Compositio Mathematica **119:** 13–40, 1999.
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Tits' Constructions of Jordan Algebras and F_4 Bundles on the Plane

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(Received: 12 August 1997; accepted in final form: 15 June 1998)

Abstract. In this paper, constructions of Jordan algebras over commutative rings are given which place, within a general set-up, the classical Tits constructions of exceptional central simple Jordan algebras over fields. These are used to exhibit nontrivial Jordan algebra bundles over the affine plane with a given exceptional Jordan division algebra over k as the fibre. The associated principal F_4 bundles are shown to admit no reduction of the structure group to any proper connected reductive subgroup.

Mathematics Subject Classifications (1991): Primary: 17C40; Secondary: 14L10.

Key words: exceptional Jordan algebras, F_4 bundles.

Introduction

Let k be a field of characteristic not 2 or 3. The classical constructions of Tits give all exceptional central simple Jordan algebras over k. In this paper, we give constructions of Jordan algebras over any commutative domain (in which 2 and 3 are invertible), which place, in a general set up, Tits constructions of exceptional Jordan algebras. We use these to produce nontrivial Jordan algebra bundles on the affine plane $\mathbb{A}_k^2$, whose fibre is a given exceptional Jordan division algebra J over k. If $G =$Aut J, we further show that the associated principal G-bundles on $\mathbb{A}_k^2$ admit no reduction of the structure group to any proper connected reductive subgroup of G. This theorem, along with the results of ([PST]), completes the case $G = F_4$ which is left out in a theorem of Raghunathan on the existence of principal G-bundles on $\mathbb{A}_k^2$ for a connected reductive anisotropic group G over k, whose structure group has no reduction to any proper connected reductive subgroup ([R], 4.9).

Let R be a commutative domain in which 2 and 3 are invertible. Let A be an Azumaya algebra over R of degree 3. Let P be a projective module of rank 1 over A. We assume that the reduced norm $\mathcal{N}(P)$ of P (cf. [KOS]) is free. Let $\mu\colon \mathcal{N}(P) \simeq R$ be an isomorphism of R-modules. We associate to the pair (P, μ), in a functorial way, a Jordan algebra $J(P, \mu)$ (Section 1), whose underlying R-module is $A \oplus P \oplus P^{(*)}$, where $P^{(*)} = \mathrm{Hom}_A(P, A)$. We call a Jordan algebra a *Tits*

https://doi.org/10.1023/A:1001507928187 Published online by Cambridge University Press

first construction algebra if it is isomorphic to $J(P,\mu)$ for some pair (P,μ) over a degree 3 Azumaya algebra A over R. We show that such an algebra is *exceptional* in the sense of ([PST]), containing a Jordan subalgebra isomorphic to A_+. We prove conversely that if J is an exceptional Jordan algebra and A is an Azumaya algebra of degree 3 over R such that the special Jordan algebra A_+ is contained in J as a Jordan subalgebra, then J is a Tits first construction Jordan algebra.

Let B be an Azumaya algebra of degree 3 over an étale quadratic extension S of R and with an involution σ of second kind. Let (P,h) be a projective module of rank 1 over B with a Hermitian form h over (B,σ). Suppose that the discriminant $\mathrm{disc}(h)$ of (P,h) is trivial. Let $\delta\colon \mathrm{disc}(h)\simeq(S,\langle 1\rangle)$ be a trivialization. Then to the triple (P,h,δ), we associate, functorially, a Jordan algebra $J(P,h,\delta)$ (Section 2) with the underlying R-module $B^+\oplus P$, where B^+ is the Jordan algebra consisting of symmetric elements in B for σ. We call a Jordan algebra a *Tits second construction algebra* if it is isomorphic to $J(P,h,\delta)$ for some triple (P,h,δ) over an Azumaya algebra (B,σ) of degree 3 over an étale quadratic extension S/R with an involution of second kind. We show that these Jordan algebras are exceptional and contain a Jordan subalgebra isomorphic to B^+. We prove that if J is an exceptional Jordan algebra and (B,σ) as above such that B^+ is contained in J as a Jordan subalgebra, then J is a Tits second construction Jordan algebra. These results for Jordan algebras over fields are due to McCrimmon, which in fact we use in our proofs.

The idea behind the construction of nontrivial Jordan algebra bundles over $\mathbb{A}_k^2$ is the following. Let $J=J(D,\mu)$ be the Jordan algebra over k associated to the pair (D,μ) arising from Tits first construction, where D is a degree 3 central division algebra over k and $\mu\in k^*$ is not a norm from D. We choose a nonfree projective $D[X,Y]$-module P of rank 1 together with a trivialization $\widetilde{\mu}$ of the reduced norm such that $(P,\widetilde{\mu})$ specializes to (D,μ) at a rational point. We then construct $J(P,\widetilde{\mu})$ as indicated before. Using these algebras as prototypes on open sets and through a patching argument, one gets an infinite family of mutually nonisomorphic Jordan algebras over $\mathbb{A}_k^2$ with J as the fibre.

Let D be a central division algebra of degree 3 over a quadratic extension K of k, together with an involution σ of second kind. Let $J=J(D,\sigma,u,\mu)$ be a Tits second construction algebra with $u\in D^*$, $\mu\in K^*$, $Nrd(u)=\mu\sigma(\mu)$ and μ not a norm from D^*. We choose nonfree projective $D[X,Y]$-modules P of rank 1 with Hermitian forms h and trivializations $\widetilde{\mu}\colon \mathrm{disc}(h)\simeq(S,\langle 1\rangle)$ of their discriminants. The existence of such triples $(P,h,\widetilde{\mu})$ is guaranteed by a theorem of Raghunathan ([R], 4.9). These give rise to Jordan algebras $J(P,h,\widetilde{\mu})$ over $k[X,Y]$ as explained earlier. Once again, using these as prototypes on open sets and by patching arguments, one gets an infinite family of Jordan algebra bundles over $\mathbb{A}_k^2$ with J as the fibre.

In Section 3, we prove some results (which are of independent interest) on the rigidity of bundles on $\mathbb{P}_k^2$ with a D-structure obtained by extending nonfree projective $D[X,Y]$-modules, for a central division algebra D over k of prime degree.

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Using these results, we show that the principal F_4-bundles associated to the Jordan algebras, constructed as above, do not admit reduction of the structure group to any proper connected reductive subgroup of G.

As is well known, every exceptional central simple Jordan algebra over a field arises from the first or the second construction of Tits and, hence, the associated F_4-bundle admits reduction of the structure group to $\mathrm{SL}_1(D)$ or $\mathrm{SU}(D, \sigma)$ for some D and σ. However, the above constructions of Jordan algebra bundles over $\mathbb{A}_k^2$ yield F_4-bundles with no reduction of the structure group to any proper connected reductive subgroup. In particular, they *do not arise* from a generalised Tits construction.

It has been pointed out to us by Prof. H. P. Petersson that the (unpublished) thesis of G. Achhammer also contains a generalized construction of Jordan algebras.

1. A General Tits First Construction of Jordan Algebras

Let R be a commutative domain in which 2 and 3 are invertible. Let A be an Azumaya algebra of degree 3 over R, i.e., $A \otimes_R \widetilde{R} \simeq M_3(\widetilde{R})$ for some faithfully flat extension $\widetilde{R}$ of R. Let $\mu \in R^*$ and

$$J(A, \mu) = A_0 \oplus A_1 \oplus A_2,$$

where $A_i = A$, $i = 0, 1, 2$. Following Tits, we define a multiplication on $J(A, \mu)$ by

$$\begin{aligned}
&(a_0, a_1, a_2)\,(a_0', a_1', a_2')\\
&\quad = (a_0.a_0' + \overline{a_1 a_2'} + \overline{a_1' a_2},\ \overline{a_0} a_1' + \overline{a_0'} a_1 + \mu^{-1} a_2 \times a_2',\\
&\qquad a_2' \overline{a_0} + a_2 \overline{a_0'} + \mu a_1 \times a_1'),
\end{aligned}$$

where $a_i, a_i' \in A_i$, $0 \leqslant i \leqslant 2$. Here, for $x, y \in A$, if $\mathrm{tr}\colon A \to R$ is the reduced trace map,

$$x.y = \tfrac{1}{2}(xy + yx), \qquad \overline{x} = \tfrac{1}{2}(\mathrm{tr}(x) - x),$$

$$x \times y = x.y - \tfrac{1}{2}\,\mathrm{tr}(x)y - \tfrac{1}{2}\,\mathrm{tr}(y)x + \tfrac{1}{2}(\mathrm{tr}(x)\,\mathrm{tr}(y) - \mathrm{tr}(x.y)).$$

With this multiplication, $J(A, \mu)$ is a Jordan algebra over R of rank 27 and $A_0 = A_+$ is a Jordan subalgebra, where for any associative algebra C, C_+ denotes the corresponding special Jordan algebra. It can be checked that $J(A, \mu)$ is an *exceptional* Jordan algebra i.e., locally for the étale topology, $J(A, \mu)$ is isomorphic to the split 27-dimensional exceptional Jordan algebra (cf. [PST]). We would like to give a construction of exceptional Jordan algebras, which we shall call Tits' first construction, which includes $J(A, \mu)$ as a subclass.

