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ON THE IDEAL-TRIANGULARIZABILITY OF SEMIGROUPS OF QUASINILPOTENT POSITIVE OPERATORS ON $C(\mathcal{K})$

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ABSTRACT. It is known that a semigroup of quasinilpotent integral operators, with positive lower semicontinuous kernels, on $L^2(X,\mu)$, where X is a locally compact Hausdorff-Lindelöf space and μ is a σ-finite regular Borel measure on X, is triangularizable. In this article we use the Banach lattice version of triangularizability to establish the ideal-triangularizability of a semigroup of positive quasinilpotent integral operators on $C(\mathcal{K})$ where $\mathcal{K}$ is a compact Hausdorff space.

1**. Introduction.** By Proposition V.6.1 of [6], each quasinilpotent positive operator T on $C_0(X)$, where X is a locally compact Hausdorff space, is decomposable and by Theorem 3.14 of [2], T is ideal-triangularizable. It is, therefore, interesting to ask whether or not a semigroup of quasinilpotent positive operators on $C_0(X)$ is decomposable or ideal-triangularizable. Some partial answers are given in [2]. In Section 3 we use similar techniques to those used in [1] to prove the decomposability of a semigroup of quasinilpotent integral operators on $C_0(X)$, whose kernels are positive and lower semicontinuous. Then, in Section 4, we prove some facts, concerning the compression of an integral operator, and use Theorem 3.13 of [2], to establish the ideal-triangularizability of a semigroup of quasinilpotent integral operators on $C(\mathcal{K})$, where $\mathcal{K}$ is a compact Hausdorff space and the kernel of each operator in the semigroup is positive and lower semi-continuous.

2**. Preliminaries.** In what follows X is a locally compact Hausdorff-Lindelöf space. By an operator on $C_0(X)$ we mean a bounded linear transformation on $C_0(X)$.

We assume familiarity with basic results concerning the Banach lattice $C_0(X)$. When $\mathcal{K}$ is a compact space we know that $C_0(\mathcal{K}) = C(\mathcal{K})$ and J is a closed ideal of $C(\mathcal{K})$ if and only if there exists a closed subset $\mathcal{K}_0$ of $\mathcal{K}$ such that

$$J = \left\{ f \in C(\mathcal{K}) : f(t) = 0 \text{ for all } t \in \mathcal{K}_0 \right\},$$

(*e.g.* see [6, Example III.1.1]).

By $\mathcal{S}$ we always mean a semigroup of operators on $C_0(X)$ and by Ilat$(\mathcal{S})$ we mean the collection of all closed ideals of $C_0(X)$ which are invariant under $\mathcal{S}$. We say that $\mathcal{S}$ is *decomposable* if there exists a non-trivial $J \in$ Ilat$(\mathcal{S})$. $\mathcal{S}$ is said to be *ideal-triangularizable* if Ilat$(\mathcal{S})$ contains a nontrivial maximal chain.

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If $J_1, J_2 \in \mathrm{Ilat}(\mathcal{S})$ and if $J_1 \subseteq J_2$, by the compression of an element $T \in \mathcal{S}$ to J_2/J_1 we mean an operator $\hat{T}: J_2/J_1 \longrightarrow J_2/J_1$ defined by

$$\hat{T}(f + J_1) = Tf + J_1 \quad \forall f \in J_2.$$

The collection of all compressions of operators in $\mathcal{S}$ to J_2/J_1 will be denoted by $\hat{\mathcal{S}}$.

Let μ be a σ-finite, regular Borel measure on X. By the Lindelöf property we may assume that $\mu(U) > 0$ for every non-empty open subset U of X (*cf.* [1, Section 3]). Now suppose $\mathcal{U}$ is a non-empty open subset of X and consider the restriction $\mu|_{\mathcal{U}}$ of μ to $\mathcal{U}$. Since every open subset W_u of $\mathcal{U}$ is an open subset of X we also have

$$\mu|_{\mathcal{U}}(W_u) = \mu(W_u) > 0$$

for every non-empty open subset W_u of $\mathcal{U}$.

