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Klyachko Models for General Linear Groups of Rank 5 over a p-Adic Field

Chufeng Nien

Abstract. This paper shows the existence and uniqueness of Klyachko models for irreducible unitary representations of $\mathrm{GL}_5(\mathcal{F})$, where $\mathcal{F}$ is a p-adic field. It is an extension of the work of Heumos and Rallis on $\mathrm{GL}_4(\mathcal{F})$.

1 Introduction

In 1984, A. A. Klyachko [Kl] initiated the investigation of a class of models for $\mathrm{GL}(n)$ over a finite field, which we refer to as Klyachko models (also known as Whittaker–symplectic models). These models consist of a series of representations $\mathcal{M}_{n,\,k}, 0 \leq k \leq [\frac{n}{2}]$ with the following properties.

1. Existence of models: every irreducible representation of $\mathrm{GL}(n, \mathbb{F}_q)$ is a subrepresentation of $\mathcal{M}_{n,\,k}$ for some k.
2. Uniqueness of models: for each irreducible representation, the multiplicity in $\mathcal{M}_{n,\,k}$ is at most one.
3. Disjointness of models: $\mathcal{M}_{n,\,i}$ and $\mathcal{M}_{n,\,j}$ are disjoint for $i \neq j$, that is, no irreducible representation can be embedded in both $\mathcal{M}_{n,\,i}$ and $\mathcal{M}_{n,\,j}$, for $i \neq j$.

These models generalize the usual Whittaker models by adding a symplectic component to the inducing subgroup. When $k = 0$, $\mathcal{M}_{n,\,0}$ is the famous Whittaker model, a representation induced from a generic character on the unipotent radical of standard Borel subgroups of GL_n. When n is even and $k = \frac{n}{2}$, $\mathcal{M}_{n,\,\frac{n}{2}}$ is induced from the trivial character of Sp_n and is called a symplectic model. The other "mixed" models $\mathcal{M}_{n,\,k}, 0 < k < [\frac{n}{2}]$ are induced from characters of subgroups with smaller unipotent and symplectic components.

Klyachko's work was followed by that of Michael J. Heumos and Stephen Rallis [HR] who in 1990 considered the realization of these models on GL_n over p-adic fields. At first, as in the finite fields case, disjointness and uniqueness of these models are expected for all irreducible representations, but soon after they found an irreducible *non-unitary* representation of $\mathrm{GL}_3(\mathcal{F})$ which does not have any such models. Then they restricted the discussion to irreducible unitary representations, and proved the uniqueness of the symplectic models and the disjointness for unitary representations of the different models. Moreover, for $n \leq 4$ they proved that any unitary irreducible representation admits a unique Klyachko model. Following their work,

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a unique model for each irreducible unitary representation of $GL_5(\mathcal{F})$ is explicitly classified in Theorem 5.7.

O. Offen and E. Sayag [OS] showed that a certain family of irreducible unitary representations of GL_{2n} has symplectic models by embedding a local problem into a global setting. Recently, they further proved that every irreducible unitary representation of GL_n admits a $\mathcal{M}_k$ model for a unique $0 \le k \le [\frac{n}{2}]$.

2 Notation and Terminology

For notation and terminology, we follow [HR, BZ1, BZ2]. Throughout, $\mathcal{F}$ denotes a p-adic field, and G_n denotes $GL(n, \mathcal{F})$.

The standard (upper triangular) parabolic subgroups of G_n are parameterized by ordered partitions $(n_1, \dots, n_k)$ of $n = n_1 + \cdots + n_k$. Let $P_{n_1,\dots,n_k}$ denote the associated *parabolic subgroups* and $N_{n_1,\dots,n_k}$ denote its *unipotent radical*. $\mathcal{J}_n$ denotes the $2n \times 2n$ matrix $\left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$ and the associated *symplectic form* $\mathcal{J}(x, y) = {}^t x \mathcal{J}_n y$ is denoted by $\mathcal{J}$. The *symplectic groups* Sp_{2n} are the set of elements preserving the form $\mathcal{J}$ in G_{2n}. Let U_n denote the group of *upper triangular unipotent matrices* in G_n. With U_{n-2k} embedded in the upper left and Sp_{2k} in the lower right, let $M_k = (U_{n-2k} \times Sp_{2k})N_k$, where $N_k = N_{n-2k,2k}$.

We denote by ν the homomorphism $g \mapsto |\det g|$ and by δ_P the *modular function* of the group P. *A character* (one-dimensional representation) of G_n is of the form $g \mapsto \chi(\det g)$ for some character χ of $\mathcal{F}^*$. We sometimes write χ_n to indicate the group G_n involved. Induction is always *normalized* unless otherwise stated, with Ind (ind respectively) denoting *full* (*compact* respectively) *induction*. Given representations σ_i of G_{n_i}, $i = 1, \dots, k$, extend $\sigma_{n_1} \otimes \cdots \otimes \sigma_{n_k}$ to $P_{n_1,\dots,n_k}$, so that it is trivial on $N_{n_1,\dots,n_k}$. Denote

$$\mathrm{Ind}_{P_{n_1,\dots,n_k}}^{G_{n_1+\cdots+n_k}} \sigma_{n_1} \otimes \cdots \otimes \sigma_{n_k} \text{ by } \sigma_{n_1} \times \cdots \times \sigma_{n_k}.$$

Given a unipotent radical $N_{n_1,\dots,n_k}$ and a representations π of G_n, the *Jacquet functor* $r_{n_1,\dots,n_k}$ is defined to be the functor mapping π to the quotient space

$$V_\pi / \{\pi(n)v - v | v \in V_\pi, n \in N_{n_1,\dots,n_k}\}.$$

The quotient space is a $G_{n_1} \times \cdots \times G_{n_k}$ module and is called the *Jacquet module* of π. Let $\widetilde{r}$ denote the *normalized Jacquet functor* (refer to [BZ2]). Let ψ be any nontrivial, complex, additive character of $\mathcal{F}$. Define the character ψ_n of U_n by

$$\psi_n(u) = \psi(u_{1,2} + \cdots + u_{n-1,n}), u = (u_{ij}).$$

A *generic* (or *nondegenerate*, *Whittaker*) character is a character which is nontrivial on all the simple root groups in U_n. For $1 \le k \le [\frac{n}{2}]$, define a series of models for G_n to be representations $\mathcal{M}_{n,k} = \mathrm{Ind}_{M_k}^{G_n}(\psi_{n-2k} \otimes 1 \otimes 1)$. Denote $\psi_{n-2k} \otimes 1 \otimes 1$ by $\hat{\psi}_{n-2k}$. When n is understood, we simply write $\mathcal{M}_k$. We call $\mathcal{M}_0$ a *Whittaker model*. The Whittaker models for any two Whittaker characters are equivalent, since the diagonal torus of G_n normalizes U_n and acts transitively on the set of Whittaker characters. The Weyl group of G_n is the symmetric group S_n, and we use cycle forms

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$(i_1, i_2, \dots, i_k)$ of permutations to denote the corresponding Weyl elements in W. For example, in G_4,

$$(1,2,3) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We also use the notations $\mathrm{Ind}_{n_1,\dots,n_k}^{n_1+\cdots+n_k} = \mathrm{Ind}_{\mathrm{G}_{n_1}\times\cdots\times\mathrm{G}_{n_k}}^{\mathrm{G}_{n_1+\cdots+n_k}}$, $\widetilde{\mathrm{r}}_{n_1,\dots,n_k}^{n_1+\cdots+n_k} = \widetilde{\mathrm{r}}_{\mathrm{G}_{n_1}\times\cdots\times\mathrm{G}_{n_k}}^{\mathrm{G}_{n_1+\cdots+n_k}}$, and $\mathrm{Hom}_{n_1,\dots,n_k} = \mathrm{Hom}_{\mathrm{G}_{n_1}\times\cdots\times\mathrm{G}_{n_k}}$.

3 Known Results on GL_n

Denote by Alg$G$ the set of all smooth representations of an algebraic group G.

Proposition 3.1 **(Proposition 1.9 [BZ2])** *Let M, U be closed subgroups of G_n such that M normalizes U, $M \cap U = \{e\}$, and the subgroup $P = MU \subset \mathrm{G}_n$ is closed. Then*

1. *The functors Ind_P, ind_P are exact.*
2. *The functor $\widetilde{\mathrm{r}}_M$ is left adjoint to Ind_P, that is, $\mathrm{Hom}_M(\widetilde{\mathrm{r}}_M(\pi), \rho) \simeq \mathrm{Hom}_{\mathrm{G}_n}(\pi, \mathrm{Ind}_P)$.*
3. *Induction by stages: let S, T be subgroups of M and $H = ST$ such that the functors Ind_H, $\mathrm{ind}_H \colon \mathrm{Alg}S \mapsto \mathrm{Alg}M$ and $\widetilde{\mathrm{r}}_S \colon \mathrm{Alg}M \mapsto \mathrm{Alg}S$ are well defined. Then*

$$\mathrm{ind}_P^{\mathrm{G}_n} \circ \mathrm{ind}_H^M = \mathrm{ind}_H^{\mathrm{G}_n},\ \mathrm{Ind}_P^{\mathrm{G}_n} \circ \mathrm{Ind}_H^M = \mathrm{Ind}_H^{\mathrm{G}_n},\ \widetilde{\mathrm{r}}_S^M \circ \widetilde{\mathrm{r}}_M^{\mathrm{G}_n} = \widetilde{\mathrm{r}}_S^{\mathrm{G}_n}.$$

Theorem 3.2 **(Jacquet's Theorem [BZ1])** *Let $\pi \in \mathrm{AlgG}_n$ be irreducible. Then there exists a parabolic triple (P, M, U) of G_n and an irreducible cuspidal representation $\rho \in \mathrm{Alg}M$ such that π can be embedded into $\mathrm{ind}_P^{\mathrm{G}_n}(\rho)$. In particular, π is admissible.*

Let $\alpha = (n_1, \dots, n_r)$ be an ordered partition of n, and let $\mathrm{G}_\alpha = \mathrm{G}_{n_1} \times \cdots \times \mathrm{G}_{n_r}$ be the subgroup of G_n, embedded as the subgroup of block-diagonal matrices. By blocks of α we mean the sets of indices

$$I_1 = \{1, \dots, n_1\}, I_2 = \{n_1 + 1, \dots, n_1 + n_2\}, \dots, I_r = \{n_1 + \cdots + n_{r-1} + 1, \dots, n\}.$$

For two partitions β, γ with blocks $I_1, \dots, I_r$ and $J_1, \dots, J_s$ respectively, set

$$W^{\beta,\gamma} = \{w \in W \mid w(k) < w(l) \text{ if } k < l \text{ and both } k,\ l \text{ belong to the same } I_i\ ;\quad w^{-1}(k) < w^{-1}(l) \text{ if } k < l \text{ and both } k \text{ and } l \text{ belong to the same } J_j\}.$$

Let $F_w = \mathrm{ind}_{\gamma\cap w(\beta)}^{\gamma} \circ w \circ \widetilde{\mathrm{r}}_{\beta\cap w^{-1}(\gamma)}^{\beta}$.