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Let, for the moment, A denote any Azumaya algebra over R. We recall the notion of a reduced norm functor $\mathcal{N}\colon {}_A\mathrm{Mod} \to_R \mathrm{Mod}$, defined in ([KOS]). This functor associates to a projective A-module P, a projective R-module $\mathcal{N}(P)$ with the following properties:

(1) $\mathcal{N}(A) = R$.
(2) The map $\mathcal{N}_P\colon P = \mathrm{Hom}_A(A, P) \to \mathrm{Hom}_R(\mathcal{N}(A), \mathcal{N}(P)) = \mathcal{N}(P)$, induced by the functoriality, has the property $\mathcal{N}_P\ (ax) = Nrd_A(a)\mathcal{N}_P(x)$ for $a \in A, x \in P$, where Nrd_A is the reduced norm on A.
(3) If P has rank 1, then $\mathcal{N}(P)$ is invertible. Further, $\mathcal{N}_A\colon A \to R$ is the reduced norm map of A.

There is also a functor $\mathcal{N}\colon \mathrm{Mod}_A \to_R \mathrm{Mod}$ which has similar properties. We have, for a projective left A-module P, $P^{(*)} = \mathrm{Hom}_A(P, A)$ is a projective right A-module, $\mathcal{N}(P^{(*)}) = \mathcal{N}(P)^* = \mathrm{Hom}_R(\mathcal{N}(P), R)$ and the map $\mathcal{N}_{P^{(*)}}\colon P^{(*)} \to \mathcal{N}(P)^*$ is the composite

$$P^{(*)} = \mathrm{Hom}_A(P, A) \to \mathrm{Hom}_R(\mathcal{N}(P), \mathcal{N}(A)) = \mathrm{Hom}_R(\mathcal{N}(P), R) = \mathcal{N}(P)^*,$$

induced by the functoriality of $\mathcal{N}$ on ${}_A\mathrm{Mod}$. We abbreviate $\mathcal{N}_P = \mathcal{N}$.

Let A, B be Azumaya algebras over R and P, Q projective left A-modules over A and B respectively. Let $f\colon A \to B$ be an isomorphism of R-algebras and $\tilde{f}\colon P \to Q$ an f-semilinear isomorphism. Then there is an R-linear isomorphism $\mathcal{N}(\tilde{f})\colon \mathcal{N}(P) \to \mathcal{N}(Q)$ such that $\mathcal{N}(\tilde{f})\mathcal{N}_P = \mathcal{N}_Q\tilde{f}$. The map $\mathcal{N}(\tilde{f})$ is constructed by descent.

PROPOSITION 1.1. *Let A be an Azumaya algebra over R and P a projective left A-module of rank 1. Suppose that $N\colon P \to R$ is a map such that*

(1) $N(ax) = \mathcal{N}_A(a)N(x)$, $a \in A, x \in P$.
(2) *The values of N generate the unit ideal in R.*

Then there exists a unique isomorphism $\eta\colon \mathcal{N}(P) \simeq R$ such that $N = \eta\mathcal{N}$. Further, if $v \in P$ is such that $N(v)$ is a unit, then P is free and v is a basis element for P as an A-module.

Proof. Since the values of $\mathcal{N}$ generate $\mathcal{N}(P)$, if η exists, it is unique. Hence it is enough to show that η exists if R is local. In this case P is free. Let e be an A-basis element for P. Since $N(ae) = Nrd_A(a)N(e)$, by (2), $N(e)$ is a unit in R. The isomorphism $\phi_e\colon A \to P, a \mapsto ae$ induces an isomorphism $\mathcal{N}(\phi_e)\colon \mathcal{N}(A) = R \to \mathcal{N}(P)$, which gives a generator $\gamma_e = \mathcal{N}(\phi_e)(1)$ of $\mathcal{N}(P)$ as an R-module and by definition, $\mathcal{N}(e) = \gamma_e$. We define $\eta\colon \mathcal{N}(P) \to R$ by setting $\eta(\gamma_e) = N(e)$. Then

$$\eta\mathcal{N}(ae) = \eta(Nrd_A(a)\gamma_e) = Nrd_A(a)N(e) = N(ae).$$

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Thus $\eta\mathcal{N} = N$. To prove the last assertion of the proposition, it suffices to show that v generates P locally on R. We assume therefore that R is local and e is a basis element for P. We have seen above that $N(e)$ is a unit in R. Let $v = ae$ with $a \in A$. Then $N(v) = Nrd_A(a)N(e)$ and hence $Nrd_A(a)$ is a unit in R so that a is a unit in A. This proves that v is an A-basis element for P. $\square$

Let now A be an Azumaya algebra of degree 3 over R. Let P be a projective A-module of rank 1. The map $\mathcal{N}\colon P \to \mathcal{N}(P)$ yields an R-trilinear map $\mathcal{N}\colon P \times P \times P \to \mathcal{N}(P)$ with the properties:

(1) $\mathcal{N}(x, x, x) = \mathcal{N}(x)$ for $x \in P$.
(2) If P is free with e as a basis element, $\mathcal{N}(ae, be, ce) = \mathcal{N}_A(a, b, c)\mathcal{N}(e)$, where $\mathcal{N}_A\colon A \times A \times A \to R$ is the linearization of the reduced norm of A, namely $\mathcal{N}_A(a, b, c) = \mathrm{tr}((a \times b)c)$, $a, b, c \in A$, tr being the reduced trace of A.

We assume that $\mathcal{N}(P)$ is free. Let $\mu\colon \mathcal{N}(P) \simeq R$ be an isomorphism of R-modules. To (P, μ) we associate a Jordan algebra $J(P, \mu)$ as follows: The R-trilinear map $\mathcal{N}$ induces an R-bilinear map ϕ given by the composite

$$P \times P \to \mathrm{Hom}_R(P, \mathcal{N}(P)) \xrightarrow{\mu} \mathrm{Hom}_R(P, R) \overset{\mathrm{tr}^{-1}}{\simeq} \mathrm{Hom}_A(P, A) = P^{(*)}$$

where, tr: $P^{(*)} = \mathrm{Hom}_A(P, A) \to P^* = \mathrm{Hom}_R(P, R)$ is the isomorphism given by $\mathrm{tr}(f)(x) = \mathrm{tr}(f(x))$. Similarly, we have $\phi_*\colon P^{(*)} \times P^{(*)} \to P$ defined as the composite

$$\begin{aligned} P^{(*)} \times P^{(*)} \quad &\to \quad \mathrm{Hom}_R(P^{(*)}, \mathcal{N}(P^{(*)})) \\ &\xrightarrow{(\mu^{t^{-1}})} \mathrm{Hom}_R(P^{(*)}, R) \overset{\mathrm{tr}^{-1}}{\simeq} \mathrm{Hom}_A(P^{(*)}, A) \simeq P. \end{aligned}$$

Let $J(P, \mu) = A \oplus P \oplus P^{(*)}$. We define a multiplication on $J(P, \mu)$ by

$$\begin{aligned} &(a, x, f)(a', x', f') \\ &\quad = (a.a' + \overline{f'(x)} + \overline{f(x')}, \overline{a}x' + \overline{a'}x + \phi_*(f, f'), f'\overline{a} + f\overline{a'} + \phi(x, x')), \end{aligned}$$

for all $a, a' \in A$, $x, x' \in P$, $f, f' \in P^{(*)}$.

