THE EXPRESSION OF TRIGONOMETRICAL SERIES IN FOURIER FORM

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1. Introduction. In a paper published in 1936 Burkill **(2)** proved that, if the trigonometrical series

$$\sum_{n=1}^{\infty} c_n \exp(int), \qquad c_n = a_n - ib_n, \tag{1.1}$$

is bounded except on a countable set and if the series obtained by integrating series (1.1) once converges everywhere, then the coefficients can be written in Fourier form using the C_1P-integral. In §3 of this paper an analogous result is shown to be true when (1.1) is bounded (C, k), $k > 0$. The proof of this depends on generalizations of theorems by Verblunsky and Zygmund and both of these generalizations are obtained in §2.

For convenience, the definitions of the de la Valée Poussin derivative **(**cf. **6**, p. 59**)** and the C_rP-integral **(1)** are given here.

DEFINITION 1.1. *Let $g(t)$ be a function defined in the closed interval $[a, b]$. If, for a given t_0 in $[a, b]$,*

$$g(t_0 + h) = c_0 + c_1 h + c_2 h^2/2! + \ldots + c_k h^k/k! + o(h^k),$$

as $h \to 0$, where the numbers $c_j = c_j(t_0)$ are independent of h, then c_k is called the k*th de la Vallée Poussin derivative of g at the point t_0 and is denoted by $g_{(k)}(t_0)$. If $\phi(t) = f(t) + ig(t)$, then $\phi_{(k)}(t) = f_{(k)}(t) + ig_{(k)}(t)$ wherever $f_{(k)}(t)$ and $g_{(k)}(t)$ are defined.*

The C_nP-integral is defined by induction. Suppose that for $n \geqslant 1$ the $C_{n-1}P$-integral has been defined taking as the C_0P-integral the Perron integral **(4**, p. 201**)**. Assuming that $u(t)$ is $C_{n-1}P$-integrable, let

$$C_n(u, t, t+h) = (n/h^n) C_{n-1}P \int_t^{t+h} (t + h - \xi)^{n-1} u(\xi) d\xi.$$

DEFINITION 1.2. *The function $u(t)$ is said to be C_n-continuous at t_0 if $C_n(u, t_0, t_0 + h) \to u(t_0)$ as $h \to 0$.*

DEFINITION 1.3. *The upper and lower C_n-derivates of $u(t)$ denoted by $C_nD^*u(t)$ and $C_nD_*u(t)$, respectively, are defined to be the lim sup and the lim inf, respectively, as $h \to 0$ of the expression*

$$\left(n + \frac{1}{h}\right)(C_n(u, t, t+h) - u(t)).$$

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DEFINITION 1.4. *If $C_nD^*u(t) = C_nD_*u(t)$, their common value is defined to be the C_n-derivative of $u(t)$ and is denoted by $C_nD\,u(t)$.*

DEFINITION 1.5. *The function $M(t)$ is said to be a C_n-major function of $u(t)$ over $[a, b]$ if*

(1.2.1) $M(t)$ *is C_n-continuous;*

(1.2.2) $M(a) = 0$;

(1.2.3) $C_nD_*M(t) \geqslant u(t)$, p.p. *in* $[a, b]$;

(1.2.4) $C_nD_*M(t) > -\infty$ *in* $[a, b]$.

A C_n-minor function $m(t)$ is defined in a similar way.

DEFINITION 1.6. *If, for every $\epsilon > 0$, there is a pair $M(t)$, $m(t)$ satisfying the conditions of Definition 1.5 and such that $|M(b) - m(b)| < \epsilon$, then $u(t)$ is said to be C_nP-integrable over $[a, b]$.*

DEFINITION 1.7. *Let $I(b)$ = lower bound of all $M(b)$ and $J(b)$ = upper bound of all $m(b)$. For a C_nP-integrable function $u(t)$ the bounds have a common limit* **(1)** *which is called the definite C_nP-integral of $u(t)$ over $[a, b]$.*

If $\phi(t) = f(t) + ig(t)$ then the definition of the C_n-derivative is extended to $\phi(t)$ in the usual way, and

$$C_nP\int \phi(t)dt \equiv C_nP\int f(t)dt + i\,C_nP\int g(t)dt,$$

whenever the integrals on the right-hand side are defined.

