) + \xi\rho'(s).$$

Then $0 < \xi < 1$ and

$$\lambda(t) = (1 - \xi)\rho(t) + \xi\rho'(t), \quad 0 < t < 1,$$

has the required properties.

4.9.1. The set

$$\bigcup_{u \in I} A(u)$$

contains no point of $\tilde{M}$. By 4.6, it is open. Each point of this set lies on exactly one of the arcs

$$C_\xi = \{P(t) = f(\lambda(t), t) | \lambda(t) = (1 - \xi)\rho(t) + \xi\rho'(t)\},$$

$0 < \xi < 1$. Thus this set is homeomorphic to $I \times I$.

4.9.2. Let $0 < s < t < 1$. Then the set

$$\bigcup_{s<u<t} A(u)$$

is again homeomorphic to $I \times I$.

4.10. (i) *Let* s $< t$. *Let* $\rho(u) > 0$ *for* $s \leqq u \leqq t$. *Then*

(4.10.1) $\quad R(u) = R(s) \quad$ *for* $s \leqq u \leqq t$.

(ii) *Let* $s < t$. *Assume* (4.10.1). *Let* N *be a neighbourhood of* $R(s)$ *in* G. *Then*

(4.10.2) $\quad N \cap A(u) \neq \emptyset \quad$ *for* $s \leqq u \leqq t$.

Proof. (i) Suppose the set

$$\{u | s \leqq u \leqq t; R(u) \neq R(s)\}$$

is not void. Let v_0 denote its infimum. As $R(u)$ is continuous, we have

$$R(v_0) = R(s) \quad \text{and} \quad s \leqq v_0 < t.$$

There are parameter values v_1 arbitrarily close to v_0 such that

(4.10.3) $\quad R(v_1) \neq R(v_0)$.

Let B denote the connected component of $\tilde{M}$ which contains $R(s)$. Since $\rho(u) > 0$ for $s \leqq u \leqq t$, 4.7.4 and (4.8.2) imply that

$$R(u) \in E_u \cap B \quad \text{for } s \leqq u \leqq t.$$

Choose a closed neighbourhood $\bar{N}$ of $R(s)$ such that $\bar{N} \cap [K_{v_0}] \subset E_{v_0}$. If u is close enough to v_0, no edge $\neq E_u$ of K_u can meet $\bar{N}$. Thus $R(v_0) \in E_u$. Choose v_1 according to (4.10.3) and sufficiently close to v_0. Then for $v_0 \leqq u \leqq v_1$, both $R(v_0)$ and $R(v_1)$ lie on E_u. For every such u, if v increases from v_0 to v_1, $R(v)$ moves continuously from $R(v_0)$ on E_u to $R(v_1)$. So the functions $\sigma_0(u)$ and $\sigma_1(u)$ are well defined by

$$R(v_0) = f(\sigma_0(u), u) \quad \text{and} \quad R(v_1) = f(\sigma_1(u), u) \quad \text{for } v_0 \leqq u \leqq v_1.$$

By (4.8.2) and (4.10.3), we have

$$\sigma_1(v_0) < \rho(v_0) = \sigma_0(v_0) \quad \text{and} \quad \sigma_0(v_1) < \rho(v_1) = \sigma_1(v_1).$$

Since σ_0 and σ_1 are continuous and $\sigma_0(u) \neq \sigma_1(u)$ for all u, this yields a contradiction.

(ii) Let D_ξ denote the following subarc of C_ξ:

$$(4.10.4) \quad D_\xi = \{f(\lambda(u), u) | s \leqq u \leqq t; \lambda(u) = (1 - \xi)\rho(u) + \xi\rho'(u)\};$$

(cf. 4.9.1). It suffices to show that $D_\xi \subset N$ if ξ is sufficiently small.

Suppose this assertion is false. Then there exists a sequence of positive numbers ξ converging to zero and for each ξ a parameter u, $s \leqq u \leqq t$, such that

$$P(u) = f((1 - \xi)\rho(u) + \xi\rho'(u), u) \notin N.$$

Let u_0 be an accumulation point of the u's. Since $\rho(v)$ and $\rho'(v)$ are continuous, the parameter values $(1 - \xi)\rho(u) + \xi\rho'(u)$ have the accumulation point $\rho(u_0)$. Since f is continuous, the points $P(u)$ converge to

$$f(\rho(u_0), u_0) = R(u_0) \quad \in N,$$

a contradiction.

5. Global decompositions.

5.1. *Let $P_s \in E_s \backslash \tilde{M}$. Construct the arc*

$$(5.1.1) \quad \{P(u) | u \in I\}$$

with $P(s) = P_s$ and $P(u) \in E_u \backslash \tilde{M}$ for all u, according to 4.9. *Let $s < s'$, $P(s') \in \tilde{K}_s^\alpha$. Then*

$$(5.1.2) \quad P(u) \in \tilde{K}_s^\alpha \quad \textit{for all } u > s,$$

$$(5.1.3) \quad P(u) \in \tilde{K}_s^{-\alpha} \quad \textit{for all } u < s.$$

Proof. The arc $\{P(t) | t > s\}$ does not meet $[K_s]$. Hence it lies entirely in $\tilde{K}_s^\alpha$.

Let $A(s)$ denote the connected component of $E_s \backslash \tilde{M}$ containing $P(s)$. By 4.9.1, the set

$$(5.1.4) \quad \bigcup_{t \in I} A(t)$$

is homeomorphic to $I \times I$, the homemorphism being given by the parameters t and ξ of 4.9. In particular, $A(s) \subset E_s$ decomposes (5.1.4) into two subsets, one in $\tilde{K}_s^\alpha$, the other in $\tilde{K}_s^{-\alpha}$ (cf. 2.7).

5.2. *Let $P_s \in E_s \backslash \tilde{M}$. Construct the arc* (5.1.1). *Let $t \neq s$. Then $P_s = P(s) \in \tilde{K}_t^\alpha$ if and only if $P(t) \in \tilde{K}_s^{-\alpha}$.*

Proof. Suppose $s < t$ and $P(s) \in \tilde{K}_t^\alpha$. Choose $u < s$. Then by 3.5, $P(u) \in \tilde{K}_s^\alpha$, and, by 5.1, $P(t) \in \tilde{K}_s^{-\alpha}$.

5.3. *If t and u lie on the same side of s in* $I = (0, 1)$, *then*

$$(5.3.1) \qquad [K_s] \cap K_t^\alpha = [K_s] \cap K_u^\alpha, \quad \alpha = \pm 1.$$

In particular,

$$[\tilde{K}_s] \cap \tilde{K}_t^\alpha = [\tilde{K}_s] \cap \tilde{K}_u^\alpha, \quad \alpha = \pm 1 \quad (\text{cf. } 4.2).$$

Proof. We may assume that $0 < t < u < s < 1$ and $[K_s] \cap K_t^\alpha \neq \emptyset$. Let $P \in [K_s] \cap K_t^\alpha$. Then $P \notin [K_t]$ and hence $P \notin M$ and $P \notin [K_u]$, for all $u \neq s$ (cf. 4.4.1). We can now apply 3.5 with $J = (0, s)$ and conclude that $P \in K_t^\alpha$ if and only if $P \in K_u^\alpha$. Since P was chosen arbitrarily in $[K_s] \cap K_t^\alpha$, this proves (5.3.1).