If S is a closed subset of X then we know that S is also a Lindelöf space as well as a locally compact Hausdorff space. Consider the restriction $\mu|_S$ of μ to S and suppose $\mu|_S > 0$. Once again we may assume that

$$\mu|_S(W_s) > 0$$

for every non-empty open subset W_s of S.

Suppose $K_T: X \times X \longrightarrow C$ is a $\mu \times \mu$ -measurable function such that for each $f \in C_0(X)$ the function Tf defined by

$$(Tf)(x) = \int K_T(x, y) f(y)\, d\mu(y),$$

belongs to $C_0(X)$. Then T is called an *integral operator* on $C_0(X)$ by way of μ.

REMARK. According to [3, Section 12] there are suitable conditions under which certain class of operators on $C(X)$ can be represented as integral operators. As an example it is known that each locally compact and locally continuous operator on $C(X)$ can be represented as an integral operator by way of a regular measure (*cf.* [3, Theorem 12.2]). However, it is not known whether or not we can find a unique regular measure, by way of which, a semigroup of such operators can be represented as an integral.

3**. A decomposability theorem.** In this section we establish a decomposability theorem for a certain semigroup of quasinilpotent positive integral operators on $C_0(X)$.

LEMMA 3.1. *Suppose U is a non-empty open subset of X. Then there exists a measurable subset G of U of nonzero finite measure such that for any integral operator T on $C_0(X)$ with a non-negative kernel K_T:*

$$\|T\| \geq k\mu(G),$$

provided $K_T(x, y) \geq k > 0$ on $E \times U$ for some non-empty measurable subset E of X.

PROOF. Since X is σ-finite and $\mu(U) > 0$ we can choose a measurable subset A of U with $0 < \mu(A) < \infty$. Let $f = \chi_A$ and apply the techniques used in the proof of Lusin's Theorem (*cf.* [4, Theorem 2.23]) to find a function g in $C_c(X)$ with the following properties:

(i) $g(x) \geq 0\ \forall x \in X$,

(ii) $\mu(B) < \mu(A)/2$, where $B = \{x \in X : g(x) \neq f(x)\}$, and

(iii) $\|g\|_\infty \leq \|f\|_\infty = 1$.

Since $C_c(X) \subseteq C_0(X)$ we have

$$Tg(x) = \int K_T(x,y)g(y)\,d\mu(y) \geq \int_A K_T(x,y)g(y)\,d\mu(y).$$

So for $x \in E$,

$$Tg(x) \geq k\Big\{\int_{A_1} g(y)\,d\mu(y) + \int_{A_2} g(y)\,d\mu(y)\Big\},$$

where $A_1 = \{y \in A : g(y) = f(y) = 1\}$ and $A_2 = \{y \in A : g(y) \neq f(y)\}$. Since $A_2 \subseteq B$, $\mu(A_2) \leq \mu(B) < \mu(A)/2$. Hence $\mu(A_1) = \mu(A) - \mu(A_2) > \mu(A)/2 > 0$, and

$$Tg(x) \geq k\Big\{\mu(A_1) + \int_{A_2} g(y)\,d\mu(y)\Big\} \geq k\mu(A_1) \quad \forall x \in E,$$

as $\int_{A_2} g(y)\,d\mu(y) \geq 0$, and hence $Tg(x) \geq k\mu(A_1)$ for all $x \in E$. Therefore;

$$\|Tg\|_\infty = \sup\big\{Tg(x) : x \in X\big\} \geq \sup\big\{Tg(x) : x \in E\big\} \geq k\mu(A_1).$$

So with $G = A_1$ we obtain

$$\|T\| = \sup\big\{\|Th\|_\infty : \|h\|_\infty \leq 1\big\} \geq k\mu(G). \qquad \blacksquare$$

LEMMA 3.2. *Suppose T is an integral operator on $C_0(X)$ with a non-negative kernel K_T. If $K_T(x,y) \geq k > 0$ on a rectangle $U \times U$, where U is a non-empty open subset of X, then there exists a measurable subset G of U of nonzero finite measure such that $r(T) \geq k\mu(G)$, where $r(T)$ refers to the spectral radius of T.*

PROOF. Use Lemma 3.1 to find a measurable subset G with the stated properties given in that Lemma.