Theorem 3.3 **(Theorem 1.2 [Ze])** *The functor $F = \widetilde{\mathrm{r}}_\gamma^n \circ \mathrm{ind}_\beta^n \colon \mathrm{AlgG}_\beta \mapsto \mathrm{AlgG}_\gamma$ is glued together from those F_w where $w \in W^{\beta,\gamma}$. That is, for π a representation of G_n and β, γ partitions of n the set of composition factors of $\widetilde{\mathrm{r}}_\gamma^n \circ \mathrm{ind}_\beta^n(\pi)$ is $\{F_w(\pi) | w \in W^{\beta,\gamma}\}$.*

In the following, $C_c^\infty(X)$ denotes the space of smooth, compactly supported functions on a p-adic space X, and $\mathfrak{D}(X)$ denotes the space of complex-valued linear functionals on $C_c^\infty(X)$. Elements of $\mathfrak{D}(X)$ are called distributions. Given a Lie group G, define the left and right translations l_g and r_g on $G; C_c^\infty(G)$ and $\mathfrak{D}(G)$ as the following:

$$l_g \cdot x = gx; \quad r_g \cdot x = xg^{-1};$$

$$(l_g \cdot f)(x) = f(g^{-1}x); \quad (r_g \cdot f)(x) = f(xg);$$

$$(l_g \cdot T)(f) = T(l_{g^{-1}} \cdot f); \quad (r_g \cdot T)(f) = T(r_{g^{-1}} \cdot f),$$

where $g,\ x \in G; f \in C_c^\infty(G)$ and $T \in \mathfrak{D}(G)$.

Lemma 3.4 **(Bernstein's localization principle, Theorem 6.9 [BZ1])** *Assume that a p-adic group G acts on a p-adic space X by $q\colon X \mapsto X$ constructively, which means that the graph $\{(x, gx) \mid g \in G, x \in X\}$ of G is the union of finitely many locally closed subsets of $X \times X$. If every fiber $X_y = q^{-1}(y)$ is G-invariant and if $\mathfrak{D}(X_y)^G = 0$ for every $y \in X$, then $\mathfrak{D}(X)^G = 0$.*

A *segment* $\triangle$ is a representation of G_n of the form of $\rho \times \nu\rho \times \cdots \times \nu^k\rho$, where $k \in \mathbb{N}, mk = n$ and ρ is an irreducible cuspidal representation of G_m. We write $\triangle = [\rho, \nu^k\rho]$ to indicate the beginning and the end of a segment. Two segments $\triangle_1, \triangle_2$ are *linked* if $\triangle_1 \not\subset \triangle_2, \triangle_2 \not\subset \triangle_1$, and $\triangle_1 \cup \triangle_2$ is also a segment. Let $\triangle_1 = [\rho_1, \nu^s\rho_1], \triangle_2 = [\rho_2, \nu^t\rho_2]$. $\triangle_1$ *precedes* $\triangle_2$ if $\triangle_1$ and $\triangle_2$ are linked and $\rho_2 = \nu^m\rho_1$ for some $m > 0$.

If π is a representation, we denote by $\langle\pi\rangle$ (respectively $L(\pi)$) the *unique irreducible submodule* (respectively the *unique irreducible quotient* module) of π, when it exists. A sufficient condition for existence is explained in the following theorem, and, in many useful cases, unique submodules and unique quotients do exist.

Theorem 3.5 **(Theorem 6.1, [BZ2])** *Let $\triangle_1, \dots, \triangle_r$ be segments of G_n such that for each pair of indices $i < j$, $\triangle_i$ does not precede $\triangle_j$. Let the same condition hold for segments $\triangle'_1, \dots, \triangle'_s$.*

1. *The representation $\langle\triangle_1\rangle \times \cdots \times \langle\triangle_r\rangle$ has a unique irreducible submodule, denoted by $\langle\triangle_1, \dots, \triangle_r\rangle$.*
2. *$\langle\triangle_1, \dots, \triangle_r\rangle$ and $\langle\triangle'_1, \dots, \triangle'_s\rangle$ are isomorphic if and only if the sequences of segments $\{\triangle_1, \dots, \triangle_r\}$ and $\{\triangle'_1, \dots, \triangle'_s\}$ are equal up to a chain of transpositions of two non-linked neighbors.*
3. *Any irreducible representation in $\mathrm{Alg}\mathrm{G}_n$ is isomorphic to some representation of the form $\langle\triangle_1, \dots, \triangle_r\rangle$.*
4. *The representation $\langle\triangle_1\rangle \times \cdots \times \langle\triangle_r\rangle$ is irreducible if and only if $\triangle_i$ and $\triangle_j$ are not linked for each pair $i, j = 1, \dots, r$.*

Theorem 3.6 **(Theorem 1.2.5, [Ku])** *Let $\triangle_1, \dots, \triangle_r$ be segments of G_n such that for each pair of indices $i < j$, $\triangle_i$ does not precede $\triangle_j$. Let the same condition hold for segments $\triangle'_1, \dots, \triangle'_s$.*

1. *The representation $L(\triangle_1) \times \cdots \times L(\triangle_r)$ has a unique irreducible quotient, denoted by $L(\triangle_1, \dots, \triangle_r)$.*

2. $L(\triangle_1, \ldots, \triangle_r)$ *and* $L(\triangle'_1, \ldots, \triangle'_s)$ *are isomorphic if and only if the sequences of segments* $\{\triangle_1, \ldots, \triangle_r\}$ *and* $\{\triangle'_1, \ldots, \triangle'_s\}$ *are equal up to a chain of transpositions of two non-linked neighbors.*
3. *Any irreducible representation in* AlgG_n *is isomorphic to some representation of the form* $L(\triangle_1, \ldots, \triangle_r)$.
4. *The representation* $L(\triangle_1) \times \cdots \times L(\triangle_r)$ *is irreducible if and only if* $\triangle_i$ *and* $\triangle_j$ *are not linked for each pair* $i, j = 1, \ldots, r$.

Theorem 3.7 **(Theorem 9.3, [Ze])** *An irreducible representation* π *in* AlgG_n *is quasi-square-integrable if and only if it is isomorphic to* $L(\triangle)$ *for some segment* $\triangle = [\rho, \nu^m \rho]$, *where* ρ *is an irreducible cuspidal representation of* G_k, $km = n, k, m \in \mathbb{N}$. *That is, it is the unique quotient of some segment. In particular, every irreducible cuspidal representation of* G_n *is quasi-square-integrable.*

Lemma 3.8 **(Lemma 3.2, [Ta1])** *Let* $\triangle_1, \triangle_2$ *be two segments.*

1. *If* $\triangle_1$ *and* $\triangle_2$ *are not linked, then* $L(\triangle_1) \times L(\triangle_2) = L(\triangle_1, \triangle_2)$.
2. *If* $\triangle_1$ *and* $\triangle_2$ *are linked, then*

$$L(\triangle_1) \times L(\triangle_2) = L(\triangle_1, \triangle_2) + L((\triangle_1 \cap \triangle_2), (\triangle_1 \cup \triangle_2)),$$

*where the summation is in the sense of semi-simplification (*i.e., $L(\triangle_1, \triangle_2)$ *and* $L((\triangle_1 \cap \triangle_2), (\triangle_1 \cup \triangle_2))$ *are composition factors of* $L(\triangle_1) \times L(\triangle_2)$.*)*

Theorem 3.9 **(Theorem 9.7, [Ze])** 1. *For any k segments* $\triangle_1, \ldots, \triangle_k$, *the representation* $\pi = L(\triangle_1) \times \cdots \times L(\triangle_k)$ *has a nontrivial Whittaker functional. In particular, every irreducible quasi-square-integrable representation of* G_n *is generic.*

2. *Any generic representation* π *of* AlgG_n *can be decomposed as a product*

$$\pi = L(\triangle_1) \times \cdots \times L(\triangle_k),$$

for some segments $\triangle_1, \ldots, \triangle_k$, *such that no two of them are linked. Moreover the set* $\{\triangle_1, \ldots, \triangle_k\}$ *is uniquely determined by* π *up to isomorphisms of representations.*

Theorem 3.10 **(Theorem 2, [Ro])** *Let* π_i *be irreducible representations of* G_{n_i}, $i = 1, \ldots, k$, *and* $n = n_1 + \cdots + n_k$. *Then*

$$\mathrm{Hom}_n(\pi_1 \times \cdots \times \pi_{n_k}, \mathcal{M}_{n,0}) \simeq \mathrm{Hom}_{n_1,\ldots,n_k}(\pi_1 \otimes \cdots \otimes \pi_{n_k}, \mathcal{M}_{n_1,0} \otimes \cdots \otimes \mathcal{M}_{n_k,0}).$$

Now we recall the classification of irreducible unitary representations of G_n due to M. Tadić [Ta2].

Let $\mathrm{D}_0(n)$ denote the set of isomorphism classes of irreducible representations of G_n which are *square-integrable modulo the center* and $\mathrm{D}_0 = \bigcup_{n \geq 0} \mathrm{D}_0(n)$. Let $\mathrm{D}(n)$ be the set of representations of the form $\nu^\alpha \delta$, where α is real and $\delta \in \mathrm{D}_0(n)$; let $\mathrm{D} = \bigcup_{n \geq 0} \mathrm{D}(n)$ and let $\mathrm{M}(D)$ be the collection of all finite (unordered) multisets on D.

For ρ an irreducible cuspidal representation and $n \in \mathbb{N}$, let

$$\triangle[n]^\rho = \nu^{\frac{-n+1}{2}} \rho \times \nu^{\frac{-n+3}{2}} \rho \times \cdots \times \nu^{\frac{n-1}{2}} \rho.$$

That is, $\triangle[n]^\rho$ is a segment with exponents of ν symmetric around 0.

Given $a = (\delta_1, \dots, \delta_n) \in \mathrm{M(D)}$, $\delta_i = \nu^{\alpha_i}\delta_0^i$, $\delta_0^i \in \mathrm{D}_0$, we may assume that $\alpha_1 \geq \cdots \geq \alpha_n$. The induced representation $\delta_1 \times \cdots \times \delta_n$ has a unique irreducible quotient $L(a)$.

Given an irreducible representation σ, let σ^+ denote its *Hermitian* (complex conjugate) *contragradient*. Set $\prod(\sigma, \alpha) = \nu^\alpha\sigma \times \nu^{-\alpha}\sigma^+$ for α positive. For a positive integer n and $\delta \in \mathrm{D}_0$, set $u(\delta, n) = L(\nu^p\delta \times \nu^{p-1}\delta \times \cdots \times \nu^{-p}\delta)$, where $p = \frac{n-1}{2}$. Thus if δ is a representation of G_m, $u(\delta, n)$ is a representation of G_{nm}. We sometimes write $u(\delta_m, n)$ to emphasize the rank of δ.

Theorem 3.11 **(Theorem 7.5, [Ta2])** *Let*

$$\mathfrak{B} = \{u(\delta, n), \textstyle\prod(u(\delta, n), \alpha) | \delta \in \mathrm{D}_0, 0 < \alpha < \frac{1}{2}\},$$

$$a(n, d)^\rho = (\nu^{\frac{n-1}{2}}\triangle[d]^\rho, \nu^{\frac{n-3}{2}}\triangle[d]^\rho, \dots, \nu^{\frac{n+1}{2}}\triangle[d]^\rho),$$

where $n, d \in \mathbb{N}$, and ρ is an irreducible cuspidal representation.