5.4. We note:

$$5.4.1. \qquad \begin{aligned}[\tilde{K}_s] \cap K_t^\alpha &= [\tilde{K}_s] \cap [K_s] \cap K_t^\alpha \\ &= [\tilde{K}_s] \cap [K_s] \cap K_u^\alpha \\ &= [\tilde{K}_s] \cap K_u^\alpha\end{aligned}$$

is independent of t; $t \in (0, s)$ *or* $t \in (s, 1)$.

$$5.4.2. \qquad \begin{aligned}[\tilde{K}_s] \cap [K_t] &= [\tilde{K}_s] \cap [K_s] \cap [K_t] \\ &= [\tilde{K}_s] \cap M\end{aligned}$$

is independent of t; $t \neq s$ (cf. 4.4.1).

5.4.3. *By* 5.4.2 *and* 4.4.2,

$$\begin{aligned}([\tilde{K}_s] \cap [K_t]) \backslash [\tilde{K}_t] &= ([\tilde{K}_s] \cap [K_t]) \backslash ([\tilde{K}_s] \cap [\tilde{K}_t]) \\ &= [\tilde{K}_s] \cap M \backslash \tilde{M}\end{aligned}$$

is independent of t; $t \neq s$.

5.5. *Let* $0 < u < s < v < 1$. *Then by* (5.3.1)

$$(5.5.1) \qquad \begin{aligned}[K_u] &= ([K_u] \cap K_s^1) \cup ([K_u] \cap K_s^{-1}) \cup ([K_u] \cap [K_s]) \\ &= ([K_u] \cap K_s^1 \cap K_v^1) \cup ([K_u] \cap K_s^{-1} \cap K_v^{-1}) \cup M \\ &\subset (K_s^1 \cap K_v^1) \cup (K_s^{-1} \cap K_v^{-1}) \cup M.\end{aligned}$$

More generally, if $0 < t_0 < t_1 < \ldots < t_h < 1$, *then*

$$[K_{t_0}] \subset (K_{t_1}^1 \cap \ldots \cap K_{t_h}^1) \cup (K_{t_1}^{-1} \cap \ldots \cap K_{t_h}^{-1}) \cup M.$$

In particular,

$$(5.5.2) \qquad [\tilde{K}_{t_0}] \subset (\tilde{K}_{t_1}^1 \cap \ldots \cap \tilde{K}_{t_h}^1) \cup (\tilde{K}_{t_1}^{-1} \cap \ldots \cap \tilde{K}_{t_h}^{-1}) \cup \tilde{M}.$$

5.6. *Let* $0 < u < s < v < 1$. *Then*

$$(5.6.1) \qquad [\tilde{K}_s] \subset (\tilde{K}_u^1 \cap \tilde{K}_v^{-1}) \cup (\tilde{K}_u^{-1} \cap \tilde{K}_v^1) \cup \tilde{M}.$$

Proof. Let $P_s \in [\tilde{K}_s] \backslash \tilde{M}$. Assume at first that $P_s \in \operatorname{int} G$. Construct the arc (5.1.1). Suppose $P(s) \in \tilde{K}_u^\alpha$. Applying consecutively 5.1, 5.2 and again 5.1, we obtain $P(v) \in \tilde{K}_u^\alpha$, $P(u) \in \tilde{K}_v^{-\alpha}$, $P(s) \in \tilde{K}_v^{-\alpha}$. This yields (5.6.1) if $P_s \in \operatorname{int} G$.

If $P_s' \in \operatorname{bd} G \cap [\tilde{K}_s] \backslash \tilde{M}$, choose $P_s \in \operatorname{int} G \cap [\tilde{K}_s] \backslash \tilde{M}$ close to P_s'. Thus

$$P_s \in \tilde{K}_v^{-\alpha} \cap \tilde{K}_u^\alpha.$$

As this applies to every such P_s, we obtain

$$\begin{aligned} P_s' &\in \overline{\tilde{K}_v^{-\alpha} \cap \tilde{K}_u^\alpha} \cap [\tilde{K}_s] \backslash \tilde{M} \\ &\subset \overline{\tilde{K}_v^{-\alpha}} \cap \overline{\tilde{K}_u^\alpha} \cap [\tilde{K}_s] \backslash \tilde{M} \\ &= (\tilde{K}_v^{-\alpha} \cup [\tilde{K}_v]) \cap (\tilde{K}_u^\alpha \cup [\tilde{K}_u]) \cap [\tilde{K}_s] \backslash \tilde{M} \\ &\subset \tilde{K}_v^{-\alpha} \cap \tilde{K}_u^\alpha. \end{aligned}$$

5.6.1. From (5.6.1) and (3.5.1), we obtain

$$\bigcup_{u<s<v} [\tilde{K}_s] = (\tilde{K}_u^1 \cap \tilde{K}_v^{-1}) \cup (\tilde{K}_u^{-1} \cap \tilde{K}_v^1) \cup \tilde{M}, \quad \text{for } u < v.$$

5.7. From (5.5.2) and (5.6.1), we obtain the following results.

5.7.1. *If* $0 < t_1 < \ldots < t_h < 1$, *then*

$$(5.7.1) \qquad [\tilde{K}_{t_i}] \subset \left(\bigcap_{j=1}^{i-1} \tilde{K}_{t_j}{}^1 \cap \bigcap_{j=i+1}^{h} \tilde{K}_j{}^{-1} \right) \cup \left(\bigcap_{j=1}^{i-1} \tilde{K}_{t_j}{}^{-1} \cap \bigcap_{j=i+1}^{h} \tilde{K}_{t_j}{}^1 \right) \cup \tilde{M};$$

$i = 2, \ldots, h - 1$. In the cases $i = 1$ and $i = h$, we interpret (5.7.1) by means of (5.5.2). Thus (5.7.1) remains valid for $i = 1$ and $i = h$ if we define

$$\bigcap_{j=1}^{0} \tilde{K}_{t_j}{}^\alpha = \bigcap_{j=h+1}^{h} \tilde{K}_{t_j}{}^\alpha = G.$$

5.7.2. COROLLARY. *If* $K_0, K_1, \ldots, K_h$ *are distinct quasigraphs of* $\mathfrak{A}$, *then there exist* $\alpha_i = \pm 1$; $i = 1, \ldots, h$ *such that*

$$(5.7.2) \qquad [\tilde{K}_0] \subset \left(\bigcap_{i=1}^{h} \tilde{K}_i^{\alpha_i} \right) \cup \left(\bigcap_{i=1}^{h} \tilde{K}_i^{-\alpha_i} \right) \cup \tilde{M}.$$

5.7.3. *If* $K_0, K_1, \ldots, K_h$ *are distinct quasigraphs of* $\mathfrak{A}$ *and*

$$[\tilde{K}_0] \cap \tilde{K}_1^{\alpha_1} \cap \ldots \cap \tilde{K}_h^{\alpha_h} \neq \emptyset,$$

then (5.7.2) *holds.*

Proof. By 5.7.2, there exist $\beta_1, \ldots, \beta_h$ such that

$$[\tilde{K}_0] \subset \left(\bigcap_{i=1}^{h} \tilde{K}_i^{\beta_i} \right) \cup \left(\bigcap_{i=1}^{h} \tilde{K}_i^{-\beta_i} \right) \cup \tilde{M}.$$

Thus any point $P \in [\tilde{K}_0] \backslash \tilde{M}$ lies either in $\bigcap_{i=1}^h \tilde{K}_i^{\beta_i}$ or in $\bigcap_{i=1}^h \tilde{K}_i^{-\beta_i}$. Let $P \in [\tilde{K}_0] \cap \bigcap_{i=1}^h \tilde{K}_i^{\alpha_i}$. Suppose, for instance, that $P \in \bigcap_{i=1}^h \tilde{K}_i^{\beta_i}$. Then

$$P \in \bigcap_{i=1}^{h} (\tilde{K}_i^{\alpha_i} \cap \tilde{K}_i^{\beta_i}).$$

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In particular, $\tilde{K}_t^{\alpha_i} \cap \tilde{K}_t^{\beta_i} \neq \emptyset$; $i = 1, \ldots, h$. Hence $\alpha_i = \beta_i$; $i = 1, \ldots, h$.

