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DUAL INTEGRAL EQUATIONS WITH A TRIGONOMETRIC KERNEL*

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In this paper, we solve the following dual integral equations

$$\int_0^\infty \left[1 - \frac{2\xi\delta(1+\xi\delta)+1-e^{-2\xi\delta}}{2\xi\delta+\sinh 2\xi\delta}\right]\xi A(\xi)\cos \xi x\, d\xi = f(x), \quad 0 < x < a, \tag{1}$$

$$\int_0^\infty A(\xi)\cos \xi x\, d\xi = 0, \quad x > a, \tag{2}$$

where δ is a real positive constant and $f(x)$ is a continuous and integrable function of x in $[0, a]$. The dual integral equations (1) and (2) arise in a crack problem of elasticity.

Let us rewrite the above integral equations in the form:

$$\int_0^\infty \xi\psi(\xi)(1-\xi^2\delta^2 \operatorname{cosech}^2 \xi\delta)\cos \xi x\, d\xi = f(x), \quad 0 < x < a, \tag{3}$$

$$\int_0^\infty \psi(\xi)(\coth \xi\delta + \xi\delta \operatorname{coesch}^2 \xi\delta)\cos \xi x\, d\xi = 0, \quad x > a, \tag{4}$$

where

$$\psi(\xi) = (\coth \xi\delta + \xi\delta \operatorname{cosech}^2 \xi\delta)^{-1} A(\xi). \tag{5}$$

Equations (3) and (4) may be further put in the form

$$\int_0^\infty \psi(\xi)\frac{\partial}{\partial\delta}\left(-\frac{1}{\delta}+\xi\coth \xi\delta\right)\cos \xi x\, d\xi = \frac{f(x)}{\delta^2}, \quad 0 < x < a, \tag{6}$$

$$\int_0^\infty \psi(\xi)\frac{\partial}{\partial\delta}\left(\frac{1}{\delta}\coth \xi\delta\right)\cos \xi x\, d\xi = 0, \quad x > a. \tag{7}$$

Integrating equations (6) and (7) with respect to δ, we obtain

$$\int_0^\infty \xi\psi(\xi)\left(-\frac{1}{\delta}+\xi\coth \xi\delta\right)\cos \xi x\, d\xi = -\frac{f(x)}{\delta}+g(x), \quad 0 < x < a, \tag{8}$$

$$\int_0^\infty \psi(\xi)\coth \xi\delta \cos \xi x\, d\xi = 0, \quad x < a, \tag{9}$$

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where the limits of integration have been taken from δ to ∞ for integrating equation (7) and $g(x)$ is an arbitrary function of x.

Integrating equation (8) with respect to x between 0 to x, we obtain

$$\int_0^\infty \psi(\xi)\left(-\frac{1}{\delta}+\xi \coth \xi\delta\right) \sin \xi x \, d\xi = -\frac{F(x)}{\delta}+G(x), \quad 0<x<a, \tag{10}$$

where

$$F(x)=\int_0^x f(x)\,dx, \qquad G(x)=\int_0^x g(x)\,dx. \tag{11}$$

If we assume the representation

$$\psi(\xi)=\frac{2}{\pi}\xi^{-1}\tanh \xi\delta \int_0^a \phi(t)\sin \xi t\,dt, \tag{12}$$

the integral equation (9) is identically satisfied. Now rewriting equation (10) in the form

$$-\frac{1}{\delta}\int_0^\infty \psi(\xi)\sin \xi x\,d\xi-\frac{\partial}{\partial x}\int_0^\infty \psi(\xi)\coth \xi\delta \cos \xi x\,d\xi$$
$$= -\frac{1}{\delta}F(x)+G(x), \quad 0<x<a, \tag{13}$$

and then substituting for $\psi(\xi)$ from (12), we find that ϕ is the solution of the integral equation

$$-\frac{1}{\pi\delta}\int_0^a \phi(t)\log\left|\frac{\sinh cx+\sinh ct}{\sinh cx-\sinh ct}\right| dt-\phi(x)$$
$$=-\frac{1}{\delta}F(x)+G(x), \quad 0<x<a, \tag{14}$$

where $c=\pi/2\delta$ and we have used the following integral

$$\int_0^\infty \xi^{-1}\tanh \xi\delta \sin \xi x \sin \xi t\,d\xi=\frac{1}{2}\log\left|\frac{\sinh cx+\sinh ct}{\sinh cx-\sinh ct}\right|, \quad \delta>0, \tag{15}$$

for obtaining integral equation (14). Letting $\delta\to\infty$ in equation (14), we find that

$$G(x)=-\phi(x) \tag{16}$$

and equation (14) simplifies to

$$\int_0^a \phi(t)\log\left|\frac{\sinh cx+\sinh ct}{\sinh cx-\sinh ct}\right| dt=\pi F(x), \quad 0<x<a. \tag{17}$$

With the help of (1) or (2), the solution of the above integral equation is obtained in the following form:

$$\phi(t)=-\frac{2c}{\pi}\frac{\cosh ct}{(\sinh^2 ca-\sinh^2 ct)^{1/2}}$$
$$\times\left[\sinh ct\int_0^a \frac{(\sinh^2 ca-\sinh^2 cx)^{1/2}}{\sinh^2 cx-\sinh^2 ct}F'(x)\,dx-\frac{F(0)\sinh ca}{\sinh ct}\right], \quad 0<t<a, \tag{18}$$

where prime denotes the derivative with respect to the argument. If $f(x)$ is a constant, say,

$$f(x) = p_0, \tag{19}$$

we find from (18) and (19) that

$$\phi(t) = -\frac{2cp_0}{\pi}\frac{\sinh ct\cosh ct}{(\sinh^2 ca - \sinh^2 ct)^{1/2}}\int_0^a \frac{(\sinh^2 ca - \sinh^2 cx)^{1/2}}{\sin^2 cx - \sinh^2 ct}\,dx, \quad 0<t<a. \tag{20}$$

If we let $\delta \to \infty$ (or $c \to 0$) in equation (20), we find that

$$\phi(t) = p_0 t(a^2 - t^2)^{-1/2} \tag{21}$$

and hence from (5), (12) and (21), we have

$$A(\xi) = \psi(\xi) = ap_0\xi^{-1}J_1(a\xi), \tag{22}$$

which is the solution (see Sneddon (3), pp. 103–104) of the dual integral equations

$$\int_0^\infty \xi A(\xi)\cos \xi x\, d\xi = p_0, \quad 0<x<a, \tag{23}$$

$$\int_0^\infty A(\xi)\cos \xi x\, d\xi = 0, \quad x>a. \tag{24}$$

The integral equations (1) and (2) reduce to (23) and (24) for $f(x) = p_0$ and $\delta \to \infty$.

By evaluating the integral in equation (20), we find that $\phi(t)$ may be put in the following form.

$$\phi(t) = \frac{p_0}{\pi}\frac{\sinh^2 ct}{\cosh ca(\sinh^2 ca - \sinh^2 ct)^{1/2}}$$
$$\times\left[F(\pi/2 \tanh ca) - \Pi\left(\pi/2, \frac{\sinh^2 ca}{\sinh^2 ca - \sinh^2 ct}, \tan ca\right)\right], \quad 0<t<a, \tag{25}$$

where F and Π, respectively, denote elliptic integrals of the first and third kind. Now $A(\xi)$ may be obtained from equations (5), (12) and (25).

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