1. *If $\sigma_1, \dots, \sigma_r \in \mathfrak{B}$,* then *$\sigma_1 \times \cdots \times \sigma_r$ is irreducible and unitary.*
2. *If π is an irreducible unitarizable representation then there exist $\tau_1, \dots, \tau_s \in \mathfrak{B}$, unique up to permutations, such that $\pi = \tau_1 \times \cdots \times \tau_s$.*
3. *$L(a(n, d)^\rho) = \langle a(d, n)^\rho \rangle = u(\delta(\rho, d), n)$, where $\delta(\rho, d) = L(\triangle[d]^\rho)$.*

For this part of notation and results, we refer to [KV] and [Wa]. Let G denote a unimodular p-adic group.

Definition 3.12 Let the Frechet spaces V_{π_1}, W_{π_2} be representations of G. A separately continuous bilinear form $B\colon V_{\pi_1} \times W_{\pi_2} \mapsto \mathbb{C}$ is said to be a (π_1, π_2)-*intertwining form* if $B \circ (\pi_1 \otimes \pi_2)(g) = B, g \in G$. We denote the linear space of these forms by $\mathrm{I}(\pi_1, \pi_2)$.

If W_{π_2} admits an inner product $\langle \cdot\, , \cdot \rangle_{\pi_2}$, we define

$$B_T(v, w) = \langle Tv, w\rangle_{\pi_2}, v \in V_{\pi_1}, w \in W_{\pi_2}$$

for any given intertwining operator $T \in \mathrm{Hom}_G(V_{\pi_1}, W_{\pi_2})$. Then $B_T \in \mathrm{I}(\pi_1, \pi_2)$, and

$$\dim \mathrm{Hom}_G(V_{\pi_1}, W_{\pi_2}) \leq \dim \mathrm{I}(\pi_1, \pi_2).$$

Theorem 3.13 **(Theorem 4.7, [KV])** *Assume that G is a unimodular p-adic group and R, Q are closed subgroups of G. Let $\pi_1 = \mathrm{ind}_R^G\chi_1$, $\pi_2 = \mathrm{ind}_Q^G\chi_2$, where χ_1 (respectively χ_2) is a character of R (respectively Q). Then there exists a linear isomorphism between the linear space $\mathrm{I}(\pi_1 \otimes \pi_2)$ and the linear space $\mathfrak{D}(G)^{R\times Q}$ of $R \times Q$-invariant distributions on G. Here the $R \times Q$-action on $T \in \mathfrak{D}(G)$ is given by*

$$(r, q) \cdot T = (\delta_R(r)\delta_Q(q))^{-1}\chi_1^{-1}(r)\chi_2(q)(l_r \circ r_q) \cdot T,$$

for $r \in R, q \in Q$.

Proposition 3.14 *Assume that G is a unimodular p-adic group and R, Q are closed-unimodular subgroups of G. Let* $\pi_1 = \mathrm{Ind}_R^G \chi_1$, $\pi_2 = \mathrm{Ind}_Q^G \chi_2$, *where* χ_1 *(respectively* χ_2*) is a character of R (respectively Q). Then* $\dim \mathrm{Hom}_G(\pi_1, \pi_2) \leq \dim \mathfrak{D}(G)^{R\times Q}$.

Proof The result follows the above theorem and the following facts:

1. $\mathrm{Hom}_G(\mathrm{Ind}_R^G \chi_1, \mathrm{Ind}_Q^G \chi_2) \cong \mathrm{Hom}_G(\widetilde{\mathrm{Ind}}_Q^G \chi_2, \widetilde{\mathrm{Ind}}_R^G \chi_1)$.
2. $\widetilde{\mathrm{Ind}}_Q^G \chi_2 \cong \mathrm{ind}_Q^G \chi_2^{-1}$, when Q is unimodular (refer to [BZ1], 2.25). ■

4 The Work of Heumos and Rallis

For G_2, there are only three types of irreducible representations: cuspidal representations, submodules of segments $[\rho, \nu\rho]$, and quotients of segments $[\rho, \nu\rho]$, where ρ is an irreducible cuspidal representation on $\mathrm{G}_1 = \mathcal{F}^*$. Submodules of segments are in fact characters, and hence have symplectic models (refer to Lemma 5.5). Cuspidal representations and quotients of segments both satisfy the criterion of Theorem 3.7 and admit Whittaker models. Therefore for G_2 every irreducible representation has either a Whittaker model or a symplectic model.

Theorem 4.1 **(Theorem 2.4.2, [HR])** *Let* π *be an irreducible representation of* G_{2n}, *then* $\dim \mathrm{Hom}_{2n}(\pi, \mathcal{M}_n) \leq 1$.

Theorem 4.2 **(Theorem 3.2.2, [HR])** *An irreducible representation of* G_n *cannot have both a Whittaker model and a symplectic model.*

Theorem 4.3 **(Theorem 3.1, [HR])** *Let* π *be an irreducible unitary representation of* G_n. *Then* $\mathrm{Hom}_n(\pi, \mathcal{M}_i)$ *is nonzero for at most one integer i,* $0 \leq i \leq [\frac{n}{2}]$.

Theorem 4.4 **(Theorem 9.1.1, [HR])** *Let* $I = \mathrm{Ind}_{2,1}^3 \langle 1 \times \nu \rangle \otimes \nu^{-1}$. *The representation* $\langle \mathrm{I} \rangle$ *(the unique irreducible submodule of* I*) has neither a Whittaker model nor a mixed model.*

Theorem 4.5 **(Theorem 8.1, [HR])** *Let* π *be an irreducible unitary representation of* G_3. *Then* π *can be uniquely embedded as a submodule of Whittaker model* $\mathcal{M}_0$ *or mixed model* $\mathcal{M}_1$.

Theorem 4.6 **(Theorem 11.5, [HR])** *Let* π *be an irreducible unitary representation of* G_4. *Then* π *can be uniquely embedded as a submodule of Whittaker model* $\mathcal{M}_0$, *mixed model* $\mathcal{M}_1$, *or symplectic model* $\mathcal{M}_2$.

Theorem 4.7 **([OS])** *Let* $\pi = \sigma_1 \times \cdots \times \sigma_t \times \tau_{t+1} \times \cdots \times \tau_s$ *be an irreducible unitary representation of* $\mathrm{GL}_{2n}(\mathcal{F})$, *with* $\sigma_i = u(\delta_{k_i}, 2m_i) \in \mathfrak{B}$ *and* $\tau_i = \prod(u(\delta_{k_i}, 2m_i), \alpha_i) \in \mathfrak{B}$. *Then* π *admits a symplectic model.*

In the same paper, Offen and Sayag also made the following conjecture.

Conjecture 4.8 **([OS])** *If* π *is an irreducible unitary representation of* $\mathrm{GL}_{2n}(\mathcal{F})$ *admitting a symplectic model, then* $\pi = \sigma_1 \times \cdots \times \sigma_t \times \tau_{t+1} \times \cdots \times \tau_s$ *for some* $\sigma_i = u(\delta_{k_i}, 2m_i) \in \mathfrak{B}$ *and* $\tau_i = \prod(u(\delta_{k_i}, 2m_i), \alpha_i) \in \mathfrak{B}$.

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5 Klyachko Models on GL_5

Lemma 5.1 *For $i \neq j$, $\mathrm{Hom}_{\mathrm{G}_4}(\mathcal{M}_i, \mathcal{M}_j) = 0$.*

Proof By Proposition 3.14, it suffices to show the following claim: if a distribution T on G_4 satisfies

$$\hat{\psi}_{4-2i}(m_i)\hat{\psi}^{-1}_{4-2j}(m_j)T((l_{m_i} \circ r_{m_j}) \cdot f) = T(f) \tag{5.1}$$

$$\text{for all } f \in C^\infty_c(\mathrm{G}_4),\ m_i \in \mathrm{M}_i,\ m_j \in \mathrm{M}_j,\ i \neq j,$$

then T is trivial.

First note that all $\mathrm{M}_i, i = 0, 1, 2$ involved here are unimodular.

Let $H = \mathrm{M}_i \times \mathrm{M}_j$, for $i \neq j$. The action of H on G_4 is given by

$$(m_i, m_j) \cdot g = m_i g m_j^{-1} \text{ for } (m_i, m_j) \in H, g \in \mathrm{G}_4.$$

This action is constructive by Theorem A in 6.15 of [BZ1]. Then by Bernstein's localization principle (Lemma 3.4), it is enough to show that T_x is trivial for all $x \in \mathrm{M}_i \backslash \mathrm{G}_4/\mathrm{M}_j$, where T_x is a distribution on $\mathrm{M}_i x \mathrm{M}_j$ satisfying equation (5.1).

Define a character ψ_H on H by

$$\psi_H(m_i, m_j) = \hat{\psi}_{4-2i}(m_i)\hat{\psi}_{4-2j}(m_j) \text{ for } (m_i, m_j) \in H$$

and the action of $(m_i, m_j) \in H$ on $C^\infty_c(\mathrm{G}_4)$ by

$$(m_1, m_2) \cdot \eta(g) = \psi_H^{-1}((m_1^{-1}, m_2))\eta(m_1^{-1} g m_2), \text{ for } \eta \in C^\infty_c(\mathrm{G}_4).$$

Let T_x be a nonzero H-invariant on an H-orbit $Y_x = \mathrm{M}_i x \mathrm{M}_j$, *i.e.*,

$$T_x((m_i, m_j) \cdot \eta) = \psi_H^{-1}((m_1^{-1}, m_2))T_x((l_{m_i} \circ r_{m_j}) \cdot \eta) = T_x(\eta)$$

for $(m_i, m_j) \in H$ and $\eta \in C^\infty_c(Y_x)$. Equivalently, T_x satisfies equation (5.1). Let H_x denote the stabilizer of x in H. Then $Y_x \cong H/H_x$. Note that $C^\infty_c(Y_x) \cong \underline{\mathrm{ind}}^H_{H_x} 1$ (un-normalized compact induction) and

$$T_x \in \mathrm{Hom}_H(\underline{\mathrm{ind}}^H_{H_x} 1, \psi_H) \cong \mathrm{Hom}_{H_x}(\delta_H \delta^{-1}_{H_x}, Res_{H_x} \psi_H)$$

by Frobenius reciprocity, where δ_H (respectively δ_{H_x}) is the modular function of H (respectively H_x). Since the absolute value of $\psi_H \equiv 1$ and $\delta_H \delta^{-1}_{H_x}$ is positive, by Schur's Lemma we have

$$\dim \mathrm{Hom}_{H_x}(\delta_H \delta^{-1}_{H_x}, Res_{H_x} \psi_H) = 0 \text{ or } \delta_H \delta^{-1}_{H_x} = Res_{H_x} \psi_H \equiv 1.$$