5.7.4. Let

$$[\tilde{K}_0] \cap \bigcap_{i=1}^{h} \tilde{K}_t^{\alpha_i} \neq \emptyset.$$

Applying 2.7 to $S = \bigcap_{i=1}^{h} \tilde{K}_t^{\alpha_i}$, we obtain

$$\bigcap_{i=1}^{h} \tilde{K}_t^{\alpha_i} \cap \tilde{K}_0^{\alpha} \neq \emptyset, \quad \alpha = \pm 1.$$

5.8. *If* $0 < s \leqq t < u \leqq v < 1$, *then*

$$K_t^{\alpha} \cap K_u^{-\alpha} \subset K_s^{\alpha} \cap K_v^{-\alpha}, \quad \alpha = \pm 1.$$

Proof. If w lies between u and 1, then by (5.5.1)

$$[K_w] \subset (K_t^1 \cap K_u^1) \cup (K_t^{-1} \cap K_u^{-1}) \cup M.$$

Hence $[K_w]$ has no point in $K_t^{\alpha} \cap K_u^{-\alpha}$.

Let $P \in K_t^{\alpha} \cap K_u^{-\alpha}$. Thus $P \notin [K_w]$. Since $P \in K_u^{-\alpha}$, 3.5 yields $P \in K_w^{-\alpha}$ for all w with $u \leqq w < 1$. Hence $P \in K_t^{\alpha} \cap K_v^{-\alpha}$. Thus we obtain

$$K_t^{\alpha} \cap K_u^{-\alpha} \subset K_t^{\alpha} \cap K_v^{-\alpha}.$$

A similar argument yields

$$K_t^{\alpha} \cap K_v^{-\alpha} \subset K_s^{\alpha} \cap K_v^{-\alpha}$$

and hence

$$K_t^{\alpha} \cap K_u^{-\alpha} \subset K_s^{\alpha} \cap K_v^{-\alpha}.$$

5.8.1. *Let* $s < u < t$. *Then*

$$\tilde{K}_s^1 \cap \tilde{K}_t^1 \subset \tilde{K}_u^1.$$

Proof. By 5.6, no point of $[\tilde{K}_u]$ is in $\tilde{K}_s^1 \cap \tilde{K}_t^1$ and thus

$$\tilde{K}_s^1 \cap \tilde{K}_t^1 = (\tilde{K}_s^1 \cap \tilde{K}_t^1 \cap \tilde{K}_u^1) \cup (\tilde{K}_s^1 \cap \tilde{K}_t^1 \cap \tilde{K}_u^{-1}). \tag{5.8.1}$$

But, by 5.8, $\tilde{K}_s^1 \cap \tilde{K}_u^{-1} \subset \tilde{K}_s^1 \cap \tilde{K}_t^{-1}$, so that

$$\tilde{K}_s^1 \cap \tilde{K}_u^{-1} \cap \tilde{K}_t^1 \subset \tilde{K}_s^1 \cap \tilde{K}_t^{-1} \cap \tilde{K}_t^1 = \emptyset.$$

Hence (5.8.1) becomes

$$\tilde{K}_s^1 \cap \tilde{K}_t^1 = \tilde{K}_s^1 \cap \tilde{K}_t^1 \cap \tilde{K}_u^1.$$

This proves our assertion.

5.9. Let $0 < t_1 < \ldots < t_h < 1$. Then at most $2h$ of the sets

$$\bigcap_{1}^{h} \tilde{K}_{t_\lambda}^{\alpha_\lambda}, \quad \alpha_1, \ldots, \alpha_h = 1, -1 \tag{5.9.1}$$

are non-void.

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We may assume that $[\tilde{K}_t] \neq \tilde{M}$ for one and therefore for every t. We wish to show that only the $2h$ sets

$$(5.9.2) \qquad \bigcap_1^i \tilde{K}_{t_\lambda}{}^{\alpha} \cap \bigcap_{i+1}^h \tilde{K}_{t_\lambda}{}^{-\alpha}, \quad i = 0, 1, \ldots, h, \quad \alpha = 1, -1$$

can be non-void (cf. 5.7.1).

The cases $h < 3$ are trivial. Let $h = 3$. We have to show that

$$(5.9.3) \qquad \tilde{K}_{t_1}{}^{\alpha} \cap \tilde{K}_{t_2}{}^{-\alpha} \cap \tilde{K}_{t_3}{}^{\alpha} = \emptyset \quad \text{for } \alpha = \pm 1.$$

Replacing in 5.8 s and t by t_1, u by t_2 and v by t_3, we obtain

$$\tilde{K}_{t_1}{}^{\alpha} \cap \tilde{K}_{t_2}{}^{-\alpha} \subset \tilde{K}_{t_1}{}^{\alpha} \cap \tilde{K}_{t_3}{}^{-\alpha}.$$

This implies (5.9.3).

Suppose $h > 3$. Let T be one of the sets (5.9.1) which does not belong to the sets (5.9.2). Then there are three indices $\lambda_1, \lambda_2, \lambda_3$ such that $1 \leqq \lambda_1 < \lambda_2 < \lambda_3 \leqq h$ and $\alpha_{\lambda_1} = -\alpha_{\lambda_2} = \alpha_{\lambda_3}$. But then

$$T \subset \tilde{K}_{t_{\lambda_1}}{}^{\alpha_{\lambda_1}} \cap \tilde{K}_{t_{\lambda_2}}{}^{-\alpha_{\lambda_1}} \cap \tilde{K}_{t_{\lambda_3}}{}^{\alpha_{\lambda_1}}.$$

By our discussion of the case $h = 3$, T must be void. Hence only the 2h sets (5.9.2) may be non-void.

6. Local decompositions.

6.1. Two quasigraphs K_1 and K_2 *support* [*intersect*] each other at Q if exactly one [none] of the four open sets

$$K_1^{\pm 1} \cap K_2^{\pm 1} \cap N$$

is void for every sufficiently small neighbourhood N of Q. Thus $Q \in [\tilde{K}_1] \cap [\tilde{K}_2]$ in either case and $[\tilde{K}_1] \cap N \neq [\tilde{K}_2] \cap N$ for every small neighbourhood N of Q.

Note that

$$(6.1.1) \qquad K_1^{\alpha_1} \cap K_2^{\alpha_2} \cap N \neq \emptyset \quad \Leftrightarrow \quad \tilde{K}_1^{\alpha_1} \cap \tilde{K}_2^{\alpha_2} \cap N \neq \emptyset.$$

More generally,

$$(6.1.2) \qquad \bigcap_1^h K_j^{\alpha_j} \cap N \neq \emptyset \Leftrightarrow \bigcap_1^h \tilde{K}_j^{\alpha_j} \cap N \neq \emptyset, \quad h \geqq 2;$$

cf. (2.3.3).