THEOREM 1.2. *The multiplication above makes $J(P, \mu)$ an exceptional Jordan algebra.*

Proof. It suffices to check this after a faithfully flat base change. We therefore assume that P is free. Let $e \in P$ be a basis element of P and $e^* \in P^{(*)}$ its dual, i.e., $e^*(e) = 1$. Let $\phi'\colon P \times P \to P^{(*)}$ be defined by $\phi' = \mathrm{tr}\,\phi$, tr and ϕ being defined as above. We have,

$$\begin{aligned} \phi'(ae, be)(ce) &= \mu(\mathcal{N}_A(a, b, c)\mathcal{N}(e)) \\ &= \mathcal{N}_A(a, b, c)\mu(N(e)) = \mathrm{tr}((a \times b)c)\mu_e, \end{aligned}$$

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where $\mu_e = \mu(\mathcal{N}(e))$. Therefore, $\phi(ae, be) = e^*\mu_e(a \times b)$. Since $\mu^{t^{-1}}(\mathcal{N}(e^*)) = \mu_e^{-1}$, by a similar calculation one has $\phi_*(e^*a, e^*b) = \mu_e^{-1}(a \times b)e$. It is easily checked, using the above identities, that the map $J(A, \mu_e) \to J(P, \mu)$ given by $(a_0, a_1, a_2) \mapsto (a_0, a_1e, e^*a_2)$, is an isomorphism of Jordan algebras. □

PROPOSITION 1.3 (Functoriality). *Let A and B be Azumaya algebras of degree 3 over R and P, Q be projective modules of rank 1 over A and B respectively, with isomorphisms $\mu: \mathcal{N}(P) \to R$ and $\nu: \mathcal{N}(Q) \to R$ of R-modules. Let $g: (A, P, \mu) \to (B, Q, \nu)$ be an isomorphism i.e., $g: A \simeq B$ an isomorphism of R-algebras, $\widetilde{g}: P \simeq Q$ a g-semilinear isomorphism of R-modules such that the diagram*

$$\begin{tikzcd} \mathcal{N}(P) \arrow[rr, "\mathcal{N}(\widetilde{g})"] \arrow[dr, "\mu"'] & & \mathcal{N}(Q) \arrow[dl, "\nu"] \\ & R & \end{tikzcd} \qquad (*)$$

commutes. Then the map $J(g): J(P, \mu) \to J(Q, \nu)$ given by

$$J(g)((a, x, f)) = (g(a), \widetilde{g}(x), \widetilde{g}^{*^{-1}}(f))$$

is an isomorphism of Jordan algebras.

Proof. It is enough to show that $J(g)$ is an isomorphism after a faithfully flat base change of R. We may therefore assume that $P = Ae$ is free. Then $Q = Be'$ with $e' = \widetilde{g}(e)$. Let $\mu_e = \mu(\mathcal{N}(e))$ and $\nu_{e'} = \nu(\mathcal{N}(e'))$. Then, by $(*)$, we have $\mu_e = \nu_{e'}$. Therefore the isomorphism $g: A \to B$ induces a map $J(g): J(A, \mu_e) \to J(B, \nu_{e'})$ given by $J(g)(a_0, a_1, a_2) = (g(a_0), g(a_1), g(a_2))$, which is clearly an isomorphism of Jordan algebras. It can be easily checked that the following diagram

$$\begin{tikzcd} J(A, \mu_e) \arrow[r] \arrow[d, "J(g)"'] & J(P, \mu) \arrow[d, "J(g)"] \\ J(B, \nu_{e'}) \arrow[r] & J(Q, \nu), \end{tikzcd}$$

is commutative, where the horizontal maps are the isomorphisms given in the proof of (1.2). Thus $J(g): J(P, \mu) \to J(Q, \nu)$ is an isomorphism of Jordan algebras. □

COROLLARY 1.4. *Let θ be an invertible element of A. Then $J(P, \mu)$ and $J(P, \mathcal{N}_A(\theta)\mu)$ are isomorphic.*

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Proof. Let $g: A \to A$ be the inner automorphism given by θ^{-1}. Let $\widetilde{g}: P \to P$ be defined by $\widetilde{g}(x) = \theta^{-1}x$. Then $\widetilde{g}$ is g-semilinear and $\widetilde{g}^{*^{-1}}(f) = f\theta$. Further, the diagram

$$\begin{tikzcd} \mathcal{N}(P) \arrow[rr, "\mathcal{N}(\widetilde{g})"] \arrow[dr, "\mu"'] & & \mathcal{N}(P) \arrow[dl, "Nrd_A(\theta)\mu"] \\ & P & \end{tikzcd}$$

is commutative. Thus by (1.3), g induces an isomorphism

$$J(g): J(P, \mu) \to J(P, Nrd_A(\theta)\mu)$$

given by $(a, x, f) \mapsto (\theta^{-1}a\theta, \theta^{-1}x, f\theta)$. □

We call a Jordan algebra isomorphic to $J(P, \mu)$ a *Tits first construction Jordan algebra*. This construction can be globalised to yield Tits' first construction Jordan algebra bundles over any integral scheme for which 2 and 3 are invertible.

Let k be a field of characteristic different from 2 and 3 and J an exceptional central simple Jordan algebra over k. Let D be a central simple algebra of degree 3 over k such that the special Jordan algebra D_+ is a subalgebra of J. We record the following result of McCrimmon ([M-1], Theorem 8).

PROPOSITION 1.5. *Let J, D, k be as above and $D_+ \hookrightarrow J$. Then there exists $\mu \in k^*$ such that $J \simeq J(D, \mu)$.* □

We shall prove a similar result in a more general setting. Let R be a commutative domain in which 2 and 3 are invertible. Let A be an Azumaya algebra over R. Let M be a projective R-module which is an A_+-module, i.e., there is a Jordan algebra homomorphism $\phi: A_+ \to (\mathrm{End}_R M)_+$. Since the unital special universal envelope $\mathrm{SU}(A_+)$ of A_+ is $A \times A^{op}$ (cf. [J], p. 143, Corollary 2, [JR], Theorem 4), there exists a homomorphism $\widetilde{\phi}: A \times A^{op} \to \mathrm{End}_R M$ of associative R-algebras making the diagram

$$\begin{tikzcd} A_+ \arrow[rr, "\delta"] \arrow[dr, "\phi"'] & & A \times A^{op} \arrow[dl, "\widetilde{\phi}"] \\ & \mathrm{End}_R M & \end{tikzcd}$$

commutative, where $\delta: A_+ \to A \times A^{op}$ is given by $\delta(a) = (a, a)$. Let e_1, e_2 denote the images of the idempotents $(1, 0), (0, 1)$ respectively under the map $\widetilde{\phi}$, so that $e_1 + e_2 = 1$. Let $M_1 = e_1(M)$, $M_2 = e_2(M)$. Then M_1 is a left and M_2 a right A-module through the restriction of $\widetilde{\phi}$ to A and A^{op} respectively and $M = M_1 \oplus M_2$. In fact,

$$M_1 = \{m \in M | \phi(a)(\phi(b)(m)) = \phi(ab)(m), a, b \in A\},$$

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$$M_2 = \{m \in M | \phi(a)(\phi(b)(m)) = \phi(ba)(m), a, b \in A\}.$$

Let J be an exceptional Jordan algebra over R. Let A be an Azumaya algebra of degree 3 over R such that $A_+ \hookrightarrow J$ as a Jordan subalgebra. Let $M \subset J$ be the orthogonal complement of A_+ in J with respect to the trace form of J. Then $J = A_+ \oplus M$. For $x, y \in J$ let

$$x \times y = xy - \tfrac{1}{2}T(x)y - \tfrac{1}{2}T(y)x + \tfrac{1}{2}(T(x)T(y) - T(xy)),$$

where T is the trace map on J. The map $A_+ \to (\mathrm{End}_R(M))_+$ given by $a \mapsto S_a$, $S_a(m) = -2a \times m$, is a Jordan algebra homomorphism (cf. [PR-1], 3.2) and thus factors through a homomorphism $\mathrm{SU}(A_+) = A \times A^{op} \to \mathrm{End}_R M$. We have a decomposition $M = M_1 \oplus M_2$ as described above, where

$$M_1 = \{m \in M | a \times (b \times m) = -\tfrac{1}{2}(ab) \times m, a, b \in, A\},$$

$$M_2 = \{m \in M | a \times (b \times m) = -\tfrac{1}{2}(ba) \times m, a, b \in A\}.$$

Since J is a projective R-module, each M_i is R-projective and since A is an Azumaya algebra over R, each M_i is A-projective. We write $a.x$ for any $a \in A$, $x \in M_1$ (resp. $x.a$ for $x \in M_2$) for the module action of A on M_1 (resp. M_2). We note that by ([M-2], proof of Theorem 8), $M_i \otimes_R K$ has rank 1 over $A \otimes_R K$, K denoting the quotient field of R. Hence M_i is of rank 1 over A.