Proposition 1.3 in [Kl] shows that there are no admissible double cosets between $(\mathrm{M}_i, \hat{\psi}_{4-2i})$ and $(\mathrm{M}_j, \hat{\psi}_{4-2j})$, so $Res_{H_x} \psi_H \neq 1$ and $\mathrm{Hom}_{H_x}(\delta_H \delta^{-1}_{H_x}, Res_{H_x} \psi_H) = 0$ for all $x \in \mathrm{M}_i \backslash \mathrm{G}_4/\mathrm{M}_j$. Therefore $\mathfrak{D}(\mathrm{G}_4)^H = 0$ and $\mathrm{Hom}_{\mathrm{G}_4}(\mathcal{M}_i, \mathcal{M}_j) = 0$ follows. ■

Theorem 5.2 *$\mathcal{M}_{4,i}$ and $\mathcal{M}_{4,j}$ are disjoint for $i \neq j$. That is, an irreducible representation of G_4 cannot have both a nontrivial $\mathcal{M}_i$ model and a nontrivial $\mathcal{M}_j$ model for $i \neq j$.*

Proof $\mathcal{M}_0$ and $\mathcal{M}_2$ are disjoint by Theorem 4.2 and it remains to show that $\mathcal{M}_0$ and $\mathcal{M}_1$ (respectively, $\mathcal{M}_1$ and $\mathcal{M}_2$) are disjoint. Let π be an irreducible representation of G_4. Assume that π have both $\mathcal{M}_0$ (Whittaker) model and $\mathcal{M}_1$ model. By Proposition 3.2.1 of [HR], the contragradient $\widetilde{\pi}$ of π also admits a Whittaker model. The dual of $\mathrm{Hom}_{\mathrm{G}_4}(\widetilde{\pi}, \mathcal{M}_0) \neq 0$ gives $\mathrm{Hom}_{\mathrm{G}_4}(\mathrm{ind}_{\mathrm{M}_0}^{\mathrm{G}_4} \hat{\psi}_4^{-1}, \pi) \neq 0$ (refer to [GK] or [BZ1]). The composition of nontrivial

$$T_1 \in \mathrm{Hom}_{\mathrm{G}_4}(\mathrm{ind}_{\mathrm{M}_0}^{\mathrm{G}_4} \hat{\psi}_4^{-1}, \pi) \text{ and } T_2 \in \mathrm{Hom}_{\mathrm{G}_4}(\pi, \mathcal{M}_1)$$

produces a nontrivial intertwining operator (since π is irreducible) in

$$\mathrm{Hom}_{\mathrm{G}_4}(\mathrm{ind}_{\mathrm{M}_0}^{\mathrm{G}_4} \hat{\psi}_4^{-1}, \mathcal{M}_1).$$

The right action of M_1 on $\mathrm{M}_0\backslash\mathrm{G}_4$ is constructive by [BZ1, Theorem A, 6.15]. The restriction of T to the coset $\mathrm{M}_0 w \mathrm{M}_1$ is associated with $\mathrm{ind}_{\mathrm{M}_1 \cap w^{-1}\mathrm{M}_0 w}^{\mathrm{M}_1} \hat{\psi}_4^{-w}$, where $\hat{\psi}_4^{-w}(g) = \hat{\psi}_4^{-1}(wgw^{-1})$, for $g \in \mathrm{M}_1 \cap w^{-1}\mathrm{M}_0 w$. Frobenius reciprocity gives

$$\mathrm{Hom}_{\mathrm{M}_1}(\mathrm{ind}_{\mathrm{M}_1 \cap w^{-1}\mathrm{M}_0 w}^{\mathrm{M}_1} \hat{\psi}_4^{-w}, \hat{\psi}_2) \cong \mathrm{Hom}_{\mathrm{M}_1 \cap w^{-1}\mathrm{M}_0 w}(\hat{\psi}_4^{-w}, \hat{\psi}_2).$$

By the result of [Kl], there exists no admissible double coset for the pair $(\mathrm{M}_0, \hat{\psi}_4^{-1})$ and $(\mathrm{M}_1, \hat{\psi}_2)$, so $\mathrm{Hom}_{\mathrm{M}_1 \cap w^{-1}\mathrm{M}_0 w}(\hat{\psi}_4^{-w}, \hat{\psi}_2) = 0$ for all $w \in \mathrm{G}_4$. Hence by Bernstein's localization principle

$$\mathrm{Hom}_{\mathrm{G}_4}(\mathrm{ind}_{\mathrm{M}_0}^{\mathrm{G}_4} \hat{\psi}_4^{-1}, \mathrm{Ind}_{\mathrm{M}_1}^{G} \hat{\psi}_2) = 0,$$

which contradicts our assumption. And this contradicts the result of Lemma 5.1 that $\mathrm{Hom}_{\mathrm{G}_4}(\mathcal{M}_0, \mathcal{M}_1) = 0$. Hence π cannot possess both an $\mathcal{M}_0$ model and an $\mathcal{M}_1$ model. The proof for the disjointness of $\mathcal{M}_1$ and $\mathcal{M}_2$ follows the same argument, since $\widetilde{\pi}$ also admits a symplectic model if π does. ■

Lemma 5.3 *For $k \neq \pm 2$, ρ, τ unitary representations of G_1, the representation $\nu^k \rho \times \chi_3 = \nu^k \rho \times \langle \nu^{-1}\tau \times \tau \times \nu\tau \rangle$ has a unique $\mathcal{M}_1$ model.*

Proof By reciprocity,

$$\mathrm{Hom}_4\big(\mathrm{Ind}_{1,3}^4 \nu^k \rho \otimes \chi_3, \mathrm{Ind}_{\mathrm{U}_2 \times \mathrm{Sp}_2 \times \mathrm{N}_1}^{\mathrm{G}_4} \psi_2 \otimes 1 \otimes 1\big) \simeq \\ \mathrm{Hom}_{2,2}\big(\widetilde{\mathrm{r}}_{2,2}^4 \mathrm{Ind}_{1,3}^4 \nu^k \rho \otimes \chi_3, \mathrm{Ind}_{\mathrm{U}_2}^{\mathrm{G}_2} \psi_2 \otimes \mathrm{Ind}_{\mathrm{Sp}_2}^{\mathrm{G}_2} 1\big).$$

Let $\beta = \{1,3\}, \gamma = \{2,2\}$. In the notation of Theorem 3.3,

$$\mathrm{W}^{\beta,\gamma} = \{w_0 = \mathrm{id}, w_1 = (1,3,2)\}.$$

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For $w_0 = \mathrm{id}$, $\beta' = \beta \cap w_0^{-1}(\gamma) = \{1, 1, 2\}$, and $\gamma' = \gamma \cap w_0(\beta) = \{1, 1, 2\}$,

$$\begin{aligned} F_{w_0} &= \mathrm{Ind}^{2,2}_{1,1,2} \circ \mathrm{id} \circ (\widetilde{\mathrm{r}}^{1,3}_{1,1,2} \nu^k \rho \otimes \langle \nu^{-1}\tau \times \tau \times \nu\tau \rangle) \\ &= (\mathrm{Ind}^2_{1,1} \nu^k \rho \otimes \nu^{-1}\tau) \otimes \langle \tau \times \nu\tau \rangle. \end{aligned}$$

Because the representation $\mathrm{Ind}^2_{1,1}\nu^k\rho \otimes \nu^{-1}\tau$ has a unique Whittaker model and $\langle \tau \times \nu\tau \rangle = \nu^{\frac{1}{2}} \langle \nu^{-\frac{1}{2}}\tau \times \nu^{\frac{1}{2}}\tau \rangle$ has a unique $\mathcal{M}_{2,1}$ model, $\nu^k \times \chi_3$ has at least one $\mathcal{M}_1$ model.

For $w_1 = (1, 3, 2)$, $\beta' = \beta \cap w_1^{-1}(\gamma) = \{1, 2, 1\}$, and $\gamma' = \gamma \cap w_1(\beta) = \{2, 1, 1\}$,

$$\begin{aligned} F_{w_1} &= \mathrm{Ind}^{2,2}_{2,1,1} \circ w_1 \circ \widetilde{\mathrm{r}}^{1,3}_{1,2,1} \nu^k \rho \otimes \langle \nu^{-1}\tau \times \tau \times \nu\tau \rangle \\ &= \langle \nu^{-1}\tau \times \tau \rangle \otimes (\mathrm{Ind}^2_{1,1} \nu^k \rho \otimes \nu\tau). \end{aligned}$$

The representation $\mathrm{Ind}^2_{1,1}\nu^k\rho \otimes \nu\tau$ is irreducible and has a Whittaker model. Therefore F_{w_1} has no $\mathcal{M}_0 \otimes \mathcal{M}_{2,1}$ model, and $\nu^k \times \chi_3$ has a unique $\mathcal{M}_1$ model. ■

Lemma 5.4 *For $k \neq t \pm 1$, and $k, t \neq \pm\frac{3}{2}$; ρ, δ, τ unitary representations of* G_1, *the representation $\nu^k\rho \times \nu^t\delta \times \chi_2 = \nu^k\rho \times \nu^t\delta \times \langle \nu^{-\frac{1}{2}}\tau \times \nu^{\frac{1}{2}}\tau \rangle$ has a unique $\mathcal{M}_1$ model.*

Proof By reciprocity,

$$\begin{aligned} &\mathrm{Hom}_4\big(\mathrm{Ind}^4_{1,1,2}\nu^k\rho \otimes \nu^t\delta \times \chi_2, \mathrm{Ind}^{\mathrm{G}_4}_{\mathrm{U}_2 \times \mathrm{Sp}_2 \times \mathrm{N}_1} \psi_2 \otimes 1 \otimes 1\big) \simeq \\ &\qquad \mathrm{Hom}_{2,2}\big(\widetilde{\mathrm{r}}^4_{2,2}\mathrm{Ind}^4_{1,1,2}\nu^k\rho \times \nu^t\delta \times \chi_2, \mathrm{Ind}^{\mathrm{G}_2}_{\mathrm{U}_2}\psi_2 \otimes \mathrm{Ind}^{\mathrm{G}_2}_{\mathrm{Sp}_2} 1\big). \end{aligned}$$

Let $\beta = \{1, 1, 2\}$, $\gamma = \{2, 2\}$. Then

$$\mathrm{W}^{\beta,\gamma} = \{w_0 = id, w_1 = (2, 3), w_2 = (1, 3)(2, 4), w_3 = (1, 3, 2)\},$$

and the quotient

$$F_{w_0} = (\mathrm{Ind}^2_{1,1}\nu^k\rho \otimes \nu^t\delta) \otimes \chi_2.$$

Since $\mathrm{Ind}^2_{1,1}\nu^k\rho \otimes \nu^t\delta$ is irreducible and has a unique Whittaker model and χ_2 has a unique $\mathcal{M}_{2,1}$ model, $\nu^k\rho \times \chi_3$ has at least one $\mathcal{M}_1$ model. Note that

$$F_{w_1} = (\mathrm{Ind}^2_{1,1}\nu^k\rho \otimes \nu^{-\frac{1}{2}}\tau) \otimes (\mathrm{Ind}^2_{1,1}\nu^t\delta \otimes \nu^{\frac{1}{2}}\tau),$$

and $\mathrm{Ind}^2_{1,1}\nu^t\delta \otimes \nu^{\frac{1}{2}}\tau$ is irreducible and has a Whittaker model. Therefore F_{w_1} has no $\mathcal{M}_0 \otimes \mathcal{M}_{2,1}$ model.