6.2. *Suppose $Q \in [K_1] \cap [K_2]$ and K_1 and K_2 neither support nor intersect each other at Q. Then either*

$$(6.2.1) \qquad K_t^{\alpha} \cap N = \emptyset, \quad \text{i.e. } N \subset \tilde{K}_t^{-\alpha}$$

for some $i \in \{1, 2\}$, $\alpha \in \{1, -1\}$, *or*

(6.2.2) $\quad [\tilde{K}_1] \cap N = [\tilde{K}_2] \cap N$

for every small neighbourhood N *of* Q.

In the first case, at least one of the quasigraphs does not decompose G at Q. In the second, $\tilde{K}_1$ and $\tilde{K}_2$ may both decompose G at Q, but they do so in the same way or in opposite ways.

Proof. By our assumption, at least two of the four open sets

$$K_1^{\alpha_1} \cap K_2^{\alpha_2} \cap N$$

are void. Suppose

$$K_1^{\beta_1} \cap K_2^{\beta_2} \cap N = \emptyset,$$
$$K_1^{\gamma_1} \cap K_2^{\gamma_2} \cap N = \emptyset.$$

Then only two cases are essentially different: either

(6.2.3) $\quad \gamma_1 = \beta_1 \quad \text{and} \quad \gamma_2 = -\beta_2$

or

(6.2.4) $\quad \gamma_1 = -\beta_1 \quad \text{and} \quad \gamma_2 = -\beta_2.$

If (6.2.3) holds, we may assume that

$$K_1^{-1} \cap K_2^{1} \cap N = \emptyset \quad \text{and} \quad K_1^{-1} \cap K_2^{-1} \cap N = \emptyset.$$

Then

$$K_1^{-1} \cap N \subset \text{int}\,([K_2] \cap N) = \emptyset$$

and thus

$$K_1^{-1} \cap N = \emptyset \quad \text{and} \quad \overline{K_1^{-1}} \cap N = \emptyset$$

or

$$N \subset \mathscr{C}\overline{K_1^{-1}} = \tilde{K}_1^{1}.$$

This yields (6.2.1); cf. 2.6.

From now on we may assume that both K_1 and K_2 decompose G at Q. Then (6.2.4) holds and we may assume that, for some $\alpha \in \{1, -1\}$,

(6.2.5) $\quad K_1^{\alpha} \cap K_2^{1} \cup N = \emptyset$

and

(6.2.6) $\quad K_1^{-\alpha} \cap K_2^{-1} \cap N = \emptyset.$

By (6.2.5),

$$K_1^{\alpha} \cap \overline{K_2^{1}} \cap N = \emptyset \quad \text{and} \quad \overline{K_1^{\alpha}} \cap K_2^{1} \cap N = \emptyset$$

and thus, by 2.6,

$$K_1^{\alpha} \cap N \subset \tilde{K}_2^{-1} \cap N \quad \text{and} \quad K_2^{1} \cap N \subset \tilde{K}_1^{-\alpha} \cap N.$$

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Taking the relative closure on each side, we obtain

(6.2.7) $\overline{K_1{}^\alpha} \cap N \subset \overline{\tilde{K}_2{}^{-1}} \cap N$

and

(6.2.8) $\overline{K_2{}^1} \cap N \subset \overline{\tilde{K}_1{}^{-\alpha}} \cap N.$

Similarly, from (6.2.6),

(6.2.9) $\overline{K_1{}^{-\alpha}} \cap N \subset \overline{\tilde{K}_2{}^1} \cap N$

and

(6.2.10) $\overline{K_2{}^{-1}} \cap N \subset \overline{\tilde{K}_1{}^\alpha} \cap N.$

Hence, by (6.2.7) and (6.2.9),

$$\begin{aligned}[\tilde{K}_1] \cap N &= (\overline{K_1{}^\alpha} \cap N) \cap (\overline{K_1{}^{-\alpha}} \cap N) \\ &\subset (\overline{\tilde{K}_2{}^{-1}} \cap N) \cap (\overline{\tilde{K}_2{}^1} \cap N) = [\tilde{K}_2] \cap N\end{aligned}$$

and similarly, by (6.2.8) and (6.2.10),

$$[\tilde{K}_2] \cap N \subset [\tilde{K}_1] \cap N.$$

This yields (6.2.2).

6.3. The theorems which will be proved in this section are not valid without an additional restriction. We consider the following example.

After a homeomorphism, G may be assumed to be the square

$$\bar{I}^2 = \{(x, y)|0 \leqq x \leqq 1, 0 \leqq y \leqq 1\}.$$

Let

$$\begin{aligned}\tilde{M} &= \{(x, \tfrac{1}{2})|0 \leqq x \leqq 1\}, D_s = \{(s, y)|0 < y < 1\}, \\ \tilde{K}_s &= \tilde{M} \cup D_s \cup \{(s, 0), (s, 1)\}, \\ \tilde{K}_s{}^1 &= \{(x, y) \in I^2|x > s, y > \tfrac{1}{2} \text{ or } x < s, y < \tfrac{1}{2}\}, \\ \tilde{K}_s{}^{-1} &= \{(x, y) \in I^2|x > s, y < \tfrac{1}{2} \text{ or } x < s, y > \tfrac{1}{2}\}.\end{aligned}$$

Put $\tilde{K} = \tilde{K}_{1/2}$. Then $\mathfrak{A} = \{\tilde{K}_s|s \in I\}$ satisfies the requirements of § 4. Note:

(i) The vertex $Q_s = (s, \frac{1}{2})$ of $\tilde{K}_s$ is not fixed.

(ii) The quasigraphs $\tilde{K}_{1/4}$ and $\tilde{K}_{1/8}$ [$\tilde{K}_{1/4}$ and $\tilde{K}_{3/4}$] decompose $\bar{I}^2$ at $(\frac{1}{2}, \frac{1}{2})$ in the same way [in opposite ways].

(iii) $\tilde{K}_{1/2}$ and $\tilde{K}_{1/4}$ intersect each other at $(\frac{1}{2}, \frac{1}{2})$.

6.3.1. For the rest of Section 6 we make the following

Assumption. If $Q_s \in \tilde{M}$, then Q_s is a vertex either of every or of no $\tilde{K}_t$.

6.3.2. *Let* $R(s) \in \tilde{M}$. *Then* $R(u) = R(s)$ *for all* $u \in I$ (cf. 4.8).

Proof. (i) If $\rho(u) > 0$ for all $u \in I$, this assertion follows from 4.10 (i).
(ii) Suppose $\rho(s) > 0$, $\rho(t) = 0$ and e.g. $s < t$. Put

$$t_0 = \inf \{u|s \leqq u \leqq t; \rho(u) = 0\}.$$

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Since ρ is continuous, we have

(6.3.1) $\quad \rho(t_0) = 0$; thus $s < t_0 \leqq t$.

We have $\rho(v) > 0$ for $s \leqq v < t_0$. Hence by 4.10(i), $R(v) = R(s)$. As $R(v)$ depends continuously on v, this yields $R(t_0) = R(s) \in \tilde{M}$.