LEMMA 1.6. *Let N_i be the restriction of the cubic norm of J to M_i. Then $N_i\colon M_i \to R$ has the properties:*

(1) *$N_i(-2a \times x) = Nrd_A(a)N_i(x)$, for all $a \in A$ and $x \in M_i$.*
(2) *The image of N_i generates the unit ideal of R.*

Proof. By going over to the quotient field K of R, property (1) reduces to a simple computation. To prove (2), we may assume that R is local. In this case, it suffices to show that $N_i(v)$ is a unit of R for some $v \in M_i$. Let $\mathcal{M}$ denote the maximal ideal of R. Then $J \otimes_R R/\mathcal{M}$ is an exceptional simple Jordan algebra over $R/\mathcal{M}$ with a decomposition

$$J \otimes R/\mathcal{M} = (A_+ \otimes R/\mathcal{M}) \oplus (M_1 \otimes R/\mathcal{M}) \oplus (M_2 \otimes R/\mathcal{M})$$

and in view of ([M-2], proof of Theorem 8), N_i takes a nonzero value on $M_i \otimes R/\mathcal{M}$. Since $\overline{N_i(x)} = N_i(\overline{x})$, for $x \in M_i$, bar denoting modulo $\mathcal{M}$, it follows that the image of N_i contains a unit of R. $\square$

Let for $x \in J$, $x^\# = x \times x$ denote the *adjoint* of x in J, i.e., $xx^\# = N_J(x)$, N_J denoting the cubic norm on J. Then $M_i^\# \subset M_j$, $i \neq j$ and $M_1M_2 \subset A_+$. This can be shown by going to the quotient field of R (cf. [M-2], proof of Theorem 8), in view of the fact that $(A_+ \otimes K) \cap J = A_+$ and $(M_i \otimes K) \cap J = M_i$.

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LEMMA 1.7. *The map* $\phi\colon M_2 \to M_1^{(*)}$ *given by* $y \mapsto \phi_y$, *where* $\overline{\phi_y(x)} = xy$, $x \in M_1$, $y \in M_2$, *is an isomorphism of* A*-modules.*

Proof. We may assume without loss of generality that M_i are free. Let M_1 be free with e as a basis element. By (1.6), $N_1(e) = \mu$ is a unit of R. By the remarks above, $\mu^{-1}e^\# \in M_2$ and since $N_2(\mu^{-1}\, e^\#) = \mu^{-1}$, by (1.1), $M_2 = \mu^{-1}e^\# A$. We have $\phi_{\mu^{-1}e^\#.b}(a.e) = ab$. This can be seen by going over to the quotient field of R. This shows that $\phi_y \in M_1^{(*)}$ for $y \in M_2$ and that ϕ is linear. Moreover, $\overline{\phi_{(\mu^{-1}e^\#)}(e)} = e(\mu^{-1}e^\#) = 1$. Thus ϕ maps the generator $\mu^{-1}e^\#$ of M_2 to the dual basis element e^* of $M_1^{(*)}$. This shows that ϕ is an isomorphism of A-modules. $\square$

THEOREM 1.8. *Let* J *be an exceptional Jordan algebra over a commutative domain* R *in which 2 and 3 are invertible. Suppose that* A *is an Azumaya algebra over* R *of degree 3 such that* A_+ *is a Jordan subalgebra of* J*. Then there exists a projective* A*-module* P *of rank 1 and an isomorphism* $\widetilde{\mu}\colon \mathcal{N}(P) \simeq R$ *such that the inclusion* $A_+ \hookrightarrow J$ *induces an isomorphism* $J \simeq J(P, \widetilde{\mu})$.

Proof. Let M_1 and M_2 be as above. Let N_i denote the restriction of the Jordan norm on J to M_i. By (1.1) and (1.6) we have an isomorphism $\widetilde{\mu}\colon \mathcal{N}(M_1) \simeq R$ such that $N_1 = \widetilde{\mu}\mathcal{N}$. We define

$$\psi\colon J = A_+ \oplus M_1 \oplus M_2 \to J(M_1, \widetilde{\mu})$$

by $\psi(a, x, y) = (a, x, \phi_y)$. By (1.7), ψ is an isomorphism of R- modules. We show that ψ preserves multiplication. It suffices to do this after a faithfully flat base change. Therefore we assume that M_1 is free. Let $M_1 = Ae$ and $N_1(e) = \mu$. Then, as in the proof of (1.7), $M_2 = \mu^{-1}e^\# A$ and $N_2(\mu^{-1}e^\#) = \mu^{-1}$. We have

$$\begin{aligned}
&\psi((a, x.e, \mu^{-1}e^\#.y)(a', x'.e, \mu^{-1}e^\#.y'))\\
&\quad = \psi(a.a' + \overline{xy'} + \overline{x'y}, (\overline{a}x' + \overline{a'}x + \mu^{-1}y \times y').e,\\
&\qquad\quad \mu^{-1}e^\#.(y'\overline{a} + y\overline{a'} + \mu x \times x'))\\
&\quad = (a.a' + \overline{xy'} + \overline{x'y}, (\overline{a}x' + \overline{a'}x + \mu^{-1}y \times y').e,\\
&\qquad\quad \phi_{\mu^{-1}e^\#.(y'\overline{a} + y\overline{a} + \mu x \times x')}).
\end{aligned}$$

On the other hand, we have as in the proof of (1.7), $\phi_{\mu^{-1}e^\#} = e^*$. This shows that ψ is multiplicative. $\square$

Remark. Given a Jordan algebra J containing A_+, where A is an Azumaya algebra of degree 3 over R, we decompose $J = A_+ \oplus M_1 \oplus M_2$ as in the theorem and treating $\psi\colon M_2 \to M_1^{(*)}$ as an identification, we write $J = J(M_1, \widetilde{\mu})$, by an abuse of notation.

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2. A General Tits Second Construction of Jordan Algebras

Let R be as in Section 1. Let S/R be an étale quadratic algebra over R. Let B be an Azumaya algebra of degree 3 over S with an involution σ of second kind, i.e., σ restricted to S is the nontrivial automorphism τ_0 of S over R. Let $u \in B^*$ with $\sigma(u) = u$ and $Nrd_B(u) = \mu\tau_0(\mu)$ for some $\mu \in S^*$. Let $J(B, \sigma, u, \mu) = B^+ \oplus B$, B^+ denoting the Jordan algebra of symmetric elements of B. Following Tits, we define a multiplication on $J(B, \sigma, u, \mu)$ by

$$\begin{aligned}(b_0, b)(b_0', b') &= (b_0.b_0' + \overline{bu\sigma(b')} \\ &\quad + \overline{b'u\sigma(b)}, \overline{b_0}b' + \overline{b_0'}b + \sigma(\mu)(\sigma(b) \times \sigma(b'))u^{-1}),\end{aligned}$$

where $b_0, b_0' \in B^+, b, b' \in B$ and $x.y, \overline{x}, x \times y$ are defined as in Section 1. With this multiplication, $J(B, \sigma, u, \mu)$ is an exceptional Jordan algebra over R, with B^+ as a Jordan subalgebra. We give a construction of exceptional Jordan algebras, which we shall call Tits' second construction, which includes $J(B, \sigma, u, \mu)$ as a subclass.