Since $F_{w_2} = \chi_2 \otimes (\mathrm{Ind}^2_{1,1}\nu^k\rho \otimes \nu^t\delta)$, and $\mathrm{Ind}^2_{1,1}\nu^k\rho \otimes \nu^t\delta$ is irreducible and has a Whittaker model, F_{w_2} has no $\mathcal{M}_0 \otimes \mathcal{M}_{2,1}$ model.

Also $F_{w_3} = (\mathrm{Ind}^2_{1,1}\nu^t\delta \otimes \nu^{-\frac{1}{2}}\tau) \otimes (\mathrm{Ind}^2_{1,1}\nu^k\rho \otimes \nu^{\frac{1}{2}}\tau)$, and $(\mathrm{Ind}^2_{1,1}\nu^k\rho \otimes \nu^{\frac{1}{2}}\tau)$ is irreducible and has a Whittaker model. Therefore F_{w_3} has no $\mathcal{M}_0 \otimes \mathcal{M}_{2,1}$ model, and $\nu^k\rho \times \nu^t\delta \times \chi_2$ has a unique $\mathcal{M}_1$ model. ■

Lemma 5.5 *If χ is a character of G_n, $n \in \mathbb{N}$, then χ has a unique $\mathcal{M}_{[\frac{n}{2}]}$ model.*

Proof There are two cases.

1. n=2k: Since SL_{2k} is the commutator subgroup of G_{2k} and characters are trivial on the commutator subgroup, χ admits an embedding in $\mathrm{Ind}_{SL_{2k}}^{G_{2k}} 1$ and also in

$$\mathcal{M}_k = \mathrm{Ind}_{Sp_{2k}}^{G_{2k}} 1 = \mathrm{Ind}_{SL_{2k}}^{G_{2k}} \mathrm{Ind}_{Sp_{2k}}^{SL_{2k}} 1.$$

2. n=2k+1: Similarly χ admits an embedding in $\mathrm{Ind}_{SL_{2k+1}}^{G_{2k+1}} 1$, hence an embedding in

$$\mathcal{M}_k = \mathrm{Ind}_{U_1 \times Sp_{2k} \times N_1}^{G_{2k+1}} 1 = \mathrm{Ind}_{SL_{2k+1}}^{G_{2k+1}} \mathrm{Ind}_{U_1 \times Sp_{2k} \times N_1}^{SL_{2k+1}} 1.$$

The embedding is unique since χ is one-dimensional. ■

Lemma 5.6 *In $G_n = G_{m+2k}$, if (π, V_π) has a unique $\mathcal{M}_k$ model, then so does*

$$\pi' = \nu^t \otimes \pi, \ t \in \mathbb{R}.$$

Proof The existence of an $\mathcal{M}_k$ model for (π, V_π) means that for $x \in V_\pi$, there exists a function $f_x \colon G \mapsto V$, such that

1. $f_x(ug) = \psi_m(u) f_x(g)$, for $u \in U_m \times Sp_{2k} \times N_k, g \in G_n$.
2. $f_{ax+by} = af_x + bf_y$, for $a, b \in \mathbb{C}, x, y \in G_n$.
3. $\pi(s) f_x = f_{\pi(s)x}$. That is $f_x(gs) = f_{\pi(s)x}(g)$, for $g, s \in G_n, x \in V$.
4. f_x is locally constant. (That is, there exists open compact subgroup $K_{f_x} \subset G_n$ such that $f_x(gk) = f_x(g)$, for $k \in K_{f_x}, g \in G_n$.)

Let $W = \{h_x \mid h_x(g) = f_{\nu^t(g)x}(g), \forall x \in V, g \in G_n\}$. Then W is a $\mathcal{M}_k$ model of π', upon the verification of the following facts:

1. $h_x(ug) = f_{\nu^t(ug)x}(ug) = f_{\nu^t(g)x}(ug) = \psi_m(u) f_{\nu^t(g)x}(g) = \psi_m(u) h_x(g)$.
2. $h_{ax+by}(g) = f_{a\nu^t(g)x+b\nu^t(g)y}(g) = af_{\nu^t(g)x}(g) + bf_{\nu^t(g)y}(g) = ah_x(g) + bh_y(g)$.
3. $\pi'(s) h_x(g) = h_x(gs) = f_{\nu^t(gs)x}(gs) = f_{\pi(s)\nu^t(gs)x}(g) = f_{\nu^t(g)\nu^t(s)\pi(s)x}(g) = h_{\pi'(s)x}(g)$.
4. Because $\nu \colon G_n \mapsto \mathbb{R}_{>0}$ is a homomorphism, $\nu(K) = 1$ for all compact subgroup K of G_n. Given any f_x, there exists an open compact subgroup K_{f_x} in G_n such that $f_x(gk) = f_x(g)$, for $k \in K_{f_x}$. Then $h_x(gk) = f_{\nu^t(gk)x}(gk) = \nu^t(g) f_x(gk) = \nu^t(g) f_x(g) = f_{\nu^t(g)x}(g) = h_x(g)$.

By the above construction, if π' admits two different models $\mathcal{M}_k$, $\mathcal{M}_{k'}$, then so does $\pi = \nu^{-t} \otimes \pi'$. This shows the uniqueness. ■

Let δ_i, ρ_i, τ_i be square integrable representations of G_i, let χ_i be characters of G_i (we omit the subscript i if $i = 1$), and let $\alpha, \lambda \in (0, \frac{1}{2})$ be real numbers.

Theorem 5.7 *Any unitary representation on G_5 has one of the following expressions and indicted models:*

1. *δ_5, a square integrable representation of G_5, has a unique Whittaker model.*
2. *$u(\delta, 5) = L(\nu^2\delta \times \nu\delta \times \delta \times \nu^{-1}\delta \times \nu^{-2}\delta)$, a character of G_5, has a unique $\mathcal{M}_2$ model.*

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3. *Unitary representations induced from* $P_{1,4}$*:*
   (a) $\delta \times \delta_4$ *has a unique Whittaker model, since* δ *and* δ_4 *both have Whittaker models.*
   (b) $\delta \times \chi_4$ *has a unique* $\mathcal{M}_2$ *model.*
   (c) $\delta \times L(\nu^{\frac{1}{2}}\delta_2 \times \nu^{-\frac{1}{2}}\delta_2)$ *has a unique* $\mathcal{M}_2$ *model.*

4. *Unitary representations induced from* $P_{2,3}$*:*
   (a) $\delta_2 \times \delta_3$ *has a unique Whittaker model, since* δ_2 *and* δ_3 *both have Whittaker models.*
   (b) $\delta_3 \times \chi_2$ *has a unique* $\mathcal{M}_1$ *model.*
   (c) $\delta_2 \times \chi_3$ *has a unique* $\mathcal{M}_1$ *model.*
   (d) $\chi_2 \times \chi_3$ *has a unique* $\mathcal{M}_2$ *model.*

5. *Unitary representations induced from* $P_{1,1,3}$*:*
   (a) $\delta \times \tau \times \delta_3$ *has a unique Whittaker model, since* δ*,* τ*, and* δ_3 *all have Whittaker models.*
   (b) $\delta \times \tau \times \chi_3$ *has a unique* $\mathcal{M}_1$ *model.*
   (c) $\nu^{\alpha}\delta \times \nu^{-\alpha}\delta \times \delta_3$ *has a unique Whittaker model, since* δ *and* δ_3 *both have Whittaker models.*
   (d) $\nu^{\alpha}\delta \times \nu^{-\alpha}\delta \times \chi_3$*, has a unique* $\mathcal{M}_1$ *model.*

6. *Unitary representations induced from* $P_{1,1,1,2}$*:*
   (a) $\delta \times \rho \times \tau \times \delta_2$ *has a unique Whittaker model, since* δ*,* ρ*,* τ*, and* δ_2 *all have Whittaker models.*
   (b) $\delta \times \tau \times \rho \times \chi_2$ *has a unique* $\mathcal{M}_1$ *model.*
   (c) $\nu^{\alpha}\delta \times \nu^{-\alpha}\delta \times \tau \times \delta_2$ *has a unique Whittaker model, since* δ, τ*, and* δ_2 *all have Whittaker models.*
   (d) $\nu^{\alpha}\delta \times \nu^{-\alpha}\delta \times \tau \times \chi_2$ *has a unique* $\mathcal{M}_1$ *model.*

7. *Unitary representations induced from* $P_{1,2,2}$*:*
   (a) $\delta \times \delta_2 \times \delta_2'$ *has a unique Whittaker model, since* δ*,* δ_2*, and* δ_2' *all have Whittaker models.*
   (b) $\delta \times \delta_2 \times \chi_2$ *has a unique* $\mathcal{M}_1$ *model.*
   (c) $\delta \times \chi_2 \times \chi_2'$ *has a unique* $\mathcal{M}_2$ *model.*
   (d) $\delta \times \nu^{\alpha}\delta_2 \times \nu^{-\alpha}\delta_2$ *has a unique Whittaker model, since* δ *and* δ_2 *both have Whittaker models.*
   (e) $\delta \times \nu^{\alpha}\chi_2 \times \nu^{-\alpha}\chi_2$ *has a unique* $\mathcal{M}_2$ *model.*

8. *Unitary representations induced from* $P_{1,1,1,1,1}$*:*
   (a) $\delta \times \tau \times \rho \times \delta' \times \tau'$ *has a unique Whittaker model, since* δ*,* τ*,* ρ*,* δ'*, and* τ' *all have Whittaker models.*
   (b) $\nu^{\alpha}\delta \times \nu^{-\alpha}\delta \times \tau \times \rho \times \rho'$ *has a unique Whittaker model, since* δ, τ, ρ*, and* ρ' *all have Whittaker models.*
   (c) $\nu^{\alpha}\delta \times \nu^{-\alpha}\delta \times \nu^{\lambda}\tau \times \nu^{-\lambda}\tau \times \rho$ *has a unique Whittaker model, since* δ*,* τ*, and* ρ *all have Whittaker models.*

Proof 3(b): Set $\chi_4 = \langle \nu^{-\frac{3}{2}}\rho \times \nu^{-\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \times \nu^{\frac{3}{2}}\rho \rangle$, where ρ is a unitary representation of G_1. By reciprocity,

$$\mathrm{Hom}_5\big(\delta \times \chi_4, \mathrm{Ind}_{\mathrm{U}_1\times \mathrm{Sp}_4\times \mathrm{N}_2}^{\mathrm{G}_5}\psi_1 \otimes 1 \otimes 1\big) \simeq \mathrm{Hom}_{1,4}\big(\tilde{\mathrm{r}}_{1,4}^5 \mathrm{Ind}_{1,4}^5 \delta \otimes \chi_4, \mathrm{Ind}_{\mathrm{U}_1}^{\mathrm{G}_1}\psi_1 \otimes \mathrm{Ind}_{\mathrm{Sp}_4}^{\mathrm{G}_4} 1\big).$$

For $\beta = \{1,4\}, \gamma = \{1,4\}$, $\mathrm{W}^{\beta,\gamma} = \{w_0 = id, w_1 = (1,2)\}$, and the quotient $F_{w_0} = \delta \otimes \chi_4$ has a unique $\mathcal{M}_0 \otimes \mathcal{M}_{4,2}$ model. F_{w_1} has no $\mathcal{M}_0 \otimes \mathcal{M}_{4,2}$ model, because

$$\begin{aligned} F_{w_1} &= \nu^{-\frac{3}{2}}\rho \otimes \mathrm{Ind}_{1,3}^4 \delta \otimes \nu^{\frac{1}{2}}\langle \nu^{-1}\rho \times \rho \times \nu\rho\rangle \\ &= \nu^{-\frac{3}{2}}\rho \otimes \nu^{\frac{1}{2}}\big(\mathrm{Ind}_{1,3}^4 \nu^{-\frac{1}{2}}\delta \otimes \langle \nu^{-1}\rho \times \rho \times \nu\rho\rangle\big) \end{aligned}$$

and the representation $\mathrm{Ind}_{1,3}^4 \nu^{-\frac{1}{2}}\delta \otimes \langle \nu^{-1}\rho \times \rho \times \nu\rho\rangle$ has an $\mathcal{M}_{4,1}$ model (refer to Lemma 5.3). Hence $\delta \times \chi_4$ has a unique $\mathcal{M}_2$ model.