By (6.3.1), $R(t_0)$ is a vertex of K_{t_0}. Thus by 6.3.1, $R(t_0) = R(s)$ would also be a vertex of K_s. This contradicts our assumption that $\rho(s) > 0$. Hence this case can not occur.

(iii) If $\rho(s) = 0$ and $\rho(t) > 0$, then $R(t) \in \tilde{M}$ and we come back to the second case.

(iv) Finally let $\rho(u) = 0$ for all $u \in I$. Thus $R(u)$ is a vertex of K_u for all u. On the other hand, our assumption $R(s) \in \tilde{M}$ implies, on account of 6.3.1, that $R(s)$ also is a vertex of K_u, i.e. an endpoint of $\bar{E}_u$ for every u. Since $\bar{E}_u$ and $R(u)$ depend continuously on u, this yields once more our assertion.

6.3.3. *Let $s < t$, $R(s) \in \tilde{M}$. Let N be a neighbourhood of $R(s)$ in G. Then $N \cap A(u) \neq \emptyset$ for $s \leqq u \leqq t$.*

This remark follows at once from 6.3.2 and 4.10 (ii).

The proof of 4.10 (ii) shows that the arc (4.10.4) lies in N if $\xi > 0$ is small.

6.3.4. By the proof of 6.3.2, $\rho(s)$ is either always positive or always zero. Thus $R(s)$ is either always or never a vertex; $0 < s < 1$.

6.4. *Let $0 < s < t < 1$; $Q_s \in \tilde{M}$; $\alpha \in \{1, -1\}$. Suppose*

$$\tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha} \cap N \neq \emptyset$$

for every neighbourhood N of Q_s. Then there exists an edge E of $\tilde{K}$ such that $Q_s \in \bar{E}_u$ and $E_u \cap \tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha} \cap N \neq \emptyset$ for all $u \in (s, t)$.

Proof. By 3.5.1, there is a $v = v_N \in (s, t)$ *such that*

$$[\tilde{K}_v] \cap \tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha} \cap N \neq \emptyset.$$

Thus there is an edge $E = E(N)$ of $\tilde{K}$ such that

(6.4.1) $\quad E_v \cap \tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha} \cap N \neq \emptyset.$

This holds true for every choice of N. As $\tilde{K}$ has only a finite number of edges, there is an edge E of $\tilde{K}$ such that (6.4.1) applies to all neighbourhoods N and a suitable $v = v_N \in (s, t)$. Let v_0 be an accumulation point of v_N as the radius of N tends to zero. Then $Q_s \in \bar{E}_{v_0}$. If Q_s is not a vertex, assumption 6.3.1 implies that $Q_s \in E_u$ for all $u \in I$. If $Q_s = F(Q, s)$ is a vertex and Q is an end point of E, then $Q_s = Q_u$ is an end point of E_u for all $u \in I$. Thus $Q_s \in \bar{E}_u$ for all $u \in I$.

Using the notation of 4.8, let

$$Q_s = f(\sigma(u), u), \quad u \in I.$$

By 4.7.1, $\sigma : I \rightarrow \bar{I}$ is continuous. Suppose Q_s is not a vertex. Since $\sigma(u) \neq 0, 1$, there is an $\epsilon > 0$ such that

$$\epsilon < \sigma(u) < 1 - \epsilon \quad \text{for all } u \in [s, t].$$

Making ϵ smaller if necessary, we may assume that

$$(6.4.2) \qquad B(u) = f((\sigma(u) - \epsilon, \sigma(u) + \epsilon), u) \subset N \quad \text{for all } u \in [s, t].$$

The closed subset

$$\bar{E}_u \backslash B(u) = f([0, \sigma(u) - \epsilon], u) \cup f([\sigma(u) + \epsilon, 1], u)$$

has a positive distance from Q_s for every $u \in [s, t]$. Hence there is a neighbourhood $N' \subset N$ of Q_s such that

$$N' \cap \bar{E}_u \backslash B(u) = \emptyset \quad \text{for all } u \in [s, t].$$

Applying (6.4.1) with N' instead of N, we obtain

$$B(v) \cap \tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha} \cap N' \neq \emptyset.$$

By (6.4.2), there is therefore a point

$$P_v \in B(v) \cap \tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha} \cap N.$$

Let $A(v)$ denote the connected component of $E_v \backslash \tilde{M}$ containing P_v. One of the end points of $A(v)$, say the point $R(v)$ either lies on E_v between P_v and Q_s or is equal to Q_s. At any rate, $R(v) \in \tilde{M} \cap B(v) \subset \tilde{M} \cap N$ (cf. (6.4.2)).

As N is a neighbourhood of $R(v)$, we obtain, from 6.3.3 that

$$N \cap A(u) \neq \emptyset$$

both for $s < u < v$ and for $v < u < t$.

The case that Q_s is a vertex is even simpler.

6.5. *Suppose $\tilde{K}_s$ and $\tilde{K}_t$ support each other at Q_s. Then there exists a neighbourhood N of Q_s and an $\alpha \in \{1, -1\}$ such that*

$$\tilde{K}_s^{\alpha} \cap \tilde{K}_t^{-\alpha} \cap N = \emptyset.$$

Proof. Let $s < t$. Since $\tilde{K}_s$ and $\tilde{K}_t$ support each other at Q_s, every neighbourhood N of Q_s contains points $P_s \in [\tilde{K}_s] \backslash \tilde{M}$. Construct the arc $\{P(u) | u \in I\}$ according to 4.9. By the proof of 4.10 (ii), P_s can be chosen in such a way that the subarc $\{P(u) | s - \epsilon < u < t + \epsilon; \epsilon > 0\}$ lies entirely in N.

Suppose $P(s) \in \tilde{K}_t^{\alpha}$. Then, by 5.2, $P(t) \in \tilde{K}_s^{-\alpha}$. As $[\tilde{K}_s] \cap \tilde{K}_t^{\alpha} \cap N$ is not void, 2.7 implies

$$\tilde{K}_s^{\pm 1} \cap \tilde{K}_t^{\alpha} \cap N \neq \emptyset.$$

Similarly,

$$[\tilde{K}_t] \cap \tilde{K}_s^{-\alpha} \cap N \neq \emptyset \text{ implies}$$
$$\tilde{K}_s^{-\alpha} \cap \tilde{K}_t^{\pm 1} \cap N \neq \emptyset.$$

This yields

$$\tilde{K}_s^\beta \cap \tilde{K}_t^\beta \cap N \neq \emptyset \quad \text{for } \beta = \pm 1.$$

6.6. *Let* $0 < s < v < 1$, $0 < t < u < 1$. *Let* $Q_s \in \tilde{M}$; $\alpha \in \{1, -1\}$. *Let* N *be a small neighbourhood of* Q_s. *Then*

$$K_s^\alpha \cap K_v^{-\alpha} \cap N \neq \emptyset \quad \Leftrightarrow \quad K_t^\alpha \cap K_u^{-\alpha} \cap N \neq \emptyset.$$

Proof. Obviously, our assertion can be reduced to the special case

$$0 < s \leqq t < u < v < 1.$$

By (6.1.1), it suffices to consider the quasigraphs of the reduced family $\tilde{\mathfrak{A}}$.