Let $K_T^{(n)}$ denote the kernel of T^n. Then for $x, y \in U$,

$$\begin{aligned} K_T^{(n)}(x,y) &= \int K_T(x,t_1)K_T(t_1,t_2)\cdots K_T(t_{n-1},y)\,dt_1\cdots dt_{n-1} \\ &\geq \int_{U\times U\times\cdots\times U} k^n\,dt_1\,dt_2\cdots\,dt_{n-1} = k^n\mu(U)^{n-1}. \end{aligned}$$

Therefore

$$\|T^n\| \geq k^n\mu(U)^{n-1}\mu(G) \geq k^n\mu(G)^n,$$

which means $\|T^n\|^{1/n} \geq k\mu(G)$ for all n, and hence $r(T) \geq k\mu(G)$. $\blacksquare$

LEMMA 3.3. *If T is a quasinilpotent integral operator on $C_0(X)$ with non-negative, lower semicontinuous kernel K_T, then $K_T(x,x) = 0$ for all $x \in X$.*

PROOF. Suppose not and choose any x_0 with $K_T(x_0, x_0) = 2k > 0$. Lower semicontinuity implies there is an open set U such that $K_T(x, y) \geq k$ for all $(x, y) \in U \times U$. Now apply Lemma 3.2 to obtain a subset G of U of nonzero finite measure such that $r(T) \geq k\mu(G)$, which contradicts the fact that T is quasinilpotent. ∎

Suppose $\mathcal{S}$ is a semigroup of quasinilpotent integral operators on $C_0(X)$ such that every operator in $\mathcal{S}$ has a non-negative, lower semicontinuous kernel. By using Lemma 2.3 and an argument similar to the proof of [1, Theorem 3.4] we can show that there exists an open set V of finite measure such that the subspace

$$J = \left\{ f \in C_0(X) : f = 0 \text{ on } X \backslash V \right\},$$

is invariant under $\mathcal{S}$. Since $C_0(X)$ is a Banach lattice and since J is a closed ideal of $C_0(X)$, we conclude that $\mathcal{S}$ is decomposable. We summarize this observation in the following theorem and use the procedure given in the proof of [1, Theorem 3.4] to give a sketch of its proof.

THEOREM 3.4. *Let $\mathcal{S}$ be a semigroup of quasinilpotent integral operators on $C_0(X)$ by way of μ, such that every operator in $\mathcal{S}$ has a non-negative, lower semicontinuous kernel. Then $\mathcal{S}$ is decomposable.*

SKETCH OF PROOF. If $\mathcal{S} = \{0\}$, with any open subset V of X, the closed ideal $J = \{ f \in C_0(X) : f(t) = 0 \text{ for all } t \in X \setminus V\}$ is invariant under $\mathcal{S}$. Otherwise choose $T \in \mathcal{S}$, with $T(x_0, y_0) > 0$ for some $(x_0, y_0) \in X \times X$, and use the lower semicontinuity of its kernel and Lemma 3.3 to find two open subsets U_0 and V_0 of X with the following properties:

(i) $U_0 \cap V_0 = \emptyset$,

(ii) $K_S(y, x) = 0$ whenever $S \in \mathcal{S}$ and $(x, y) \in U_0 \times V_0$,

(iii) $x_0 \in U_0$ and $y_0 \in V_0$.

Now for each $x \in U_0$ define

$$W_x = \left\{ t \in X : K_S(t, x) = 0 \text{ for all } S \in \mathcal{S} \right\}$$

and observe that it is a closed subset of X that includes V_0. We distinguish two cases:

(1) $\mu(X \setminus W_x) = 0$ for every $x \in U_0$. In this case put $V = U_0$ and observe that

$$K_S(x, y) = 0 \quad \forall (x, y) \in (X \setminus V) \times V,$$

whenever $S \in \mathcal{S}$.