3(c): By reciprocity,

$$\mathrm{Hom}_5\big(\delta \times L(\nu^{\frac{1}{2}}\delta_2 \times \nu^{-\frac{1}{2}}\delta_2), \mathrm{Ind}_{\mathrm{U}_1\times \mathrm{Sp}_4\times \mathrm{N}_2}^{\mathrm{G}_5}\psi_1 \otimes 1 \otimes 1\big) \simeq \mathrm{Hom}_{1,4}\big(\tilde{\mathrm{r}}_{1,4}^5 \mathrm{Ind}_{1,4}^5 \delta \otimes L(\nu^{\frac{1}{2}}\delta_2 \times \nu^{-\frac{1}{2}}\delta_2), \mathrm{Ind}_{\mathrm{U}_1}^{\mathrm{G}_1}\psi_1 \otimes \mathrm{Ind}_{\mathrm{Sp}_4}^{\mathrm{G}_4} 1\big).$$

For $\beta = \gamma = \{1,4\}$, we have $\mathrm{W}^{\beta,\gamma} = \{w_0 = id, w_1 = (1,2)\}$, with quotient $F_{w_0} = \delta \otimes L(\nu^{\frac{1}{2}}\delta_2 \times \nu^{-\frac{1}{2}}\delta_2)$. Because $L(\nu^{\frac{1}{2}}\delta_2 \times \nu^{-\frac{1}{2}}\delta_2)$ has a unique $\mathcal{M}_2$ model by [HR, Theorem 11.1], F_{w_0} has a unique $\mathcal{M}_0 \otimes \mathcal{M}_2$ model. For

$$F_{w_1} = \mathrm{Ind}_{1,1,3}^{1,4} \circ w_1 \circ \tilde{\mathrm{r}}_{1,1,3}^{1,4} \delta \times L(\nu^{\frac{1}{2}}\delta_2 \times \nu^{-\frac{1}{2}}\delta_2),$$

with δ_2 either (i) supercuspidal or (ii) Steinberg:

(i) When δ_2 is supercuspidal, F_{w_1}=0 has no $\mathcal{M}_0 \otimes \mathcal{M}_2$ model.

(ii) When δ_2 is Steinberg, set

$$\pi = L(\nu^{\frac{1}{2}}\delta_2 \times \nu^{-\frac{1}{2}}\delta_2) = L(L(\triangle_1) \times L(\triangle_2)),$$

where $\delta_2 = \langle \nu^{\frac{1}{2}}\rho \times \nu^{-\frac{1}{2}}\rho\rangle$, and segments $\triangle_1 = [\rho, \nu\rho]$, $\triangle_2 = [\nu^{-1}\rho, \rho]$.

By Lemma 3.8, $L(\triangle_1) \times L(\triangle_2) = \langle \nu\rho \times \rho\rangle \times \langle \rho \times \nu^{-1}\rho\rangle$ has two constitutions, $L(L(\triangle_1) \times L(\triangle_2)) = \pi$ and

$$\begin{aligned} L(\triangle_1 \cup \triangle_2, \triangle_1 \cap \triangle_2) &= L(\triangle_1 \cup \triangle_2) \times L(\triangle_1 \cap \triangle_2) \\ &= L([\nu^{-1}\rho, \rho, \nu\rho]) \times \rho. \end{aligned}$$

First, $\tilde{\mathrm{r}}_{1,3}^4 \mathrm{Ind}_{2,2}^4 \langle \nu\rho \times \rho\rangle \times \langle \rho \times \nu^{-1}\rho\rangle$ has two constitutions, $F_{w_0'}$ and $F_{w_1'}$, where $F_{w_0'} = \nu\rho \otimes (\mathrm{Ind}_{1,2}^3 \rho \times \langle \rho \times \nu^{-1}\rho\rangle)$ and $F_{w_1'} = \rho \otimes (\mathrm{Ind}_{2,1}^3 \langle \nu\rho \times \rho\rangle \otimes \nu^{-1}\rho)$ are obtained from $\mathrm{W}^{\beta',\gamma'} = \{w_0' = id, w_1' = (1,2,3)\}$, with $\beta' = \{2,2\}$ and $\gamma' = \{1,3\}$. Next, $\tilde{\mathrm{r}}_{1,3}^4 \mathrm{Ind}_{3,1}^4 \langle \nu\rho \times \rho \times \nu^{-1}\rho\rangle \otimes \rho$ also has two constitutions, $F_{w_0''}$ and $F_{w_1''}$, where

$F_{w_0''} = \nu\rho \otimes \big(\mathrm{Ind}^3_{2,1}\langle \rho \times \nu^{-1}\rho\rangle \otimes \rho\big)$ and $F_{w_1''} = \rho \otimes \langle \nu\rho \times \rho \times \nu^{-1}\rho\rangle$ are obtained from $W^{\beta'',\gamma''} = \{w_0'' = id, w_1'' = (1,2,3,4)\}$, with $\beta'' = \{3,1\}$ and $\gamma'' = \{1,3\}$. Since

$$\begin{aligned}\widetilde{\mathrm{r}}^4_{1,3}L(\nu^{\frac12}\delta_2 \times \nu^{-\frac12}\delta_2) &= \widetilde{\mathrm{r}}^4_{1,3}\mathrm{Ind}^4_{2,2}\langle \nu\rho \times \rho\rangle \times \langle \rho \times \nu^{-1}\rho\rangle - \widetilde{\mathrm{r}}^4_{1,3}\langle \nu\rho \times \rho \times \nu^{-1}\rho\rangle \times \rho \\ &= \rho \otimes \big(\mathrm{Ind}^3_{2,1}\langle \nu\rho \times \rho\rangle \otimes \nu^{-1}\rho\big) - \rho \otimes \langle \nu\rho \times \rho \times \nu^{-1}\rho\rangle \\ &= \rho \otimes L(\langle \nu\rho \times \rho\rangle \times \nu^{-1}\rho),\end{aligned}$$

we have

$$\begin{aligned}F_{w_1} &= \mathrm{Ind}^{1,4}_{1,1,3} \circ w_1 \circ \big(\delta \otimes \rho \otimes L(\langle \nu\rho \times \rho\rangle \times \nu^{-1}\rho)\big) \\ &= \rho \otimes \big(\mathrm{Ind}^4_{1,3}\delta \otimes L(\langle \nu\rho \times \rho\rangle \times \nu^{-1}\rho)\big).\end{aligned}$$

We claim that $\mathrm{Ind}^4_{1,3}\delta \otimes L(\langle \nu\rho \times \rho\rangle \times \nu^{-1}\rho)$ has an $\mathcal{M}_{4,1}$ model. The quotient of

$$\widetilde{\mathrm{r}}^4_{2,2}\mathrm{Ind}^4_{1,2,1}\delta \otimes \langle \nu\rho \times \rho\rangle \otimes \nu^{-1}\rho$$

is $\lambda = \mathrm{Ind}^2_{1,1}(\delta \otimes \nu\rho) \otimes \mathrm{Ind}^2_{1,1}(\rho \otimes \nu^{-1}\rho)$. Because $\mathrm{Ind}^2_{1,1}\delta \otimes \nu\rho$ has a Whittaker model and the quotient $L(\rho \times \nu^{-1}\rho)$ of $\mathrm{Ind}^2_{1,1}\rho \otimes \nu^{-1}\rho$ has a symplectic model, λ has an $\mathcal{M}_{4,1}$ model and so does $\mathrm{Ind}^4_{1,2,1}\delta \otimes \langle \nu\rho \times \rho\rangle \otimes \nu^{-1}\rho$. Since $\mathrm{Ind}^4_{1,2,1}\delta \otimes \langle \nu\rho \times \rho\rangle \otimes \nu^{-1}\rho$ consists of two irreducible constitutions, $\mathrm{Ind}^4_{1,3}\rho \otimes \langle \nu\rho \times \rho \times \nu^{-1}\rho\rangle$ (with a Whittaker model) and $\mathrm{Ind}^4_{1,3}\delta \otimes L(\langle \nu\rho \times \rho\rangle \times \nu^{-1}\rho)$, the $\mathcal{M}_{4,1}$ model must be supported in $\mathrm{Ind}^4_{1,3}\delta \otimes L(\langle \nu\rho \times \rho\rangle \times \nu^{-1}\rho)$. This proves the claim.

By the disjointness of $\mathcal{M}_{4,1}$ and $\mathcal{M}_{4,2}$, $\mathrm{Ind}^4_{1,3}\delta \otimes L(\langle \nu\rho \times \rho\rangle \times \nu^{-1}\rho)$ has no $\mathcal{M}_{4,2}$ model, and we conclude that F_{w_1} has no $\mathcal{M}_0 \otimes \mathcal{M}_2$ model. Hence $\delta \times L(\nu^{\frac12}\delta_2 \times \nu^{-\frac12}\delta_2)$ has a unique $\mathcal{M}_2$ model.

4(b): Set $\chi_2 = \langle \nu^{-\frac12}\rho \times \nu^{\frac12}\rho\rangle$. By reciprocity,

$$\begin{aligned}&\mathrm{Hom}_5\big(\delta_3 \times \chi_2, \mathrm{Ind}^{G_5}_{U_3 \times \mathrm{Sp}_2 \times N_1}\psi_3 \otimes 1 \otimes 1\big) \simeq \\ &\qquad\qquad \mathrm{Hom}_{3,2}\big(\widetilde{\mathrm{r}}^5_{3,2}\mathrm{Ind}^5_{3,2}\delta_3 \otimes \chi_2, \mathrm{Ind}^{G_3}_{U_3}\psi_3 \otimes \mathrm{Ind}^{G_2}_{\mathrm{Sp}_2} 1\big).\end{aligned}$$

For $\beta = \gamma = \{3,2\}$, $W^{\beta,\gamma} = \{w_0 = id, w_1 = (2,4)(3,5), w_2 = (3,4)\}$, and the quotient $F_{w_0} = \delta_3 \otimes \chi_2$ has a unique $\mathcal{M}_0 \otimes \mathcal{M}_1$ model. For $F_{w_1} = \mathrm{Ind}^{3,2}_{1,2,2} \circ w_1 \circ \widetilde{\mathrm{r}}^{3,2}_{1,2,2}\delta_3 \otimes \chi_2$, with δ_3 either (i) supercuspidal or (ii) Steinberg:

(i) When δ_3 is supercuspidal, F_{w_1}=0 has no $\mathcal{M}_0 \otimes \mathcal{M}_1$ model.