Suppose $\tilde{K}_s^\alpha \cap \tilde{K}_v^{-\alpha} \cap N = \emptyset$. Thus, by 5.8,

$$\tilde{K}_t^\alpha \cap \tilde{K}_u^{-\alpha} \cap N \subset \tilde{K}_s^\alpha \cap \tilde{K}_v^{-\alpha} \cap N = \emptyset.$$

Conversely, suppose $\tilde{K}_s^\alpha \cap \tilde{K}_v^{-\alpha} \cap N \neq \emptyset$. Choose $w \in (t, u) \subset (s, v)$. Then, by 6.4, there is a point $P_w \in [\tilde{K}_w]$ such that $P_w \in \tilde{K}_s^\alpha \cap \tilde{K}_v^{-\alpha} \cap N$. Since $P_w \notin \tilde{M}$, we have $P_w \notin [\tilde{K}_r]$ for $r \in [s, t] \cup [u, v] \subset (0, w) \cup (w, 1)$, and thus, by 3.5,

$$P_w \in \tilde{K}_t^\alpha \cap \tilde{K}_u^{-\alpha} \cap N \neq \emptyset.$$

6.7. We first prove a lemma.

6.7.1. *If* $\tilde{K}_s$ *and* $\tilde{K}_t$ *neither intersect nor support each other at* $Q_s \in \tilde{M}$, *then both decompose* G *in the same way at* Q_s.

Proof. Let N denote a small neighbourhood of Q_s. Obviously, $N \subset \tilde{K}_s^{-\alpha}$ is impossible. On account of 6.2, we may therefore assume

$$[\tilde{K}_s] \cap N = [\tilde{K}_t] \cap N.$$

Thus either

$$\tilde{K}_s^1 \cap N = \tilde{K}_t^1 \cap N \quad \text{and} \quad \tilde{K}_s^{-1} \cap N = \tilde{K}_t^{-1} \cap N$$

or

$$\tilde{K}_s^1 \cap N = \tilde{K}_t^{-1} \cap N \quad \text{and} \quad \tilde{K}_s^{-1} \cap N = \tilde{K}_t^1 \cap N. \tag{6.7.1}$$

In the first case, $\tilde{K}_s$ and $\tilde{K}_t$ decompose G in the same way at Q_s. We have to show that (6.7.1) cannot occur.

Let $\alpha \in \{1, -1\}$. By (6.7.1), we have $\tilde{K}_s^\alpha \cap \tilde{K}_t^{-\alpha} \cap N \neq \emptyset$. Hence by 6.4, there is an edge E of $\tilde{K}$ such that

$$Q_s \in \bar{E}_u \quad \text{and} \quad E_u \cap \tilde{K}_s^\alpha \cap \tilde{K}_t^{-\alpha} \cap N \neq \emptyset \tag{6.7.2}$$

for all u with $s < u < t$. Choose u fixed.

The point set

$$[\tilde{K}_s] \cap N = \tilde{M} \cap N \tag{6.7.3}$$

consists of the intersection of N with one or several edges E_s' of $\tilde{K}_s$ such that $Q_s \in \bar{E}_s'$. To each of them corresponds an edge E_u' of $\tilde{K}_u$ such that $E_s' \cap N \subset E_u' \cap N$ (cf. (6.7.3)). Making N smaller, we may assume

$$E_u' \cap N = E_s' \cap N \subset \tilde{M}.$$

By (6.7.2), $E_u \cap N \not\subset \tilde{M}$. Thus the edge E_u is distinct from the edges E_u'. As $Q_s \in \bar{E}_u$, Q_s must be a vertex of $\tilde{K}_u$. Hence Q_s also is a vertex of $\tilde{K}_s$ (cf. 6.3.1). But this vertex would be the end point of more edges of $\tilde{K}_u$ than of $\tilde{K}_s$, which is impossible.

6.7.2. We note the following corollary of 6.5 and 6.7.1:
Let $Q_s \in \tilde{M}$. Let N be a neighbourhood of Q_s in G. Then

$$\tilde{K}_s^{\beta} \cap \tilde{K}_t^{\beta} \cap N \neq \emptyset \quad \textit{for } \beta = \pm 1.$$

6.7.3. From 6.6 and 6.7.2 we finally obtain

THEOREM 1. *If two given quasigraphs of an $\mathfrak{A}$-family intersect each other [support each other; both decompose G in the same way] at $Q_s \in \tilde{M}$, then so do any two quasigraphs of that family.*

6.8. *Let $s \neq t$; $s, t \in I$ and $Q_s \in \tilde{M}$. Then K_s and K_t intersect each other at Q_s if and only if*

$$(6.8.1) \qquad [\tilde{K}_s] \cap \tilde{K}_t^{\alpha} \cap N \neq \emptyset, \quad \alpha = \pm 1,$$

for every neighbourhood N of Q_s.

Proof. By 2.7, the condition (6.8.1) is sufficient. Conversely, suppose $\tilde{K}_s$ and $\tilde{K}_t$ intersect at Q_s. We may assume $s < t$. Let $u < s$. Then, by 6.7, $\tilde{K}_u$ and $\tilde{K}_t$ intersect at Q_s and, by 6.4, there is, for each $\alpha \in \{1, -1\}$ and edge E_s of K_s such that

$$E_s \cap \tilde{K}_t^{\alpha} \cap \tilde{K}_u^{-\alpha} \cap N \neq \emptyset$$

and $Q_s \in \bar{E}_s$. This implies (6.8.1).

6.9. *Let $s \neq t$; $Q_s \in \tilde{M}$. Then K_s and K_t support each other at Q_s if and only if the following conditions are satisfied:*

$$(6.9.1) \qquad [\tilde{K}_s] \cap N \neq [\tilde{K}_t] \cap N,$$

$$(6.9.2) \qquad [\tilde{K}_s] \cap N \subset \tilde{K}_t^{\alpha} \cup \tilde{M} \quad \textit{and} \quad [\tilde{K}_t] \cap N \subset \tilde{K}_s^{-\alpha} \cup \tilde{M}$$
$$\textit{for some } \alpha \in \{1, -1\}.$$

Proof. (i) Suppose K_s and K_t support at Q_s. Then (6.9.1) follows from our definitions. Also, from 6.8, there are $\alpha, \beta \in \{1, -1\}$ such that

$$[\tilde{K}_s] \cap N \subset \tilde{K}_t^{\alpha} \cup \tilde{M} \quad \text{and} \quad [\tilde{K}_t] \cap N \subset \tilde{K}_s^{\beta} \cup \tilde{M}.$$

Hence, by 2.7,

$$\tilde{K}_s^{\pm 1} \cap \tilde{K}_t^{\alpha} \cap N \neq \emptyset \quad \text{and} \quad \tilde{K}_t^{\pm 1} \cap K_s^{\beta} \cap N \neq \emptyset.$$

By 6.5, one of the two sets $\tilde{K}_s{}^1 \cap \tilde{K}_t{}^{-1} \cap N$ and $\tilde{K}_s{}^{-1} \cap \tilde{K}_t{}^1 \cap N$ must be void. Hence $\beta = -\alpha$.

(ii) Conversely, assume (6.9.1) and (6.9.2). Then 6.7.1, 6.2 and (6.9.1) imply that $\tilde{K}_s$ and $\tilde{K}_t$ either support or intersect at Q_s. Since (6.9.2) excludes (6.8.1), they cannot intersect.