(2) $\mu(X \setminus W_x) \neq 0$ for some $x \in U_0$. In this case cut U_0 down and relabel if necessary, to assume this x is x_0. Put $V = X \setminus W_{x_0}$ and show that

$$K_S(x, y) = 0 \quad \forall (x, y) \in W_{x_0} \times (X \setminus W_{x_0}),$$

whenever $S \in \mathcal{S}$.

In each case verify that the closed ideal

$$J = \left\{ f \in C_0(X) : f(t) = 0 \text{ for all } t \in X \setminus V \right\}$$

is invariant under $\mathcal{S}$. ∎

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4. **An ideal-triangularizability theorem.** Under suitable conditions, we can say more about a semigroup S, of quasinilpotent integral operators on $C_0(X)$, each of whose members has a non-negative lower semicontinuous kernel. To do this we need the following lemmas.

LEMMA 4.1. *Let X be a locally compact normal space and let μ be a finite regular Borel measure on X. Let X_0 be a nonempty compact subset of X and let $h_0 \in C(X_0)$. Then, given $\kappa > 0$ there exists a closed subset A of $B = X \setminus X_0$ and a continuous extension h of h_0 to X such that the following hold:*

(a) $\mu(B \setminus A) \leq \kappa$.
(b) $h(x) = 0$ *for all* $x \in A$.
(c) $|h(x)| \leq \|h_0\|_\infty$ *for all* $x \in X$.

PROOF. First use Tietze Extension Theorem [4, Theorem 20.4] to find a continuous extension g of h_0 to X such that $\|g\|_\infty = \|h_0\|_\infty$ for all $x \in X$. Then use the regularity of μ to find a compact subset A of B with $\mu(B \setminus A) \leq \kappa$. This can be done as μ is also a finite measure. Since X is a Hausdorff space A is a closed subset of X. Now use the normality of X and the fact that $A \cap X_0 = \emptyset$ to find a continuous function f on X such that $f(A) = \{0\}$, $f(X_0) = \{1\}$, and $0 \leq f(x) \leq 1$ for all $x \in X$. Finally define $h = fg$. Then h is a continuous function on X,

$$h(y) = f(y)g(y) = 1 \cdot h_0(y) = h_0(y) \quad \text{for all } y \in X_0,$$
$$h(t) = f(t)g(t) = 0 \cdot g(t) = 0 \quad \text{for all } t \in A,$$

and

$$|h(x)| = f(x) \cdot |g(x)| \leq |g(x)| \leq \|h_0\|_\infty \quad \text{for all } x \in X. \qquad \blacksquare$$

LEMMA 4.2. *Assume all the conditions of Lemma 4.1 and let K be a bounded integrable function on $X \times X$. Then given $\epsilon > 0$, there exists a continuous extension h of h_0 to X such that*

$$\left| \int_X K(x,t)h(t)\,d\mu(t) - \int_{X_0} K(x,t)h_0(t)\,d\mu(t) \right| \leq \epsilon$$

for all $x \in X$.

PROOF. Put $\kappa = \epsilon / (M\|h_0\|_\infty)$, where M is a bound for K, and use Lemma 4.1 to find a continuous extension h of h_0 to X with the stated properties given in Lemma 4.1. Then

$$\begin{aligned}\int_X K(x,t)h(t)\,d\mu(t) &= \int_{X_0} K(x,t)h(t)\,d\mu(t) + \int_A K(x,t)h(t)\,d\mu(t) + \int_{B\setminus A} K(x,t)h(t)\,d\mu(t) \\ &= \int_{X_0} K(x,t)h_0(t)\,d\mu(t) + \int_{B\setminus A} K(x,t)h(t)\,d\mu(t),\end{aligned}$$

for any $x \in X$, and hence

$$\begin{aligned}\left| \int_X K(x,t)h(t)\,d\mu(t) - \int_{X_0} K(x,t)h_0(t)\,d\mu(t) \right| &\leq \int_{B\setminus A} |K(t)| \cdot |h(t)|\,d\mu(t) \\ &\leq M\|h_0\|_\infty \mu(B \setminus A) \leq \kappa M \|h_0\|_\infty = \epsilon.\end{aligned}$$

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for all $x \in X$. ∎

The following lemma is known and was implicitly used in [5]. For completeness we state and prove it here.