Let P be a projective module of rank 1 over B. Let $P^\vee = \mathrm{Hom}_B(P, B)$ regarded as a left B- module through σ. Let $h: P \times P \to B$ be a nonsingular σ-Hermitian form. We regard h as an isomorphism $h: P \simeq P^\vee$, where $h(x)(y) = h(y, x)$. Then $\mathrm{disc}(h): \mathcal{N}(P) \times \mathcal{N}(P) \to S$ is a rank one Hermitian form over (S, τ_0) satisfying

$$\mathrm{disc}(h)(\mathcal{N}(x), \mathcal{N}(y)) = Nrd_B(h(x, y)).$$

We assume that this form is trivial. Let $\mu: (\mathcal{N}(P), \mathrm{disc}(h)) \simeq (S, \langle 1\rangle)$ be an isomorphism of Hermitian spaces. Let $\nu = \mu^{\vee^{-1}}: \mathcal{N}(P)^\vee \simeq S$. Let $\phi: P \times P \to P^{(*)}$ be the map defined as the composite

$$P \times P \xrightarrow{\mathcal{N}} \mathrm{Hom}_S(P, \mathcal{N}(P)) \xrightarrow{\mu} \mathrm{Hom}_S(P, S) \xrightarrow{\mathrm{tr}^{-1}} \mathrm{Hom}_B(P, B) = P^\vee,$$

where $\mathcal{N}$ is the trilinearization of the reduced norm $\mathcal{N}: P \to \mathcal{N}(P)$. With this notation, we set $J(P, h, \mu) = B^+ \oplus P$ and define multiplication by

$$(a, v)(a', v') = (a.a' + \overline{h(v, v')} + \overline{h(v', v)}, \overline{a}v' + \overline{a'}v + h^{-1}(\phi(v, v'))).$$

THEOREM 2.1. *The algebra $J(P, h, \mu)$ is an exceptional Jordan algebra.*

Proof. It suffices to check this in a faithfully flat extension of R. We therefore assume that P is free and choose a generator e for P. Let $h(e, e) = u_e$. Then u_e is a unit of B with $\sigma(u_e) = u_e$ and $h(be, b'e) = bu_e\sigma(b')$. Further, $\mu: (\mathcal{N}(P), \mathrm{disc}(h)) \simeq (S, \langle 1\rangle)$ is an isometry. Let $\mu_e = \mu(\mathcal{N}(e))$. Then $Nrd_B(u_e) = \mu_e\sigma(\mu_e)$. We show that the map $\eta: J(P, h, \mu) \to J(B, \sigma, u_e, \mu_e)$ given by $\eta(a, be) = (a, b)$, is an isomorphism of Jordan algebras. We have, as in Section 1, $P^\vee = Be^\vee$, $e^\vee(e) = 1$ and $\phi(ae, be) = \sigma(\mu_e)(\sigma(a) \times \sigma(b))e^\vee$. Thus

$$\begin{aligned}h^{-1}(\phi(ae, be)) &= h^{-1}(\sigma(\mu_e)(\sigma(a) \times \sigma(b))e^\vee) \\ &= \sigma(\mu_e)(\sigma(a) \times \sigma(b))u_e^{-1}e. \qquad (\#)\end{aligned}$$

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Hence

$$
\begin{aligned}
(a, be)(a', b'e) &= (a.a' + \overline{h(be, b'e)} + \\
&\qquad + \overline{h(b'e, be)}, \overline{a}b'e + \overline{a'}be + h^{-1}(\phi(be, b'e))) \\
&= (a.a' + \overline{bu_e\sigma(b')} + \overline{b'u_e\sigma(b)}, \\
&\qquad (\overline{a}b' + \overline{a'}b + \sigma(\mu_e)(\sigma(b) \times \sigma(b'))u_e^{-1})e),
\end{aligned}
$$

so that $\eta((a, be)(a', b'e)) = (a, b)(a', b')$. This completes the proof of the theorem. □

PROPOSITION 2.2 (Functoriality). *Let $f\colon (B, \sigma) \to (B', \sigma')$ be an isomorphism of S-algebras with involutions of second kind over S/R. Let (P, h) (resp. (P', h')) be a Hermitian (B, σ) (resp (B', σ')) space with a trivialization $\mu\colon (\mathcal{N}(P)$, disc$(h)) \simeq (S, \langle 1\rangle)$ (resp. $\mu'\colon (\mathcal{N}(P')$, disc$(h')) \simeq (S, \langle 1\rangle)$). Let $\widetilde{f}\colon (P, h) \to (P', h')$ be an f-semilinear isomorphism of Hermitian spaces such that the diagram*

$$
\begin{tikzcd}
(\mathcal{N}(P), \mathrm{disc}(h)) \arrow[rr, "\mathcal{N}(\widetilde{f})"] \arrow[dr, "\mu"'] & & (\mathcal{N}(P'), \mathrm{disc}(h')) \arrow[dl, "\mu'"] \\
& (S, \langle 1\rangle) &
\end{tikzcd}
\qquad (*')
$$

commutes. Then the map $J(f)\colon J(P, h, \mu) \to J(P', h', \mu')$ given by $J(f)(a, x) = (f(a), \widetilde{f}(x))$, is an isomorphism of Jordan algebras.

Proof. The proof runs on the same lines as that of (1.3). □

We call a Jordan algebra isomorphic to $J(P, h, \mu)$ a *Tits second construction Jordan algebra*.

PROPOSITION 2.3. *Let (B, σ) be an Azumaya algebra of degree 3 over a quadratic étale extension S of R, with an involution σ of second kind over S/R. Let $(P, h, \widetilde{\mu})$ be a Hermitian space over (B, σ) of rank 1 with a trivialization $\widetilde{\mu}$ of its discriminant. Let $J = J(P, h, \widetilde{\mu})$ be the Tits second construction Jordan algebra over R associated to this data. Then there is an isomorphism*

$$\psi\colon J_S = J(P, h, \widetilde{\mu}) \otimes_R S \simeq J(P, \widetilde{\mu})$$

of Jordan algebras over S such that the transport of the involution $1 \otimes \tau_0$ on J_S to $J(P, \widetilde{\mu})$ through ψ is given by $\tau(a, x, f) = (\sigma(a), h^{-1}(f), h(x))$.

Proof. Let $\psi(h)\colon J_S \to J(P, \widetilde{\mu})$ be given by

$$\psi((a, x) \otimes 1) = (a, x, h(x)) \quad (a, x) \in J$$

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and

$$\psi(1 \otimes \lambda) = (\lambda, 0, 0), \quad \lambda \in S.$$

Since B is generated by B^+ over S, ψ is an S-linear bijection. We show that ψ is multiplicative. It suffices to check this on J since J generates J_S over S. For (a, x), $(a', x') \in J$, we have, $\sigma(a) = a$, $\sigma(a') = a'$, $\psi((a, x)(a', x')) =$

$$\begin{aligned}&(a.a' + \overline{h(x, x')} + \overline{h(x', x)}, \overline{a}x' + \overline{a'}x + \\ &\quad + h^{-1}(\phi(x, x')), h(x')\overline{a} + h(x)\overline{a'} + \phi(x, x'))\end{aligned}$$

and

$$\begin{aligned}\psi(a, x)\psi(a', x') &= (a, x, h(x))(a', x', h(x')) \\ &= (a.a' + \overline{h(x, x')} + \overline{h(x', x)}, \overline{a}x' + \overline{a'}x + \\ &\quad + \phi_*(h(x), h(x')), h(x')\overline{a} + h(x)\overline{a'} + \phi(x, x')).\end{aligned}$$

So we need to verify that

$$\phi_*(h(x), h(x')) = h^{-1}(\phi(x, x')), \quad \text{for all } \ x, x' \in P.$$

This we may check in a faithfully flat extension. We therefore assume that P is free with e as a basis element. Let $u_e = h(e, e)$ and $\mu_e = \widetilde{\mu}\ \mathcal{N}(e)$. Then, as in the proof of (1.2),

$$\phi_*(h(ae), h(be)) = \phi_*(e^* u_e \sigma(a), e^* u_e \sigma(b)) = \mu_e^{-1}(u_e\sigma(a) \times u_e\sigma(b))e$$

and

$$h^{-1}(\phi(ae, be)) = h^{-1}(e^*\mu_e a \times b) = \sigma(\mu_e)(\sigma(a) \times \sigma(b))u_e^{-1}e.$$

By ([J], p.413, (63)), we have

$$\begin{aligned}\sigma(\mu_e)(\sigma(a) \times \sigma(b))u_e^{-1} &= \mu_e^{-1} Nrd(u_e)(\sigma(a) \times \sigma(b))u_e^{-1} \\ &= \mu_e^{-1}(u_e\sigma(a) \times u_e\sigma(b)).\end{aligned}$$

This shows that $\psi(h)$ is an isomorphism of Jordan algebras. The last assertion follows from the commutativity of the diagram

$$\begin{tikzcd} J(P, \widetilde{\mu}) \arrow[r, "\tau"] & J(P, \widetilde{\mu}) \\ J(P, h, \widetilde{\mu}) \otimes S \arrow[u, "\psi(h)"] \arrow[r, "1\otimes\tau_0"] & J(P, h, \widetilde{\mu}) \arrow[u, "\psi(h)"'] \end{tikzcd}$$

□

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Remark. Identifying $B^+ \otimes_R S$ with B_+ and treating ψ as an identification, we write $J(P, h, \widetilde{\mu}) \otimes_R S = J(P, \widetilde{\mu})$.

