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ON A FAMILY OF CYCLICALLY-PRESENTED FUNDAMENTAL GROUPS

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To Laci Kovács on his 65th birthday

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Abstract

Bounds are obtained for the minimum number of generators for the fundamental groups of a family of closed 3-dimensional manifolds. A significant role has been played by the use of computers.

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1. Introduction

In a recent paper [JKO99] Johnson, Kim and O'Brien studied two pairs of families of cyclically-presented groups which are fundamental groups of 3-manifolds. They show that the corresponding groups in each pair of these families are isomorphic. Johnson *et al.* ask (Question 4.3) which of the groups in these families are finite. The groups are all infinite (apart from degenerate cases)—see Havas, Holt and Newman [HHN01]. Johnson *et al.* also say (Remark 4.4) that there is some evidence to suggest that the groups can be generated by two elements. This paper addresses the question of finding 'small' generating sets for these groups.

It will be convenient to introduce some notation for cyclically-presented groups. Let F be a free group of countably infinite rank freely generated by $\{x_1, x_2, \dots\}$. Let θ_n be the automorphism of F which maps $x_1 \to x_2 \to \cdots \to x_n \to x_1$ and fixes all the other generators. Let w be a word on $\{x_1, \dots, x_r\}$. For $n \geq r$ let $G_w(n)$ be the group generated by $\{x_1, \dots, x_n\}$ and defined by the relators $w, w\theta_n, \dots, w\theta_n^{n-1}$. For example, when $w = x_1x_2x_3^{-1}$ these are the Fibonacci groups $F(2, n)$. It is easy to see, and well-known, that $F(2, n)$ can be generated by two elements. On the other hand for $w = x_1^2$ the group $G_w(n)$ cannot be generated by fewer than n elements because it has an elementary abelian group with order 2^n as a quotient. Let $d(G)$ be the generator number of a group G—that is, $d(G) = \min\{|X| \mid \langle X \rangle = G\}$ where $|X|$ is the cardinality of X.

The two distinct families of groups considered in [JKO99] are given by the words $u = x_2^{-1}x_3x_2^{-1}x_3x_1x_2^{-1}x_1$ and $v = x_2^{-1}x_3x_2^{-1}x_3x_2^{-2}x_1x_2^{-1}x_1$. The generator number functions $d(G_u(\ ))$ and $d(G_v(\ ))$ lie between those in the examples above. Though the families have different Alexander polynomials, the two cases are sufficiently similar in this context that details are presented only for one of them, for the word u. The corresponding result for the word v is given in Section 4.

Put $G(n) = G_u(n)$ (denoted $G_1(n)$ in [HHN01]). It is implicit in [JKO99] that $d(G(3)) = 2$. The main result of this paper is:

THEOREM 1. *The generator number of the group* $G(n)$ *is at most* $(n+1)/2$ *and at least* $C\lfloor (n-6)/4 \rfloor$ *where* $C = \log 24/\log 60 > 0.776$.

As will be seen verification is not difficult. Finding a result was more difficult. This involved, as in [JKO99] and [HHN01], significant use of computers (often a laptop) to help get a picture. Computations were done using the systems GAP and Magma and the packages **quotpic** and ACE (see references). Many of these computations are not mentioned further in this paper. The upper bound is proved in Section 2 and the lower bound in Section 3.

The lower bound is obtained by considering quotients of $G(n)$ which are direct products of alternating groups of degree 5. The package **quotpic** is well-suited to gathering evidence. It gives easily that $G(5)$ has A_5 as a quotient but does not have the direct product of 2 copies of A_5 as a quotient; and that $G(8)$ has a quotient which is the direct product of 32 copies of A_5. The well-known result of Philip Hall [Hal36] that the direct product of 20 copies of A_5 cannot be generated by 2 elements gives that $G(8)$ cannot be generated by 2 elements. Theorem 1 says, in particular, that $G(n)$ cannot be generated by 2 elements for $n \geq 18$. The gap can be filled using **quotpic** to give: $G(n)$ cannot be generated by 2 elements for $n \geq 8$. This is the best that can be achieved with this approach since the largest quotient of $G(7)$ which is the direct product of copies of A_5 is the direct product of 7 copies. More detailed study of quotients which are direct products of A_5 can be used to increase the lower bound in Theorem 1. However the increase is not enough to reach the current upper bound.

Theorem 1 shows that $G(4)$ can be generated by 2 elements; since $G(4)$ has the direct product of two cyclic groups with order 3 as a quotient, $d(G(4)) = 2$. The precise value of $d(G(n))$ is not known for $n \geq 5$.

It would be interesting to obtain more information about the generator number function $d(G_w(\ ))$ in general. For example, what types of functions arise?

2. An upper bound

In this kind of context an upper bound is usually established by exhibiting a generating set of the appropriate size.