LEMMA 4.3. *Let K be a compact Hausdorff space and let J be a closed ideal in $C(K)$. Then the quotient $C(K)/J$ can be canonically identified with $C(K_0)$ where K_0 is a suitable closed subset of K.*

PROOF. Since J is a closed ideal of $C(K)$, there exists a closed, and hence compact, subset K_0 of K such that

$$J = \{f \in C(K) : f(t) = 0 \text{ for all } t \in K_0\}.$$

Define $\rho\colon C(K_0) \longrightarrow C(K)/J$ by $\rho(f_0) = f + J$, where f is a continuous extension of f_0 to K. Tietze's Extension Theorem and the structure of J imply that ρ is well defined, and it can be easily verified that ρ is linear, one-to-one, onto, and $\rho^{-1}(f + J) = f_0$, where $f_0 = f|_{K_0}$.

We show that $\|\rho(f_0)\| = \|f_0\|_\infty$. First observe that for each $f \in C(K)$ and $g \in J$

$$\sup\{|(f+g)(x)| : x \in K\} = \sup\Big\{\{|(f+g)(x)| : x \in K \setminus K_0\} \cup \{|f(x)| : x \in K_0\}\Big\},$$

and hence $\|f_0\|_\infty \leq \|f + g\|_\infty$ for all $g \in J$. This shows that $\|f_0\|_\infty \leq \|f + J\|$. On the other hand, if we use Tietze's Extension Theorem to find a continuous extension h of f_0 to K with $\|h\|_\infty = \|f_0\|_\infty$, then

$$\|f + J\| = \|h + J\| \leq \|h\|_\infty = \|f_0\|_\infty.$$

Thus ρ is an isometric isomorphism from $C(K_0)$ to $C(K)/J$. ∎

LEMMA 4.4. *Suppose K is a compact Hausdorff space and μ is a regular Borel measure on K. Let T be an integral operator on $C(K)$ with a bounded kernel K_T. If $J \in \mathrm{Ilat}(T)$, then the operator $\hat{T}\colon C(K)/J \longrightarrow C(K)/J$ can be identified with an integral operator.*

PROOF. Suppose K_0 is a closed, and hence a compact, subset of K such that

$$J = \{f \in C(K) : f(t) = 0 \text{ for all } t \in K_0\}.$$

Since K_0 is a Borel subset of K, the restriction μ_0 of μ to K_0 is well defined. Since K_T is also bounded and measurable on $K_0 \times K_0$, we can define T_0 on $C(K_0)$ by

$$T_0 f_0(y) = \int_{K_0} K_T(y,t) f_0(t)\, d\mu_0(t) \quad \forall y \in K_0.$$

We claim that $T_0 = \rho^{-1}\hat{T}\rho$, where ρ is as in Lemma 4.3, and hence $\hat{T}$ can be identified with the kernel operator T_0. To prove the claim, let $f_0 \in C(K_0)$. Then $\rho^{-1}\hat{T}\rho(f_0) = (Tf)|_{K_0}$, where f is any continuous extension of f_0 to K. Let $\epsilon > 0$ and use Lemma 4.2, with $X = K$, $X_0 = K_0$, and $h_0 = f_0$, to find an extension h of f_0 to K such that

$$\left| \int_K K_T(y,t)h(t)\,d\mu(t) - \int_{K_0} K_T(y,t) f_0(t)\,d\mu(t) \right| \le \epsilon,$$

for all $y \in K_0$. Since

$$(Tf)|_{K_0}(y) = (Th)|_{K_0}(y) = \int_K K_T(y,t)h(t)\,d\mu(t)$$

and

$$T_0 f_0(y) = \int_{K_0} K_T(y,t) f_0(t)\,d\mu_0(t) = \int_{K_0} K_T(y,t) f_0(t)\,d\mu(t),$$

for each $y \in K_0$, $\|\rho^{-1}\hat{T}\rho(f_0) - T_0(f_0)\|_\infty \le \epsilon$, and hence $\rho^{-1}\hat{T}\rho = T_0$, as desired. ■