2. The integrated series.

THEOREM 2.1. *Let the series*

$$\sum_{n=1}^{\infty} c_n \exp(int) \tag{2.1}$$

be bounded (C, k) for a fixed $k = 0, 1, 2, \ldots$, and $t \in E$, $|E| > 0$. If $r = k + 2$, then for each $t \in E$,

$$\sum_{n=1}^{\infty} [c_n \exp(int)/(in)^j] = H^{r-j}(t), \qquad (C, r - j - 1) \tag{2.2.j}$$

for $j = 1, 2, \ldots, r$. Further the series (2.2.r) converges absolutely and uniformly to $H^0(t)$ [$= H(t)$, say] for all $t \in E$ and $H_{(s)}(t)$ exists and is finite $0 \leqslant s \leqslant r - 1$, $t \in E$ and

$$H_{(s)}(t) = H^s(t). \tag{2.3}$$

Furthermore for all $t \in E$

$$H(t + h) = H(t) + hH_{(1)}(t) + \ldots + \frac{h^{r-1}}{(r-1)!}H_{(r-1)}(t) + \frac{w(t, h)}{r!}h^r \tag{2.4}$$

where $w(t, h) = O(1)$ as $h \to 0$, and in particular $H_{(r)}(t)$ exists p.p. *in E.*

Proof. It is clear that $c_n \exp(int) = O(n^k)$ and this is sufficient to guarantee the convergence property of series (2.2.r). The summability (C, k) of series (2.2.1) and the summability of series (2.2.2), (2.2.3), . . . , (2.2.r − 1) follows from two theorems by Hardy (**3**, Theorem 71, p. 128) and (**3**, Theorem 76, p. 131).

To obtain (2.3) and (2.4) it may be assumed without loss of generality that $t = 0$. Let

$$\gamma(u) = \frac{\exp(iu)}{(iu)^r},$$

$$P(h) = \sum_{\nu=0}^{r-1} \frac{(ih)^\nu}{\nu!}, \lambda(h) = \frac{\exp(ih) - P(h)}{(ih)^r},$$

and for any sequence $\{u_m\}$ let $\Delta u_n = \Delta^1 u_n = u_n - u_{n+1}$, $\Delta^j u_n = \Delta(\Delta^{j-1}u_n)$. Then Zygmund's proof (**6**, p. 66), with condition $s_n{}^k = o(n^k)$ replaced by $s_n{}^k = O(n^k)$ yields

$$H(h) = \sum_{\nu=0}^{r-1} \left(\frac{A_\nu}{\nu! h^\nu}\right) + h^r R(h), \tag{2.5}$$

where $A_\nu = \sum s_n{}^k \Delta^{k+1}(in)^{\nu-r}$ and $R(h) = \sum s_n{}^k \Delta^{k+1}\lambda(nh)$ both converge absolutely, and $R(h) = O(1)$ as $h \to 0$. Thus

$$H(t + h) = H(t) + hH_{(1)}(t) + \ldots + \frac{h^{r-1}}{(r-1)!}H_{(r-1)}(t) + \frac{w(t, h)}{r!}h^r,$$

where $w(t, h) = O(1)$ as $h \to 0$. It follows from a theorem due to Marcinkiewicz and Zygmund (**6**, p. 76) that $H_{(r)}(t)$ exists p.p. in E.

Equation (2.5) gives $H_{(r-j)}(O) = A_{r-j} = \sum s_n{}^k \Delta^{k+1}(in)^{-j}$, and since the (C, k) sum of the series $\sum c_n/(n)^j$ equals the (C, O) sum of the series $\sum s_n{}^k \Delta^{k+1}(n^{-j})$, (**3**, p. 128), (2.3) is established.

THEOREM 2.2. *If under the hypothesis of Theorem* 2.1 *the set E is an open interval and*

$$C_0 DH = \frac{dH}{dt}$$

then

$$C_s DH_{(s)}(t) = H_{(s+1)}(t) \qquad 0 \leqslant s \leqslant k, t \in E, \tag{2.6}$$

$$C_{k+1} DH_{(k+1)}(t) = H_{(r)}(t) \qquad \text{p.p. in } E. \tag{2.7}$$

Further, if $H_{(s)}(t) = F_{(s)}(t) + iG_{(s)}(t)$, $0 \leqslant s \leqslant r$, then for all $t \in E$,

$$|C_{k+1}D^* F_{(k+1)}(t)| < \infty, |C_{k+1}D_* F_{(k+1)}(t)| < \infty, |C_{k+1}D^* G_{(k+1)}(t)| < \infty, \tag{2.8}$$
$$|C_{k+1}D_* G_{(k+1)}(t)| < \infty$$

The following lemma is required for the proof.

LEMMA. *If*

$$\sum_{n=1}^{\infty} a_n$$

is summable $(C, r + 1)$, *where* $r > -1$, *then a necessary and sufficient condition that it should be bounded* (C, r) *is that* $B_n^r = O(n^{r+1})$ *where* $b_n = na_n$ *and* B_n^0, B_n^1, B_n^2, ..., *are formed from the* b_n *as* A_n^0, A_n^1, A_n^2, ... *are from the* a_n (cf. **3**, p. 96).