Let J be an exceptional Jordan algebra over a domain R in which 2 and 3 are invertible. Let (B, σ) be an Azumaya algebra of degree 3 over a quadratic étale extension S of R, with an involution σ of second kind. Assume that B^+ is a Jordan subalgebra of J. Let M denote the orthogonal complement of B^+ in J with respect to the trace form of J, so that $J = B^+ \oplus M$. For any $b \in B$ and $x \in M$, $b \times x \in M$ and the map $g\colon B^+ \to (\mathrm{End}\, M)_+$ given by $g(b)(x) = -2b \times x$, is a homomorphism of Jordan algebras. This is verified by going over to the quotient field of R and using ([PR-1], 3.2). Since the unital special universal envelope $SU(B^+)$ of B^+ is B (cf. [J], p. 141, Theorem 6, [JR], Theorem 4), g factors through a homomorphism $g\colon B \to \mathrm{End}\, M$, of associative algebras, thus making M into a left B-module. Since M is S-projective and B is Azumaya, M is B-projective. Obviously M has rank 1 over B. For $b \in B$ and $x \in M$, we write $b.x = g(b)(x)$.

LEMMA 2.4. *Let J and (B, σ) be as above. Let M denote the orthogonal complement of B^+ in J with respect to the trace form. There exists a map $N\colon M \to S$ making the diagram*

$$\begin{tikzcd} M \arrow[r, "N"] \arrow[dr, "N_J"'] & S \arrow[d, "\mathrm{tr}"] \\ & R \end{tikzcd}$$

commutative, N_J denoting the cubic norm on J. Further, N has the following properties:

(1) $N(b.x) = Nrd_B(b)N(x)$, $b \in B$, $x \in M$.
(2) *The values of N generate the unit ideal of S.*

Proof. Identifying $B^+ \otimes_R S$ with B_+ and as in the proof of (1.8) we have

$$J \otimes_R S = (B^+ \otimes_R S) \oplus M_1 \oplus M_2,$$

with M_1 a left and M_2 a right projective B-module. Let $\tau = 1 \otimes \tau_0$. Then τ is an involution on $J_S = J \otimes_R S$ which coincides with σ on B_+. We have, for $x \in M_1$ and $a, b \in B$,

$$\tau(a) \times (\tau(b) \times \tau(x)) = \tau(a \times (b \times x))$$

$$= -\tfrac{1}{2}\tau((ab) \times x) = -\tfrac{1}{2}((\tau(b)\tau(a)) \times \tau(x)),$$

which shows that $\tau(x) \in M_2$. Thus $\tau(M_1) \subset M_2$; similarly, $\tau(M_2) \subset M_1$, so that $\tau(M_1) = M_2$ and $\tau(M_2) = M_1$. Identifying M with the fixed points of τ in $M \otimes S = M_1 \oplus M_2$, we have

$$M = \{(v_1, \tau(v_1)) | v_1 \in M_1\}.$$

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Let $\theta\colon M \to M_1$ be given by $\theta(v) = v_1$, where $v \otimes 1 = (v_1, \tau(v_1))$. It is easily verified that θ is an isomorphism of B-modules. We define $N\colon M \to S$ by

$$N(v) = N_{J_S}(0, v_1, 0).$$

We observe that N_{J_S} restricted to M_1 is simply the reduced norm of M_1 for a choice of an isomorphism $\mathcal{N}(M_1) \simeq S$ (1.1, 1.6), which gives the two listed properties of N. Further,

$$\begin{aligned}\mathrm{tr}(N(v)) &= \mathrm{tr}(N_{J_S}(0, v_1, 0)) = N_{J_S}(0, v_1, 0) + \tau_0(N_{J_S}(0, v_1, 0))\\ &= N_{J_S}(0, v_1, 0) + N_{J_S}(0, 0, \tau(v_1)),\\ &= N_{J_S}(0, v_1, \tau(v_1)) = N_{J_S}(v \otimes 1) = N_J(v).\end{aligned}$$

which by ([M-1], §5).

Thus the diagram of the lemma is commutative. This proves the lemma. □

THEOREM 2.5. *Let J be an exceptional Jordan algebra over a domain R in which 2 and 3 are invertible. Let (B, σ) be an Azumaya algebra of degree 3 over a quadratic étale extension S of R, with an involution of second kind. Suppose that $B^+ \hookrightarrow J$ is a Jordan subalgebra. Then there exists a projective B-module M of rank 1 together with a Hermitian form h of trivial discriminant and a trivialization η of* $\mathrm{disc}(h)$ *such that the inclusion $B^+ \hookrightarrow J$ induces an isomorphism $J \simeq J(M, h, \eta)$.*

Proof. Let M denote the orthogonal complement of B^+ in J with respect to the trace form so that $J = B^+ \oplus M$. We have seen that M is a projective B-module of rank 1. We construct a Hermitian form h on M with a trivialization for the discriminant. For $v \in J$, let v^+ denote the component of v in B^+. We have an R-quadratic map $\widetilde{h}\colon M \to B^+$ given by $\overline{2\widetilde{h}(x)} = (x^2)^+$. We show that there exists a hermitian form $h\colon M \times M \to B$ such that $h(x, x) = \widetilde{h}(x)$. By (1.8), we have

$$J \otimes_R S = B_+ \oplus M_1 \oplus M_2.$$

We have a B-isomorphism $\theta\colon M \simeq M_1$ given by $\theta(v) = v_1$ (cf. proof of 2.4), where $v \otimes 1 = (0, v_1, \tau(v_1)) \in J_S$. Suppose that M is free with a basis element e'. Let $e = \theta(e')$, so that $M_1 = Be$. If $N_{J_S}(0, e, 0) = \mu \in S^*$, then, $M_2 = e^*B$ with $e^* = \mu^{-1}e^\#$. Let $\tau = 1 \otimes \tau_0$. Since $\tau(M_i) = M_j$, $i \neq j$ (cf. proof of 2.4), we have

$$\tau(0, e, 0) = (0, 0, e^*.u),\ \tau(0, 0, e^*) = (0, v.e, 0),$$

for some $u, v \in B^*$. Further, by (1.6, (1)), $N_2(e^*.u) = \mu^{-1}Nrd(u)$ and we have

$$\begin{aligned}\mu^{-1}Nrd(u) &= N_{J_S}(0, 0, e^*.u) = N_{J_S}(\tau(0, e, 0))\\ &= \tau_0(N_{J_S}(0, e, 0)) = \tau_0(\mu).\end{aligned}$$

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Thus $Nrd(u) = \mu\tau_0(\mu)$. Further,

$$\begin{aligned}(1,0,0) &= \tau(1,0,0) = \tau((0,e,0)(0,0,e^*)) \\ &= \tau(0,e,0)\tau(0,0,e^*) = (0,0,e^*.u)(0,v.e,0) = (\overline{vu},0,0),\end{aligned}$$

so that $vu = 1$. Moreover, multiplicativity of τ gives

$$\begin{aligned}\tau(0,b.e,0) &= \tau(-2(b,0,0)\times(0,e,0)) \\ &= -2(\sigma(b),0,0)\times(0,0,e^*.u) = (0,0,e^*.u\sigma(b)).\end{aligned}$$