There are various ways one might search for generating sets. In this case using a computer to apply Tietze transformations to the given presentation gave useful evidence. As is well-known two finite presentations of the same group can be obtained from one another by a series of Tietze transformations. However this may involve going via presentations with more generators than either of the given presentations. Programs which start from a given presentation and search for a 'simpler' presentation employ various strategies to keep the search under control. For example, they limit the extent to which the generating set is allowed to grow. Most commonly they allow no growth. Applying such programs to the given presentations for $G(n)$ (for small n) yielded nothing.

Observe that

$$u\theta_n^{n-2} \cdots u\theta_n\, u\, u\theta_n^{n-1} = (x_n^{-1}x_1)^2 x_n \cdots x_1 (x_1^{-1}x_n)^2.$$

Adding the relator $x_n \cdots x_1$ to the given presentation allowed the process to get started. The simplification program in **quotpic** (used for a number of small values of n) then suggested that it might be possible to find presentations with $\lfloor (n+2)/2 \rfloor$ generators. Moreover, going behind the graphical interface, it suggested a generating set like that used below (Holt, private communication). Behind the scenes information also gave a hint for a proof. Havas (private communication) has shown that $\{x_1^{-1}x_2, x_4\}$ generates $G(4)$. This suggested $\lfloor (n+1)/2 \rfloor$ as an upper bound for the generator number. This was fairly easily confirmed for $n \leq 1000$ using coset enumeration. A proof turns out to be easy.

Proof of the upper bound in Theorem 1.

LEMMA 2. *The group* $\langle a, \dots, d \mid u(a,b,c), u(b,c,d) \rangle$ *can be generated by* $\{a, c, d\}$.

PROOF. Observe that $u(b,c,d).u(a,b,c) = 1$ implies $b = ac^{-1}dc^{-1}dca$. □

THEOREM 3. *The group* $\langle x_1, \dots, x_n \mid u, u\theta_n, \dots, \theta_n^{n-1} \rangle$ *can be generated by the set* $\{x_1, x_2, x_4, x_6, \dots\}$ *which contains* $\lfloor (n+2)/2 \rfloor$ *elements.*

PROOF. Applying Lemma 2 to the subgroup generated by $\{x_2, x_3, x_4, x_5\}$ shows x_3 lies in the subgroup generated by $\{x_2, x_4, x_5\}$. Similarly, x_5 lies in the subgroup generated by $\{x_4, x_6, x_7\}$ and so on. □

Finally, for n even, one can find a generating set with size $n/2$ using the following observation applied to the subgroup generated by $\{x_n, x_1, x_2\}$.

LEMMA 4. *The group* $\langle a, b, c \mid u(a, b, c)\rangle$ *can be generated by* $\{b^{-1}a, b^{-1}c\}$.

PROOF. Observe $u(a, b, c) = (b^{-1}c)^2 b(b^{-1}a)^2$. □

3. A lower bound

Finding reasonable lower bounds is more difficult. The natural method is to find accessible, usually finite, quotients of $G(n)$ which have no 'small' generating sets. The lower bounds in Theorem 1 are obtained by considering quotients which are direct products of alternating groups of degree 5. Hall [Hal36] has determined the generator numbers for these direct products. Hall's proof goes via counting epimorphisms from free groups to A_5.

Proof of the lower bound in Theorem 1.

The key to the proof is to obtain a lower bound for the number of homomorphisms from $G(n)$ to the alternating group A_5.

As an illustration, it is reasonably easy to compute (for example, using **quotpic**) that there are 3840 epimorphisms from $G(8)$ to A_5. Since there are clearly at most 3600 homomorphisms from a 2-generator group to A_5, it follows that $G(8)$ cannot be generated by 2 elements.

It will be shown that for $n \geq 6$ there are at least 24^k homomorphisms from $G(n)$ to A_5 where $k = \lfloor (n-6)/4 \rfloor$. Clearly there are at most 60^d homomorphisms from a d-generator group to A_5. Hence $G(n)$ cannot be generated by fewer than $(\log 24)\lfloor (n-6)/4 \rfloor / \log 60$ elements.

Determining the lower bound for the number of homomorphisms uses some simple observations about A_5. The next two lemmas can be quickly verified, with confidence, using one of the systems GAP or Magma. Some code which does this is given in an appendix.

LEMMA 5. *Let a be a non-trivial element of order* 3 *in* A_5. *There are exactly* 31 *pairs* (c, d) *of (non-trivial) elements of* A_5 *such that* $c^3 = d^3 = (c^{-1}d)^2 = 1$ *and* $(a^{-1}c)^2 = (d^{-1}a)^2 = 1$.