(ii) When δ_3 is Steinberg, set $\delta_3 = \langle \nu\tau \times \tau \times \nu^{-1}\tau\rangle$. Then

$$F_{w_1} = (\mathrm{Ind}^3_{1,2}\nu\tau \otimes \chi_2) \otimes \langle \tau \times \nu^{-1}\tau\rangle.$$

Since $\langle \tau \times \nu^{-1}\tau\rangle$ has a Whittaker model, F_{w_1} has no $\mathcal{M}_0 \otimes \mathcal{M}_1$ model.

For $F_{w_2} = \mathrm{Ind}^{3,2}_{2,1,1,1} \circ w_2 \circ \widetilde{\mathrm{r}}^{3,2}_{1,2,2}\delta_3 \otimes \chi_2$, with δ_3 either supercuspidal or Steinberg:

(i) When δ_3 is supercuspidal, $F_{w_2}=0$ has no $\mathcal{M}_0 \otimes \mathcal{M}_1$ model.
(ii) When δ_3 is Steinberg,

$$F_{w_2} = \left(\mathrm{Ind}_{2,1}^{3}\langle \nu\tau \times \tau\rangle \otimes \nu^{-\frac{1}{2}}\rho\right) \otimes \left(\mathrm{Ind}_{1,1}^{2}\nu^{-1}\tau \otimes \nu^{\frac{1}{2}}\rho\right).$$

The representation $\mathrm{Ind}_{1,1}^{2}\nu^{-1}\tau \otimes \nu^{\frac{1}{2}}\rho$ has a Whittaker model, so F_{w_2} has no $\mathcal{M}_0 \otimes \mathcal{M}_1$ model. Hence $\delta_3 \times \chi_2$ has a unique $\mathcal{M}_1$ model.

4(c): Set $\chi_3 = \langle \nu^{-1}\rho \times \rho \times \nu\rho\rangle$. By reciprocity,

$$\mathrm{Hom}_5\left(\delta_2 \times \chi_3, \mathrm{Ind}_{U_3 \times Sp_2 \times N_1}^{G_5}\psi_3 \otimes 1 \otimes 1\right) \simeq \mathrm{Hom}_{3,2}\left(\tilde{r}_{3,2}^{5}\mathrm{Ind}_{2,3}^{5}\delta_2 \otimes \chi_3, \mathrm{Ind}_{U_3}^{G_3}\psi_3 \otimes \mathrm{Ind}_{Sp_2}^{G_2} 1\right).$$

For $\beta = \{2,3\}, \gamma = \{3,2\}$,

$$W^{\beta,\gamma} = \{w_0 = id, w_1 = (1,4,2,5,3), w_2 = (2,4,3)\}.$$

The quotient $F_{w_0} = \left(\mathrm{Ind}_{2,1}^{3}\delta_2 \otimes \nu^{-1}\rho\right) \otimes \langle \rho \times \nu\rho\rangle$ has a unique $\mathcal{M}_0 \otimes \mathcal{M}_1$ model, and $F_{w_1} = \chi_3 \otimes \delta_2$ has no $\mathcal{M}_0 \otimes \mathcal{M}_1$ model. Note that $F_{w_2} = \mathrm{Ind}_{1,2,1,1}^{3,2} \circ w_2 \circ \tilde{r}_{1,1,2,1}^{2,3}\delta_2 \otimes \langle \nu^{-1}\rho \times \rho \times \nu\rho\rangle$.

(i) When δ_2 is supercuspidal, $F_{w_2}=0$ has no $\mathcal{M}_0 \otimes \mathcal{M}_1$ model.
(ii) When δ_2 is Steinberg, set $\delta_2 = \langle \nu^{\frac{1}{2}}\tau \times \nu^{-\frac{1}{2}}\tau\rangle$. Because

$$F_{w_2} = \left(\mathrm{Ind}_{1,2}^{3}\nu^{\frac{1}{2}}\tau \otimes \langle \nu^{-1}\rho \times \rho\rangle\right) \otimes \mathrm{Ind}_{1,1}^{2}\nu^{-\frac{1}{2}}\tau \otimes \nu\rho$$

and $\mathrm{Ind}_{1,1}^{2}\nu^{-\frac{1}{2}}\tau \otimes \nu\rho$ has a Whittaker model, F_{w_2} has no $\mathcal{M}_0 \otimes \mathcal{M}_1$ model. Hence $\delta_2 \times \chi_3$ has a unique $\mathcal{M}_1$ model.

4(d): Set $\chi_3 = \langle \nu^{-1}\tau \times \tau \times \nu\tau\rangle, \chi_2 = \langle \nu^{-\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho\rangle$. By reciprocity,

$$\mathrm{Hom}_5\left(\chi_3 \times \chi_2, \mathrm{Ind}_{U_1 \times Sp_4 \times N_2}^{G_5}\psi_1 \otimes 1 \otimes 1\right) \simeq \mathrm{Hom}_{1,4}\left(\tilde{r}_{1,4}^{5}\mathrm{Ind}_{3,2}^{5}\chi_3 \times \chi_2, \mathrm{Ind}_{U_1}^{G_1}\psi_1 \otimes \mathrm{Ind}_{Sp_4}^{G_4} 1\right).$$

For $\beta = \{3,2\}, \gamma = \{1,4\}, W^{\beta,\gamma} = \{w_0 = id, w_1 = (1,2,3,4)\}$, and the quotient $F_{w_0} = \nu^{-1}\tau \otimes (\mathrm{Ind}_{2,2}^{4}\langle \nu^{-1}\tau \times \tau\rangle \otimes \langle \nu^{-\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho\rangle)$. By [HR, Proposition 11.4], $\langle \nu^{-1}\tau \times \tau\rangle \times \langle \nu^{-\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho\rangle$ has a unique $\mathcal{M}_2$ model, so F_{w_0} has a unique $\mathcal{M}_0 \otimes \mathcal{M}_2$ model.

By the disjointness of $\mathcal{M}_{4,1}$ and $\mathcal{M}_{4,2}$ and the fact that $\mathrm{Ind}_{3,1}^{4}\chi_3 \otimes \nu^{\frac{1}{2}}\rho$ has an $\mathcal{M}_{4,1}$ model,

$$F_{w_1} = \nu^{-\frac{1}{2}}\rho \otimes \left(\mathrm{Ind}_{3,1}^{4}\chi_3 \otimes \nu^{\frac{1}{2}}\rho\right)$$

has no $\mathcal{M}_0 \otimes \mathcal{M}_2$ model. Hence $\chi_3 \times \chi_2$ has a unique $\mathcal{M}_2$ model.

5(b) and (d) are both in the form of $\nu^k\delta \times \nu^t\tau \times \chi_3$, for $k \neq t \pm 1$, and $k, t \neq \pm 2$. Now we want to show that $\nu^k\delta \times \nu^t\tau \times \chi_3$ has a unique $\mathcal{M}_1$ model. Set $\chi_3 = \langle \nu^{-1}\rho \times \rho \times \nu\rho \rangle$. By reciprocity,

$$\mathrm{Hom}_5\big(\nu^k\delta \times \nu^t\tau \times \chi_3, \mathrm{Ind}^{\mathrm{G}_5}_{\mathrm{U}_3\times \mathrm{Sp}_2\times \mathrm{N}_1}\psi_3 \otimes 1 \otimes 1\big) \simeq \mathrm{Hom}_{3,2}\big(\widetilde{\mathrm{r}}^5_{3,2}\mathrm{Ind}^5_{1,1,3}\nu^k\delta \otimes \nu^t\tau \otimes \chi_3, \mathrm{Ind}^{\mathrm{G}_3}_{\mathrm{U}_3}\psi_3 \otimes \mathrm{Ind}^{\mathrm{G}_2}_{\mathrm{Sp}_2} 1\big).$$

For $\beta = \{1, 1, 3\}, \gamma = \{3, 2\}$,

$$\mathrm{W}^{\beta,\gamma} = \{w_0 = id, w_1 = (1, 4, 2, 5, 3), w_2 = (2, 4, 3)\},$$

and the quotient $F_{w_0} = (\mathrm{Ind}^3_{1,1,1}\nu^k\delta \otimes \nu^t\tau \otimes \nu^{-1}\rho) \otimes \langle \rho \times \nu\rho \rangle$. Since $\mathrm{Ind}^3_{1,1,1}\nu^k\delta \otimes \nu^t\tau \otimes \nu^{-1}\rho$ and $\mathrm{Ind}^2_{1,1}\nu^k\delta \otimes \nu^t\tau$ both have unique Whittaker models by Theorem 3.10, F_{w_0} has a unique $\mathcal{M}_0 \otimes \mathcal{M}_1$ model, and $F_{w_1} = \chi_3 \otimes \big(\mathrm{Ind}^2_{1,1}\nu^k\delta \otimes \nu^t\tau\big)$ has no $\mathcal{M}_0 \otimes \mathcal{M}_1$ model. Since $\mathrm{Ind}^3_{1,2}\nu^k\delta \otimes \langle \nu^{-1}\rho \times \rho \rangle$ is irreducible and has an $\mathcal{M}_1$ model,

$$F_{w_2} = \big(\mathrm{Ind}^3_{1,2}\nu^k\delta \otimes \langle \nu^{-1}\rho \times \rho \rangle\big) \otimes \big(\mathrm{Ind}^2_{1,1}\nu^t\tau \otimes \nu\rho\big)$$

has no $\mathcal{M}_0 \otimes \mathcal{M}_1$ model. Therefore $\nu^k\delta \times \nu^t\tau \times \chi_3$ has a unique $\mathcal{M}_1$ model.