6.10. THEOREM 2. *Suppose any two quasigraphs of $\mathfrak{A}$ support each other at Q_s. Let $0 < t_1 < t_2 < \cdots < t_h < 1$. Then for every small neighbourhood N of Q_s, exactly $h + 1$ of the 2^h open sets*

(6.10.1) $\quad K_{t_1}{}^{\pm 1} \cap \cdots \cap K_{t_h}{}^{\pm 1} \cap N$

are non-void; $h \geqq 2$.

Proof. By (6.1.2), we may replace $\mathfrak{A}$ by $\tilde{\mathfrak{A}}$. The case $h = 2$ is the definition of support for a pair of quasigraphs.

Suppose that $h > 2$ and our statement has been proved up to $h - 1$. Then exactly h of the 2^{h-1} open sets

$$\tilde{K}_{t_1}{}^{\pm 1} \cap \ldots \cap \tilde{K}_{t_{h-1}}{}^{\pm 1} \cap N$$

are non-void. Let $P \in [\tilde{K}_{t_h}] \cap N \backslash \tilde{M}$. Then $P \notin [K_r]$ for $r \in [t_1, t_{h-1}]$ so that, by 3.5, there is an $\alpha \in \{1, -1\}$ for which

$$P \in \tilde{K}_{t_1}{}^{\alpha} \cap \ldots \cap \tilde{K}_{t_{h-1}}{}^{\alpha} \cap N.$$

Thus, by 6.9,

$$[\tilde{K}_{t_h}] \cap N \backslash \tilde{M} \subset \tilde{K}_{t_1}{}^{\alpha} \cap \ldots \cap \tilde{K}_{t_{h-1}}{}^{\alpha}.$$

By 2.7, $\tilde{K}_{t_h}$ divides $\tilde{K}_{t_1}{}^{\alpha} \cap \ldots \cap \tilde{K}_{t_{h-1}}{}^{\alpha} \cap N$ into two non-void open sets, so that at least $h + 1$ of the sets (6.10.1) are non-void. Now suppose $\tilde{K}_{t_h}$ also divides

$$\tilde{K}_{t_1}{}^{\beta_1} \cap \tilde{K}_{t_2}{}^{\beta_2} \cap \ldots \cap \tilde{K}_{t_{h-1}}{}^{\beta_{h-1}} \cap N$$

into two parts. Then

$$\emptyset \neq \tilde{K}_{t_1}{}^{\beta_1} \cap \ldots \cap \tilde{K}_{t_{h-1}}{}^{\beta_{h-1}} \cap \tilde{K}_{t_h}{}^{\pm 1} \cap N \subset \tilde{K}_{t_1}{}^{\beta_1} \cap K_{t_h}{}^{\pm 1} \cap N.$$

However, $\tilde{K}_{t_1}{}^{\alpha} \cap \tilde{K}_{t_h}{}^{\pm 1} \cap N \neq \emptyset$. Hence $\beta_1 = \alpha$, since $\tilde{K}_{t_1}$ and $\tilde{K}_{t_h}$ support each other at Q_s. Similarly, $\beta_i = \alpha$, $i = 1, 2, \ldots, h - 1$. Thus $\tilde{K}_{t_h}$ divides exactly one of the h non-void sets determined by $\tilde{K}_{t_1}, \ldots, \tilde{K}_{t_{h-1}}$. This leads to exactly $h + 1$ non-void sets.

6.10.1. Suppose $K_s{}^{-\alpha} \cap K_t{}^{\alpha} \cap N = \emptyset$ for $0 < s < t < 1$ and $\alpha \in \{1, -1\}$. Then the $h + 1$ non-void sets obtained in 6.10 are

$$\bigcap_{j=1}^{i} K_{t_j}{}^{\alpha} \cap \bigcap_{j=i+1}^{h} K_{t_j}{}^{-\alpha} \cap N, \quad i = 0, 1, \ldots, h.$$

Here,

(6.10.2) $\quad \bigcap_{j=1}^{0} K_{t_j}{}^{\alpha} = \bigcap_{j=h+1}^{h} K_{t_j}{}^{-\alpha} = G,$

cf. 5.7.1.

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6.11. THEOREM 3. *Suppose any two quasigraphs of $\mathfrak{A}$ intersect each other at Q_s. Let $0 < t_1 < t_2 < \ldots < t_h < 1$; $h \geqq 2$. Then exactly $2h$ of the 2^h open sets*

$$\bigcap_1^h K_{t_i}{}^{\alpha_i}, \quad \alpha_i = \pm 1$$

are non-void, and every neighbourhood of Q_s contains points of each of these $2h$ sets.

Proof. By 5.9, it suffices to show that at least $2h$ of the open sets

$$\bigcap_1^h \tilde{K}_{t_i}{}^{\alpha_i} \cap N, \quad \alpha_i = \pm 1,$$

are non-void.

The case $h = 2$ is the definition of intersection for a pair of quasigraphs.

Suppose $h > 2$ and let N be a neighbourhood of Q_s. Suppose our statement has been proved up to $h - 1$. Then exactly $2(h - 1)$ of the 2^{h-1} open sets

$$\tilde{K}_{t_1}{}^{\pm 1} \cap \ldots \cap \tilde{K}_{t_{h-1}}{}^{\pm 1} \cap N$$

are non-void. By 6.8, there are points

$$P^1 \in [\tilde{K}_{t_h}] \cap \tilde{K}_{t_{h-1}}{}^1 \cap N \quad \text{and} \quad P^{-1} \in [\tilde{K}_{t_h}] \cap \tilde{K}_{t_{h-1}}{}^{-1} \cap N.$$

Then P^1, $P^{-1} \notin \tilde{M}$ and P^1, $P^{-1} \notin [\tilde{K}_r]$ for $r \in [t_1, t_{h-1}]$. Hence $P^1 \in \tilde{K}_r{}^1$ and $P^{-1} \in \tilde{K}_r{}^{-1}$ for all $r \in [t_1, t_{h-1}]$ and thus the two sets

$$\tilde{K}_{t_1}{}^{\alpha} \cap \ldots \cap \tilde{K}_{t_{h-1}}{}^{\alpha} \cap N, \quad \alpha = \pm 1,$$

are non-void; by 2.7, both are divided into two parts by $\tilde{K}_{t_h}$. This proves our theorem.

6.11.1. The $2h$ non-void sets obtained in 6.11 are the sets

$$\bigcap_{j=1}^{i} K_{t_j}{}^{\alpha} \cap \bigcap_{j=i+1}^{h} K_{t_j}{}^{-\alpha}, \quad i = 0, 1, \ldots, h - 1, \quad \alpha = \pm 1;$$

cf. (6.10.2).

7. Quasicurves.

7.1. An alternative way of introducing quasigraphs begins with quasicurves. A *quasicurve* H in G is a finite collection of Jordan arcs which meet the frontier of G at most at their endpoints, of Jordan curves which meet this frontier in at most one point, and of single points. Two or more of these *components* of H may be identical. A component A has *component multiplicity* $m = m(H, A)$ if H has m components identical with A. Thus

$$(7.1.1) \qquad H = \sum_A m(H, A)A$$

is a finite, possibly void, formal sum of components. This definition will be refined later.

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As before $[H]$ will denote the set of all the points incident with at least one component of H, and $[A]$ the set of points of the component A.