PROOF. Without loss of generality, take $a = (1, 2, 3)$. □

LEMMA 6. *Let a, b be non-trivial elements in A_5 such that $a^3 = b^3 = 1$ and $(a^{-1}b)^2 = 1$. There are at least 24 pairs (c, d) of (non-trivial) elements in A_5 such that $c^3 = d^3 = (c^{-1}d)^2 = 1$ and $(a^{-1}c)^2 = (d^{-1}b)^2 = 1$.*

PROOF. For $a^{-1}b = 1$, Lemma 5 gives the result. Otherwise, without loss of generality $a = (1, 2, 3), b = (1, 4, 2)$. □

The next lemma shows how to construct homomorphisms from $G(n+4)$ to A_5 out of certain homomorphisms from $G(n)$ to A_5. To avoid notational confusion the given generating set for $G(n+4)$ will be written $\{y_1, \ldots, y_{n+4}\}$. A homomorphism ϕ from $G(n)$ to A_5 such that

$$x_1^3\phi = 1,\ x_1\phi \neq 1,\ x_n^3\phi = 1,\ x_n\phi \neq 1,\ (x_n^{-1}x_1)^2\phi = 1,\ x_1x_2\phi = 1,\ x_{n-1}x_n\phi = 1$$

will be called *special.*

LEMMA 7. *Let ϕ be a special homomorphism from $G(n)$ to A_5. There are at least 24 distinct special homomorphisms ψ from $G(n+4)$ to A_5 such that $y_i\psi = x_i\phi$ for $i \in \{1, \ldots, n\}$.*

PROOF. Let $a = x_n\phi$ and $b = x_1\phi$. Because ϕ is special, $a^3 = b^3 = (a^{-1}b)^2 = 1$. By Lemma 6 there are at least 24 pairs (c, d) of elements of A_5 such that

$$c^3 = d^3 = (a^{-1}c)^2 = (c^{-1}d)^2 = (d^{-1}b)^2 = 1.$$

For each pair (c, d) define a mapping ψ by $y_i\psi = x_i\phi$ for $i \in \{1, \ldots, n\}$ and

$$y_{n+1}\psi = c,\ y_{n+2}\psi = c^{-1},\ y_{n+3}\psi = d^{-1},\ y_{n+4}\psi = d.$$

Then $u(y_i\psi, y_{i+1}\psi, y_{i+2}\psi) = 1$ for $i \in \{1, \ldots, n-2\}$ because ϕ is a homomorphism and for the rest from relations on $x_1, x_2, x_{n-1}, x_n, a, b, c, d$. The relations between d and b show ψ is special. □

The last lemma shows that the number of special homomorphisms from $G(n+4)$ to A_5 is at least 24 times the number of special homomorphisms from $G(n)$ to A_5. For $n \in \{6, 7, 8, 9\}$ there is at least one special homomorphism from $G(n)$ to A_5 mapping $\{x_1, \ldots, x_n\}$ to:
for $n = 6$
$\{(1, 4, 2), (1, 2, 4), (1, 5, 2), (1, 2, 5), (1, 3, 2), (1, 2, 3)\}$;
for $n = 7$
$\{(1, 2, 3), (1, 3, 2), (1, 3, 4, 2, 5), (1, 2, 3, 5, 4), (1, 3, 4, 2, 5), (1, 3, 2), (1, 2, 3)\}$;
for $n = 8$
$\{(1, 4, 2), (1, 2, 4), (1, 3, 2), (1, 2, 3), (1, 5, 2), (1, 2, 5), (1, 3, 2), (1, 2, 3)\}$;

for $n = 9$
$\{(1,2,3), (1,3,2), (1,3,4,2,5), (1,2,3,5,4), (1,3,4,2,5), (1,4,2), (1,2,4), (1,3,2), (1,2,3)\}$.

Hence the number of special homomorphisms from $G(n)$ to A_5 is at least 24^k where $k = \lfloor (n-6)/4 \rfloor$.

As remarked in the introduction greater lower bounds can be obtained by these methods. However the increase is not enough to reach the upper bound in Theorem 1.

4. The other family

Denote by $H(n)$ the groups defined by the word $v = x_2^{-1}x_3x_2^{-1}x_3x_2^{-2}x_1x_2^{-1}x_1$. Using similar methods to those above one can prove the following.

THEOREM 8. *The generator number of the group $H(n)$ is at most $2n/3$ and at least $C\lfloor (n-6)/4 \rfloor$ where $C = \log 16/\log 60 > 0.677$.*

Using **quotpic** then gives $d(H(n)) \geq 3$ for all $n \geq 8$.

Appendix. Some code

Here is some simple Magma code for verifying Lemmas 5 and 6.

```
A := AlternatingGroup(5);
a := A!(1,2,3);
for b in {A!(1,4,2),A!(1,2,3)} do
  print "For input \n " ,a," ",b, "with a^-1*b = ", a^-1*b;
  scount := 0; // solution count
  for c in A do
   for d in A do
    if
     c^3 eq Id(A) and
     d^3 eq Id(A) and
      (a^-1*c)^2 eq Id(A) and
      (c^-1*d)^2 eq Id(A) and
      (d^-1*b)^2 eq Id(A)
    then
     scount := scount+1;
     print "solution number ", scount, "\n ",c," ",d;
    end if;
   end for;
  end for;
end for;
```

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