6(b) and (d) are both in the form of $\nu^k\delta \times \nu^s\tau \times \nu^t\tau' \times \chi_2$, and none of them are linked. That is, $k, s, t \neq \pm\frac{1}{2}, \pm\frac{3}{2}$ and the difference between any pair of them is not ± 1. We want to show that $\nu^k\delta \times \nu^s\tau \times \nu^t\tau' \times \chi_2$ has a unique $\mathcal{M}_1$ model. Set $\chi_2 = \langle \nu^{-\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \rangle$. By reciprocity,

$$\mathrm{Hom}_5\big(\nu^k\delta \times \nu^s\tau \times \nu^t\tau' \times \chi_2, \mathrm{Ind}^{\mathrm{G}_5}_{\mathrm{U}_3\times \mathrm{Sp}_2\times \mathrm{N}_1}\psi_3 \otimes 1 \otimes 1\big) \simeq \mathrm{Hom}_{3,2}\big(\widetilde{\mathrm{r}}^5_{3,2}\mathrm{Ind}^5_{1,1,1,2}\nu^k\delta \otimes \nu^s\tau \otimes \nu^t\tau' \otimes \chi_2, \mathrm{Ind}^{\mathrm{G}_3}_{\mathrm{U}_3}\psi_3 \otimes \mathrm{Ind}^{\mathrm{G}_2}_{\mathrm{Sp}_2} 1\big).$$

For $\beta = \{1, 1, 1, 2\}, \gamma = \{3, 2\}$,

$$\mathrm{W}^{\beta,\gamma} = \{w_0 = id, w_1 = (1, 4, 2, 5, 3), w_2 = (3, 4), \dots\},$$

and the quotient $F_{w_0} = \big(\mathrm{Ind}^3_{1,1,1}\nu^k\delta \otimes \nu^s\tau \otimes \nu^t\tau'\big) \otimes \chi_2$ has a unique $\mathcal{M}_0 \otimes \mathcal{M}_1$ model. Any other F_{w_i} cannot keep χ_2 at the (4, 5)-th position. (If it does, then $w_i^{-1}(1) \leq w_i^{-1}(2) \leq w_i^{-1}(3)$ will force $w_i = \mathrm{id}$.) Therefore we cannot find another factor with an $\mathcal{M}_0 \otimes \mathcal{M}_1$ model. Thus $\nu^k\delta \times \nu^s\tau \times \nu^t\tau' \times \chi_2$ has a unique $\mathcal{M}_1$ model.

7(b): Set $\chi_2 = \langle \nu^{-\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \rangle$. By reciprocity,

$$\mathrm{Hom}_5\big(\delta \times \delta_2 \times \chi_2, \mathrm{Ind}^{\mathrm{G}_5}_{\mathrm{U}_3\times \mathrm{Sp}_2\times \mathrm{N}_1}\psi_3 \otimes 1 \otimes 1\big) \simeq \mathrm{Hom}_{3,2}\big(\widetilde{\mathrm{r}}^5_{3,2}\mathrm{Ind}^5_{1,2,2}\delta \otimes \delta_2 \otimes \chi_2, \mathrm{Ind}^{\mathrm{G}_3}_{\mathrm{U}_3}\psi_3 \otimes \mathrm{Ind}^{\mathrm{G}_2}_{\mathrm{Sp}_2} 1\big).$$

For $\beta = \{1, 2, 2\}, \gamma = \{3, 2\}$,

$$\mathrm{W}^{\beta,\gamma} = \{w_0 = id, w_1 = (2, 4)(3, 5), w_2 = (3, 4), w_3 = (1, 4, 3, 2)\},$$

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and the quotient $F_{w_0} = \left(\mathrm{Ind}^3_{1,2}\delta \otimes \delta_2\right) \otimes \chi_2$ has a unique $\mathcal{M}_0 \otimes \mathcal{M}_1$ model. Also

$$F_{w_1} = \left(\mathrm{Ind}^3_{1,2}\delta \otimes \chi_2\right) \otimes \delta_2$$

has no $\mathcal{M}_0 \otimes \mathcal{M}_1$ model. Note that

$$F_{w_2} = \mathrm{Ind}^{3,2}_{1,1,1,1,1} \circ w_2 \circ \widetilde{\mathrm{r}}^{1,2,2}_{1,1,1,1,1}\delta \otimes \delta_2 \otimes \chi_2,$$

where δ_2 is either (i) supercuspidal or (ii) Steinberg.

(i) When δ_2 is supercuspidal, F_{w_2}=0 has no $\mathcal{M}_0 \otimes \mathcal{M}_1$ model.

(ii) When δ_2 is Steinberg, set $\delta_2 = \langle \nu^{\frac{1}{2}}\tau \times \nu^{-\frac{1}{2}}\tau\rangle$. Then

$$F_{w_2} = \left(\mathrm{Ind}^3_{1,1,1}\delta \otimes \nu^{\frac{1}{2}}\tau \otimes \nu^{-\frac{1}{2}}\rho\right) \otimes \left(\mathrm{Ind}^2_{1,1}\nu^{-\frac{1}{2}}\tau \otimes \nu^{\frac{1}{2}}\rho\right)$$

has no $\mathcal{M}_0 \otimes \mathcal{M}_1$ model, and neither does

$$F_{w_3} = \left(\mathrm{Ind}^3_{2,1}\delta_2 \otimes \nu^{-\frac{1}{2}}\rho\right) \otimes \left(\mathrm{Ind}^2_{1,1}\delta \otimes \nu^{\frac{1}{2}}\rho\right).$$

Hence $\delta \times \delta_2 \times \chi_2$ has a unique $\mathcal{M}_1$ model.

7(c) and (e) are both in the form of $\delta \times \nu^\alpha\chi_2 \times \nu^\lambda\chi_2'$, where $\alpha \neq \lambda \pm 1; \alpha, \lambda \neq \pm\frac{1}{2}, \pm\frac{3}{2}$. Now we want to show that $\delta \times \nu^\alpha\chi_2 \times \nu^\lambda\chi_2'$ has a unique $\mathcal{M}_2$ model. By reciprocity,

$$\mathrm{Hom}_5\left(\delta \times \nu^\alpha\chi_2 \times \nu^\lambda\chi_2', \mathrm{Ind}^{\mathrm{G}_5}_{\mathrm{U}_1\times\mathrm{Sp}_4\times\mathrm{N}_2}\psi_1 \otimes 1 \otimes 1\right) \simeq$$
$$\mathrm{Hom}_{1,4}\left(\widetilde{\mathrm{r}}^5_{1,4}\mathrm{Ind}^5_{1,2,2}\delta \otimes \nu^\alpha\chi_2 \otimes \nu^\lambda\chi_2', \mathrm{Ind}^{\mathrm{G}_1}_{\mathrm{U}_1}\psi_1 \otimes \mathrm{Ind}^{\mathrm{G}_4}_{\mathrm{Sp}_4}1\right).$$

For $\beta = \{1,2,2\}, \gamma = \{1,4\}$,

$$\mathrm{W}^{\beta,\gamma} = \{w_0 = id, w_1 = (1,2), w_2 = (1,2,3,4)\},$$

and the quotient $F_{w_0} = \delta \otimes (\mathrm{Ind}^4_{2,2}\nu^\alpha\chi_2 \otimes \nu^\lambda\chi_2')$ has a unique $\mathcal{M}_0 \otimes \mathcal{M}_2$ model. Let $\chi_2 = \langle \nu^{-\frac{1}{2}}\tau \times \nu^{\frac{1}{2}}\tau\rangle$ and $\chi_2' = \langle \nu^{-\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho\rangle$. Then

$$F_{w_1} = \nu^{-\frac{1}{2}+\alpha}\tau \otimes \left(\mathrm{Ind}^4_{1,1,2}\delta \otimes \nu^{\frac{1}{2}+\alpha}\tau \otimes \nu^\lambda\chi_2'\right)$$

has no $\mathcal{M}_0 \otimes \mathcal{M}_2$ model, because $\mathrm{Ind}^4_{1,1,2}\delta \otimes \nu^{\frac{1}{2}+\alpha}\tau \otimes \nu^\lambda\chi_2'$ has an $\mathcal{M}_{4,1}$ model by Lemma 5.4. Also

$$F_{w_2} = \nu^{-\frac{1}{2}+\lambda}\rho \otimes \left(\mathrm{Ind}^4_{1,2,1}\delta \otimes \nu^\alpha\chi_2 \otimes \nu^{\frac{1}{2}+\lambda}\rho\right)$$

has no $\mathcal{M}_0 \otimes \mathcal{M}_2$ model, since $\mathrm{Ind}^4_{1,2,1}\delta \otimes \chi_2 \otimes \nu^{\frac{1}{2}+\lambda}\rho$ has an $\mathcal{M}_{4,1}$ model. Hence $\delta \times \nu^\alpha\chi_2 \times \nu^\lambda\chi_2'$ has a unique $\mathcal{M}_2$ model. ■

Then Table 1 lists models of unitary representation on G_5, where $\alpha, \lambda \in (0, \frac{1}{2})$ are real numbers; δ_i, ρ_i, and τ_i are square-integrable representations of G_i, and χ_i are characters of G_i. We omit the subscript i if $i = 1$.

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Representation	Model
δ_5	$\mathcal{M}_0$
$L(\nu^2\delta \times \nu\delta \times \delta \times \nu^{-1}\delta \times \nu^{-2}\delta)$	$\mathcal{M}_2$
$\delta \times \delta_4$	$\mathcal{M}_0$
$\delta \times \chi_4$	$\mathcal{M}_2$
$\delta \times L(\nu^{\frac{1}{2}}\delta_2 \times \nu^{-\frac{1}{2}}\delta_2)$	$\mathcal{M}_2$
$\delta_2 \times \delta_3$	$\mathcal{M}_0$
$\delta_3 \times \chi_2$	$\mathcal{M}_1$
$\delta_2 \times \chi_3$	$\mathcal{M}_1$
$\chi_2 \times \chi_3$	$\mathcal{M}_2$
$\delta \times \tau \times \delta_3$	$\mathcal{M}_0$
$\delta \times \tau \times \chi_3$	$\mathcal{M}_1$
$\nu^{\alpha}\delta \times \nu^{-\alpha}\delta \times \delta_3$	$\mathcal{M}_0$
$\nu^{\alpha}\delta \times \nu^{-\alpha}\delta \times \chi_3$	$\mathcal{M}_1$
$\delta \times \rho \times \tau \times \delta_2$	$\mathcal{M}_0$
$\delta \times \tau \times \rho \times \chi_2$	$\mathcal{M}_1$
$\nu^{\alpha}\delta \times \nu^{-\alpha}\delta \times \tau \times \delta_2$	$\mathcal{M}_0$
$\nu^{\alpha}\delta \times \nu^{-\alpha}\delta \times \tau \times \chi_2$	$\mathcal{M}_1$
$\delta \times \delta_2 \times \delta_2'$	$\mathcal{M}_0$
$\delta \times \delta_2 \times \chi_2$	$\mathcal{M}_1$
$\delta \times \chi_2 \times \chi_2'$	$\mathcal{M}_2$
$\delta \times \nu^{\alpha}\delta_2 \times \nu^{-\alpha}\delta_2$	$\mathcal{M}_0$
$\delta \times \nu^{\alpha}\chi_2 \times \nu^{-\alpha}\chi_2$	$\mathcal{M}_2$
$\delta \times \tau \times \rho \times \delta' \times \tau'$	$\mathcal{M}_0$
$\nu^{\alpha}\delta \times \nu^{-\alpha}\delta \times \tau \times \rho \times \rho'$	$\mathcal{M}_0$
$\nu^{\alpha}\delta \times \nu^{-\alpha}\delta \times \nu^{\lambda}\tau \times \nu^{-\lambda}\tau \times \rho$	$\mathcal{M}_0$

Table 1: Models of unitary representation on G_5.

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Department of Mathematics, National Cheng Kung University, Tainan 701, TAIWAN
e-mail: nienpig@mail.ncku.edu.tw