The relation (2.6) will be proved by induction. The result is trivial for $s = 0$ and in view of (2.3) reduces to a lemma of Verblunsky (**5**, p. 206). By the lemma stated above, series (2.2.1) is bounded $(C, k - 1)$, series (2.2.2) is bounded $(C, k - 2)$, ..., series (2.2.r − 1) is bounded. Assume that the relation holds for all $s < k$ and hence that $H_{(s)}(t)$ is a C_sP-integral of $H_{(s+1)}(t)$ for all $s < k$. Then $(k - 1)$ integrations by parts (**1**) gives

$$C_kDH_{(k)}(t) =$$

$$\lim_{h\to 0}\left[\frac{(k/h^k)C_{k-1}P\int_t^{t+h}(t+h-u)^{k-1}H_{(k)}(u)du - H_{(k)}(t)}{h/(k+1)}\right]$$

$$= \lim_{h\to 0}\left[\frac{(k+1)!}{h^{k+1}}\right]\left[H(t+h) - H(t) - \sum_{n=1}^{k}\left(\frac{h^n}{n!}\right)H_{(n)}(t)\right],$$

and, by Theorem 2.1, this limit equals $H_{(k+1)}(t)$.

It can be shown similarly that

$$C_{k+1}DH_{(k+1)}(t) =$$

$$\lim_{h\to 0}\left[\frac{(k+2)!}{h^{k+2}}\right]\left[H(t+h) - H(t) - \sum_{n=1}^{k+1}\left(\frac{h^n}{n!}\right)H_{(n)}(t)\right]$$

if the limit on the right-hand side exists. By Theorem 2.1, therefore, $C_{k+1}DH_{(k+1)}(t)$ exists p.p. in E and is equal to $H_{(r)}(t)$.

Finally, it follows from (2.4) that

$$\frac{(k+2)!}{h^{k+2}}\left[H(t+h) - H(t) - \sum_{n=1}^{k+1}\left(\frac{h^n}{n!}\right)H_{(n)}(t)\right] = O(1) \tag{2.9}$$

as $h \to 0$. This establishes (2.8).

3. The expression of coefficients in terms of the $C_{k+1}P$-integral.

This section contains the main result of the paper.

THEOREM 3.1. *Under the hypothesis and with the notation of Theorem* 2.1 *and Theorem* 2.2, *if* $E = [-\pi, \pi]$, *then*

$$c_n = \frac{1}{2\pi}C_{k+1}P\int_{-\pi}^{\pi}H_{(r)}(t)\exp(-int)dt.$$

Proof. To fix ideas take $k = 2$. In virtue of (2.6) it is clear that

$$H_{(s)}(t) - H_{(s)}(-\pi) = C_sP\int_{-\pi}^{t}H_{(s+1)}(x)dx, \qquad 0 \leqslant s \leqslant 2.$$

Furthermore, since $H_{(4)}(t) = F_{(4)}(t) + iG_{(4)}(t)$, it follows from (2.7) and (2.8) and the C_3-continuity of $F_{(3)}(t)$ and $G_{(3)}(t)$ that

$$H_{(3)}(t) - H_{(3)}(-\pi) = C_3P\int_{-\pi}^{t} H_{(4)}(x)dx.$$

Hence, using the property of integration by parts for the C_nP-integral,

$$\begin{aligned} C_3P\int_{-\pi}^{\pi} H_{(4)}(t)\exp(-int)dt &= C_2P\int_{-\pi}^{\pi} H_{(3)}(t)\ (-in)\exp(-int)dt \\ &= C_1P\int_{-\pi}^{\pi} H_{(2)}(t)\ (-in)^2\exp(-int)dt \\ &= C_0P\int_{-\pi}^{\pi} H_{(1)}(t)\ (-in)^3\exp(-int)dt \\ &= C_0P\int_{-\pi}^{\pi} H(t)\ (in)^4\exp(-int)dt = 2\pi c_n, \end{aligned}$$

since series (2.2.4) converges absolutely and uniformly to $H(t)$. This proves the theorem.

Corollary 1. *If series* (2.1) *is summable* (C, k) *for all* t *to a function* $\psi(t) = u(t) + i\,v(t)$, $|\psi(t)| < \infty$, *then the coefficients can be written in the form*

$$c_n = \frac{1}{2\pi}C_{k+1}P\int_{-\pi}^{\pi}\psi(t)\exp(-int)dt.$$

Proof. It is well known (**6**, p. 69) that if $H(t)$ is defined as in Theorem 2.1, then $H_{(r)}(t) = \psi(t)$, and the proof follows from Theorem 3.1.

Corollary 2. (The real analogue of Theorem 3.1.) *If* $\sum A_n(x)$ *and* $\sum B_n(x)$ *are bounded* (C, k) *in* $[-\pi, \pi]$ *then*

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F_{(r)}(t)\cos(nt)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} G_{(r)}(t)\sin(nt)dt,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F_{(r)}(t)\sin(nt)dt = \frac{-1}{\pi}\int_{-\pi}^{\pi} G_{(r)}(t)\cos(nt)dt,$$

where $F_{(r)}(t)$, $G_{(r)}(t)$ *are as defined in* Theorem 2.2 *and the integrals are* $C_{k+1}P$*-integrals.*

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