Similarly,

$$\tau(0,0,e^*.c) = (0,\sigma(c)u^{-1}.e,0).$$

Hence,

$$\tau(a,b.e,e^*.c) = (\sigma(a),\sigma(c)u^{-1}.e,e^*.u\sigma(b)).$$

Since

$$(0,e,0) = \tau^2(0,e,0) = \tau(0,0,e^*.u) = (0,\sigma(u)u^{-1}.e,0),$$

it follows that $\sigma(u) = u$. We have,

$$M = \{(0,b.e,e^*.u\sigma(b))|b\in B\} = Be'$$

with $e'\otimes 1 = (0,e,e^*.u)$. We define $h_{e'}\colon M\times M\to B$ by $h_{e'}(a.e',b.e') = au\sigma(b)$. Since $(e'^2)^+ = ((0,e,e^*.u)^2)^+ = 2\overline{u}$, we have, $h_{e'}(e',e') = u = \widetilde{h}(e')$. Let e'' be another basis element for M. Then $e'' = \alpha.e'$ for $\alpha\in B^*$ and $e_1 = \theta(e'') = \alpha.\theta(e') = \alpha.e$. Let $\tau(0,e_1,0) = (0,0,e_1^*.u')$. We have,

$$e_1^* = N_{J_S}(e_1)^{-1}e_1^{\#} = (\mu Nrd(\alpha))^{-1}(\alpha.e)^{\#} = (\mu Nrd(\alpha))^{-1}(e^{\#}.\alpha^{\#}) = e^*.\alpha^{-1}$$

and

$$\tau(0,e_1,0) = \tau(0,\alpha.e,0) = (0,0,e^*.u\sigma(\alpha)).$$

Therefore $u' = \alpha u\sigma(\alpha)$. Now, for $a.e' = a'.e''$, $b.e' = b'.e''$, we have $a = a'\alpha$, $b = b'\alpha$ and

$$\begin{aligned}h_{e'}(a.e',b.e') &= au\sigma(b), \\ h_{e''}(a.e',b.e') &= h_{e''}(a'.e'',b'.e'') = a'u'\sigma(b') \\ &= a'\alpha u\sigma(\alpha)\sigma(b') = (a'\alpha)u\sigma(b'\alpha) = h_{e'}(a.e',b.e').\end{aligned}$$

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Thus $h_e = h_{e'}$ is independent of the choice of e' and by patching we get a hermitian form h defined globally on M. We now give a trivialization for $\mathrm{disc}(h)$. By (2.4) and (1.1) there exists an isomorphism $\eta\colon \mathcal{N}(M) \simeq S$ such that $N = \eta\mathcal{N}$. We claim that $\eta\colon (\mathcal{N}(M), \mathrm{disc}(h)) \to (S, \langle 1\rangle)$ is an isometry. We verify this in a faithfully flat extension. We assume $M = Be'$ is free with e' as a basis element. Then $h(a.e', b.e') = au\sigma(b)$ with u a symmetric unit and $Nrd(u) = \mu\tau_0(\mu)$, where $\mu = N_{J_S}(\theta(e')) = N(e')$ (cf. 2.4). We have

$$\mathrm{disc}(h)(\alpha\mathcal{N}(e'), \beta\mathcal{N}(e')) = \alpha\tau_0(\beta)Nrd(h(e', e')) = \alpha\tau_0(\beta)\mu\tau_0(\mu)$$

and

$$\eta(\alpha\mathcal{N}(e'))\tau_0(\eta(\beta\mathcal{N}(e'))) = \alpha N(e')\tau_0(\beta N(e')) = \alpha\mu\tau_0(\beta\mu).$$

This shows that η is an isometry. We finally show that $J = B^+ \oplus M = J(M, h, \eta)$. It suffices to check that the two multiplications coincide on $B^+ \oplus M$. We may assume that $M = Be'$ is free with e' as a basis element. In $J \hookrightarrow J_S$ (cf. proof of 2.4), we have,

$$\begin{aligned}
(a, b.e')(a', b'.e') &= (a, b.e, e^*.u\sigma(b))(a', b'.e, e^*.u\sigma(b')) \\
&= (a.a' + \overline{bu\sigma(b')} + \overline{b'u\sigma(b)}, \\
&\quad (\overline{a}b' + \overline{a'}b + \phi_*(e^*.u\sigma(b), e^*.u\sigma(b'))).e, \\
&\quad e^*.(u\sigma(b')\overline{a} + u\sigma(b)\overline{a'} + \phi(be, b'e))) \\
&= (a.a' + \overline{bu\sigma(b')} + \overline{b'u\sigma(b)}, \\
&\quad (\overline{a}b' + \overline{a'}b + \mu^{-1}(u\sigma(b) \times u\sigma(b'))).e, \\
&\quad e^*.(u\sigma(b')\overline{a} + u\sigma(b)\overline{a'} + \mu(b \times b')) \\
&= (a.a' + \overline{bu\sigma(b')} + \overline{b'u\sigma(b)}, \\
&\quad (\overline{a}b' + \overline{a'}b + \sigma(\mu u^{-1}(b \times b'))).e, \\
&\quad e^*.u(\sigma(\overline{a}b' + \overline{a'}b + \sigma(\mu u^{-1}(b \times b'))) \\
&= (a.a' + \overline{bu\sigma(b')} + \overline{b'u\sigma(b)}, \\
&\quad (\overline{a}b' + \overline{a'}b + \sigma(\mu u^{-1}(b \times b'))).e').
\end{aligned}$$

However, in $J(M, h, \eta)$, we have,

$$\begin{aligned}
(a, b.e')(a', b'.e') &= (a.a' + \overline{h(b.e', b'.e')} + \\
&\qquad + \overline{h(b'.e', b.e')}, \overline{a}b'.e' + \overline{a'}b.e' + h^{-1}(\phi(b.e', b'.e')) \\
&= (a.a' + \overline{bu\sigma(b')} + \overline{b'u\sigma(b)}, \overline{a}b'.e' + \\
&\qquad + \overline{a'}b.e' + \sigma(\mu u^{-1}(b \times b')).e'),
\end{aligned}$$

by (#) in the proof of Theorem 2.1. Thus the two multiplications on $B^+ \oplus M$ coincide. This proves the theorem. □

3. Rigidity of Nonfree Projective Modules over $D[X, Y]$

Let k be a field and D a central division algebra of prime degree p over k. Let P be a nonfree projective (left) module of rank 1 over $D[X, Y]$. Then, by ([KPS], Thm. 7.1), P extends to a vector bundle $\widetilde{P}$ over $X = \mathbb{P}^2_k$ with a D structure, which is unique upto a line bundle.

LEMMA 3.1. *Let P be a nonfree projective $D[X, Y]$-module and $\widetilde{P}$ an extension of P as a vector bundle with a D-structure to $\mathbb{P}^2_k$. Then the map $D \to \mathrm{End}_{\mathcal{O}_X} \widetilde{P}$ is an isomorphism.*

Proof. Since $\mathrm{End}_{\mathcal{O}_X} \widetilde{P}$ is a finite-dimensional k-algebra and D is central simple over k with $D \hookrightarrow \mathrm{End}_{\mathcal{O}_X} \widetilde{P}$, we have ([B], 4.3, p.107) $\mathrm{End}_{\mathcal{O}_X} \widetilde{P} = D \otimes_k D'$, where $D' = \mathrm{End}_{D\otimes\mathcal{O}_X} \widetilde{P}$. Further,

$$\mathrm{End}_{D\otimes\mathcal{O}_X} \widetilde{P} \hookrightarrow \mathrm{End}_{D(X,Y)}(P \otimes k(X, Y)) \simeq D(X, Y)^{op}.$$

Thus D' is a division algebra over k. Since D is of prime degree, in view of ([P]), it follows that $\mathrm{Aut}_{D[X,Y]} P = k^*$. Since $D'^* C$ Aut $P = k^*$, $D[X, Y]$ we have $D' = k$ and $D = \mathrm{End}_{\mathcal{O}_X} \widetilde{P}$. □

THEOREM 3.2. *Let D be a central division algebra of prime degree over k. Let P be a projective module over $D[X, Y]$ and $\widetilde{P}$ an extension of P as a vector bundle with a D-structure to $\mathbb{P}^2_k$. If P is nonfree, $\widetilde{P}$ is indecomposable as a vector bundle. Let $\widetilde{P^{(*)}}$ be an extension of the right $D[X, Y]$-module $P^{(*)}$. Then, if degree D is odd, $\widetilde{P}$ is not isomorphic to $\widetilde{P^{(*)}}$ as a vector bundle.*