If μ components of H pass through a point P, then we count P with the *point multiplicity* μ in H. More precisely, if H is given by (7.1.1), P has the point multiplicity

$$\mu(H, P) = \sum_{\substack{P \in [A] \\ A \in H}} m(H, A).$$

7.2. If the component A of H decomposes G into two distinct regions, we call A a *decomposing component* and denote the regions by A^1 and A^{-1}. If A is *non-decomposing*, we define either

$$A^1 = G \backslash A, \quad A^{-1} = \emptyset$$

or

$$A^1 = \emptyset, \quad A^{-1} = G \backslash A.$$

The ordered pair of the open sets (A^1, A^{-1}) is an *orientation* of A. From now on, a component is always *oriented.*

Two components A and B are equal [opposite] if $[A] = [B]$ and $A^\alpha = \beta^\alpha [A^\alpha = \beta^{-\alpha}]$; $\alpha = \pm 1$.

Condition 7.2.1. Two distinct decomposing components of H shall have only a finite number of points in common.

By this condition, no two opposite decomposing components can occur in a given quasicurve.

Condition 7.2.2. The intersection of any two components of H shall be the union of a finite number of points and arcs.

7.3. Assume that H has the components $A_1, \ldots, A_n$, each A_i written as often as its multiplicity in (7.1.1) indicates. Thus, for $i = 1, \ldots, n$, each point of int $G \backslash H$ lies in exactly one of the sets $A_i^{\pm 1}$.
We then define

$$H^1 = \bigcup_{\prod_i \alpha_i = 1} \bigcap_{i=1}^{n} A_i^{\alpha_i}$$

and

$$H^{-1} = \bigcup_{\prod_i \alpha_i = -1} \bigcap_{i=1}^{n} A_i^{\alpha_i}.$$

Thus the point sets $[H]$, H^1, H^{-1} are mutually disjoint,

$$G = [H] \cup H^1 \cup H^{-1}$$

and

$$[H] = \mathscr{C}(H^1) \cap \mathscr{C}(H^{-1}).$$

The ordered pair (H^1, H^{-1}) is an *orientation* of H. If the orientations of the A_i's are arbitrarily chosen, H is capable of exactly two orientations.

If H is void, we can introduce two orientations of H, namely either $H^1 = G$ and $H^{-1} = \emptyset$ or $H^{-1} = G$ and $H^1 = \emptyset$.

The "global" decomposition of G by H and the decomposition of G by H at a point Q are defined as in 2.1.2. G is decomposed by H at Q if and only if Q lies on at least one decomposing component of odd multiplicity of H.

The results of Section 2.2 also apply to quasicurves.

7.4. Two distinct quasicurves H and H' can yield the same decomposition of G. We call H and H' *equivalent* and write $H \equiv H'$ if

$$H^1 = H'^1 \quad \text{and} \quad H^{-1} = H'^{-1}.$$

If two quasicurves are equivalent, they are incident with the same point set. However, they may consists of different sets of components and their points may have different multiplicities.

Let $\hat{H} = \{H' | H' \equiv H\}$ denote the set of all the quasicurves which are equivalent to H. Thus

$$H \equiv H' \Leftrightarrow \hat{H} = \hat{H}'.$$

Since $H \equiv H'$ if and only if $[H] = [H']$ and $H^\alpha = H'^\alpha$, $\alpha = \pm 1$, we may identify $\hat{H}$ with the ordered triplet

$$\hat{H} = ([H], H^1, H^{-1}).$$

If H is the void quasicurve and $H^{-1} = \emptyset [H^1 = \emptyset]$, then $\hat{H}$ contains no quasicurve except H itself and we have

$$\hat{H} = (\emptyset, G, \emptyset) \quad [\hat{H} = (\emptyset, \emptyset, G)].$$

If H decomposes G at Q and $H' \equiv H$, then H' also decomposes G at Q. We then define $\hat{H}$ to decompose G at Q.

7.5. We call a point $P \in \operatorname{int} G$ a *vertex* of $\hat{H}$ if, for every $H \in \hat{H}$, P is the end point of a component of H, or P is the intersection of two or more distinct components of H, or P is an isolated point of H.

Every point of $[H] \cap \operatorname{bd} G$ is also called a vertex of $\hat{H}$.

7.6. *The number of vertices of $\hat{H}$ is finite.*

Proof. Let $H \in \hat{H}$. Since H has only a finite number of components, only finitely many vertices are not intersections of components.

Let A and A' be any two components of H. By 7.2.2, $[A] \cap [A']$ consists of a finite number of points and arcs. If such arcs exist, 7.2.1 implies that at least one of the two components A and A', say A, is non-decomposing. Deleting the relative interior of these arcs from A, we replace A by a finite set of components, each of which has only a finite number of points in common with A'. This yields a new quasicurve of $\hat{H}$. Iterating this process, we arrive at a quasicurve of $\hat{H}$ such that the number of the intersections of its components is finite.

The proof of 7.6 yields the following corollary.

7.6.1. *$\hat{H}$ contains a quasicurve in which any two components have only a finite number of points in common.*

7.7. Let V denote the set of vertices of $\hat{H}$. Then $[H]\backslash V$ is the union of a finite number of connected sets, the *edges* of $\hat{H}$. Thus the edges and vertices of $\hat{H}$ are independent of the choice of H in $\hat{H}$ and every point of $[H]$ which is not a vertex lies on exactly one edge. Each edge has zero or one or two vertices as end points. Being a connected subset of a Jordan arc or curve, an edge also is a Jordan arc or curve.

By 7.5, no vertex of $\hat{H}$ is the common endpoint of exactly two edges.

Let H be a quasicurve in $\hat{H}$ and E an edge of $\hat{H}$. We call E *odd* if it is part of a decomposing component of odd multiplicity of H. Otherwise, E is *even*.

7.8. THEOREM 4. *$\hat{H}$ is a quasigraph. Conversely, every quasigraph can be obtained as an equivalence class of quasicurves.*

Proof. By 7.3, H^1 and H^{-1} constitute a partition of $G\backslash[H]$ such that every connected component of $G\backslash[H]$ lies entirely in H^1 or entirely in H^{-1}. As noted in 7.7, an edge of $\hat{H}$ satisfies the definition of an edge of a quasigraph. By 7.6 and 7.7, the number of vertices and edges is finite. Hence all the requirements of 2.1 are satisfied.

Conversely, let $K = ([K], K^1, K^{-1})$ be any quasigraph. We wish to con-construct a quasicurve H such that $K = \hat{H}$, i.e. $[K] = [H]$, $K^1 = H^1$ and $K^{-1} = H^{-1}$.

The non-decomposing components of H shall consist of the isolated vertices of K and of those edges of K which are not adjacent to both K^1 and K^{-1}. Each such vertex or edge is contained in the closure of K^α for exactly one $\alpha \in \{1, -1\}$. Removing it from K and transferring its points to K^α, we obtain a new quasigraph. In a finite number of steps we obtain a quasigraph $L = ([L], L^1, L^{-1})$ such that every point of $[L]$ is adjacent to both L^1 and L^{-1}.

To construct the decomposing components of H, we first note that both L^1 and L^{-1} have a finite number of connected components. Let C be one of them, say $C \subset L^\alpha$. We count the connected components of the boundary of C as decomposing components of H. Each of these components consists of a finite number of vertices and edges. Transferring the points of C and of the edges of its boundary to $L^{-\alpha}$, we obtain a new quasigraph L' such that $G\backslash[L']$ has fewer connected components. After a finite number of such steps, all the decomposing components of H have been constructed.

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