LEMMA 4.5. *Assume all the conditions of Lemma 4.4. Then $T|_J$ can be identified with an integral operator.*

PROOF. Let K_0 be as in the Proof of Lemma 4.4. Put $U = K \setminus K_0$, then U is locally compact and J is isomorphic to $C_0(U)$. In fact $\tau \colon J \longrightarrow C_0(U)$ defined by $\tau(f) = f|_U$ is an isometric isomorphism. Now for each $g \in C_0(U)$ we have

$$\tau T|_J \tau^{-1} g = \tau T|_J f = (Tf)|_U,$$

where $f \in J$ is such that $f|_U = g$. But $Tf(x) = 0$, for all $x \in K_0$, and, for each $x \in U$,

$$Tf(x) = \int_K K_T(x,t) f(t)\,d\mu(t) = \int_U K_T(x,t) g(t)\,d\mu_U(t),$$

where μ_U is the restriction of μ to U, hence $T|_J$ can be identified with an integral operator on $C_0(U)$. ■

We are now ready to state and prove the main result of this paper.

THEOREM 4.6. *Let K be a compact Hausdorff space and let μ be a regular Borel measure on K. Suppose S is a semigroup of quasinilpotent integral operators on $C(K)$ by way of μ, each of whose members has a non-negative bounded lower-semicontinuous kernel. Then S is ideal-triangularizable.*

PROOF. By Theorem 3.4, S is decomposable. Let $J_1, J_2 \in \mathrm{Ilat}(S)$ with $J_1 \subset J_2$ and $\dim(J_2/J_1) \ge 2$. Let $\hat{S}$ be the compression of S to $C(K)/J_1$. By Lemma 4.4, each $\hat{T} \in \hat{S}$ can be identified with an integral operator on $C(K_0)$ by way of the regular Borel measure $\mu|_{K_0}$, where K_0 is a closed subset of K such that

$$J_1 = \left\{ f \in C(K) : f(t) = 0 \text{ for all } t \in K_0 \right\}.$$

By Lemma 4.5, since $J_2/J_1 \in \mathrm{Ilat}\,\hat{S}$ for each $\hat{T} \in \hat{S}$, each $\hat{T}|_{(J_2/J_1)}$ can be identified with a non-negative integral operator on $C_0(U_0)$ by way of the regular Borel measure $\mu|_{U_0}$, where $U_0 = K_0 \setminus K_{00}$ and K_{00} is a closed subset of K_0 such that

$$J_2/J_1 \cong \left\{ f_0 \in C(K_0) : f_0(t) = 0 \text{ for all } t \in K_{00} \right\}.$$

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Since, for each $T \in \mathcal{S}$, the compression $\widehat{T|_{J_2}}$ of $T|_{J_2}$ to J_2/J_1 is $\hat{T}|_{(J_2/J_1)}$, and since for such T, $\hat{T}|_{(J_2/J_1)}$ is a quasinilpotent operator, the semigroup

$$S_{J_2} = \{\hat{T}|_{(J_2/J_1)} : \hat{T} \in \hat{\mathcal{S}}\}$$

can be identified with a semigroup of quasinilpotent integral operators on $C_0(U_0)$ each of whose members has a nonnegative lower-semicontinuous kernel. Therefore; $\mathcal{S}_{J_2}$ is decomposable by Theorem 3.4. This shows that $\mathcal{S}$ is compressionally decomposable. Therefore $\mathcal{S}$ is ideal-triangularizable by Theorem 3.13 of [2]. ∎

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