Proof. By (3.1), $D \simeq \mathrm{End}_{\mathcal{O}_X} \widetilde{P}$ is a division algebra, so that $\widetilde{P}$ is indecomposable. By the same argument as in (3.1), $D^{op} \simeq \mathrm{End}_{\mathcal{O}_X} \widetilde{P^{(*)}}$. If the degree of D is odd, D is not isomorphic to D^{op}, so that $\widetilde{P}$ and $\widetilde{P^{(*)}}$ are not isomorphic as vector bundles on $\mathbb{P}^2_k$. □

Remark. Let D be a central division algebra of degree 3 over k. Let $\bar{k}$ denote the algebraic closure of k. We have $\mathrm{End}_{\mathcal{O}_X} \widetilde{P}_{\bar{k}} \simeq (\mathrm{End}_{\mathcal{O}_X} \widetilde{P}) \otimes_k \bar{k} \simeq M_3(\bar{k})$, so that $\widetilde{P}_{\bar{k}} \simeq P_0 \oplus P_0 \oplus P_0$, where P_0 is a rank 3 vector bundle on $\mathbb{P}^2_{\bar{k}}$ which is simple, i.e, $\mathrm{End}_{\mathcal{O}_{X_{\bar{k}}}} P_0 = \bar{k}$.

4. Nontrivial Jordan Algebra Bundles over $\mathbb{A}^2_k$ via Tits First Construction

We begin by recalling the construction ([OS]) of nonfree projective modules of rank 1 over $D[X, Y]$, where D is a noncommutative division ring. Let $\alpha, \beta \in D^*$ be

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two noncommuting elements. Let $P_0 = P_{\alpha,\beta}$ be the projective left $D[X,Y]$-module defined as the kernel of the $D[X,Y]$-linear map $\phi_{\alpha,\beta}\colon D[X,Y]^2 \to D[X,Y]$, given by $(1,0) \mapsto X+\alpha$, $(0,1) \mapsto Y+\beta$. Then P_0 is nonfree over $D[X,Y]$ of rank 1. Suppose that D is central and finite-dimensional over k. Then there exists an irreducible polynomial $f(X) \in k[X]$, such that $\deg(f) \geqslant 2$ and $P_0 \otimes k[X]_f[Y]$ is free. (We may take for example $f(X)$ to be the minimal polynomial of $-\alpha$ over k. Since $X+\alpha$ is a unit in $k(\alpha)[X]_f \hookrightarrow D[X]_f[Y]$, $P_0 \otimes k[X]_f[Y]$ contains a unimodular element and hence is free.)

Let $\lambda \in k^*$ and P_λ be the pull back of P_0 under the automorphism $\phi_\lambda\colon D[X,Y] \to D[X,Y]$ given by $\phi_\lambda(X) = \lambda X$, $\phi_\lambda(Y) = Y$ and $\phi_\lambda|_D =$identity. Then P_λ is a nonfree rank 1 projective $D[X,Y]$-module and $P_\lambda \otimes k[X]_{f_\lambda(X)}[Y]$ is free, where $f_\lambda(X) = f(\lambda X)$. Since $f(X)$ is irreducible and $\deg(f) \geqslant 2$, one may choose a sequence $\lambda_i \in k^*$ such that $f_{\lambda_i}(X)$ and $f_{\lambda_j}(X)$ are mutually coprime for $i \neq j$. We rename $P_{\lambda_i} = P_i$, $f_{\lambda_i} = f_i$. Further, if $\{g_i\}$ is another family of mutually coprime polynomials, one may choose $\{f_i\}$ in such a way that $(f_i, g_j) = 1$ for all i, j. This can be done inductively.

Let D be a finite-dimensional central division algebra over k. Let De be the free D-module of rank 1 with e as a basis element. Let $\mu \in k^*$ and let $\mu_0\colon \mathcal{N}(De) \simeq k$ be an isomorphism such that $\mu_0 \mathcal{N}(e) = \mu$. We construct nontrivial forms for (De, μ_0) over $k[X,Y]$. Let P be a projective left $D[X,Y]$-module of rank 1. Let $\phi\colon P \simeq De \otimes k[X,Y]$ be an isomorphism of $k[X,Y]$-modules such that $\overline{\phi}\colon \overline{P} \to De \otimes k[X]$ is an isomorphism of $D[X]$-modules, bar denoting reduction modulo Y (cf. [PST], 6.1). Let $P_\phi = (De \otimes k[X,Y])^\phi$ be the $D[X,Y]$-module for the transport of the $D[X,Y]$-structure on P through ϕ. Then $\overline{P_\phi} = De \otimes k[X]$ as a $D[X]$-module and further $P \simeq P_\phi$ as $D[X,Y]$-modules. If $\widetilde{\mu}\colon \mathcal{N}(P_\phi) \simeq k[X,Y]$ is an isomorphism of $k[X,Y]$- modules, $\overline{\widetilde{\mu}}\colon \mathcal{N}(\overline{P_\phi}) = \mathcal{N}(De \otimes k[X]) \to k[X]$ is an isomorphism. Suppose that $\overline{\widetilde{\mu}}\mathcal{N}(e) = \nu$. Replacing $\widetilde{\mu}$ by $\mu\nu^{-1}\widetilde{\mu}$, we may assume without loss of generality that $\overline{\widetilde{\mu}} = \mu_0$. Thus we have constructed a pair $(P_\phi, \widetilde{\mu})$ whose reduction modulo Y coincides with $(De \otimes k[X], \mu_0)$. We record this as

PROPOSITION 4.1. *Let D and k be as above. Let De be a free module of rank 1 over D and $\mu_0\colon \mathcal{N}(De) \simeq k$ an isomorphism. Let $\{g_i\}$ be an infinite family of mutually coprime polynomials in $k[X]$. Then there exist nonfree projective modules P_i, $i \geqslant 1$, over $D[X,Y]$ of rank 1 and polynomials f_i in $k[X]$ with $(f_i, f_j) = 1, i \neq j$, $(f_i, g_j) = 1$ for all i, j and such that $P_i \otimes k[X]_{f_i}[Y]$ is free for each i. Further, there exists $\widetilde{\mu_i}\colon \mathcal{N}(P_i) \simeq k[X,Y]$ such that $(P_i, \widetilde{\mu_i})$ modulo Y is $(De, \mu_0) \otimes_k k[X]$.* □

COROLLARY 4.2. *The modules P_i in* (4.1) *are mutually nonisomorphic.*

Proof. Suppose $P_i \simeq P_j$ for some $i \neq j$. Since $(f_i, f_j) = 1$ for and $(P_i)_{f_i}$ is free over $D \otimes k[X]_{f_i}[Y]$ for all i, the corollary follows from ([BCW]). □

LEMMA 4.3. *Let $J = J(D, \mu)$ be an exceptional Jordan division algebra over a field k arising from Tits' first construction. Then there exist central division algeb-*

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ras D_1, D_2 over k of degree 3 such that $(D_1)_+, (D_2)_+$ are Jordan subalgebras of J and $(D_1)_+ \cap (D_2)_+ = k$.

Proof. Let $L \hookrightarrow D$ be a cubic cyclic extension of k. Let D_1 be the cyclic cross product (L, ι, μ), where ι is a generator of $\mathrm{Gal}(L/k)$. Then, by ([PR-2], 2.7), $(D_1)_+$ is a Jordan subalgebra of $J(D, \mu)$. Let $\theta \in D$ be such that $\theta^{-1} L\theta \neq L$ and $Nrd(\theta) = 1$. Then $\theta^{-1} L\theta \cap L = k$. Let ϕ_θ denote the automorphism of J given by $\phi_\theta(a_0, a_1, a_2) = (\theta^{-1}a_0\theta, \theta^{-1}a_1, a_2\theta)$. The subalgebra $\phi_\theta((D_1)_+)$ of $J(D, \mu)$ is equal to $(D_2)_+$, where $D_2 = (L', \iota', \mu')$ and $L' = \theta^{-1}L\theta$, $\iota' \in \mathrm{Gal}(L'/k)$ a generator and $\mu' \in k$. We show that $(D_1)_+ \cap (D_2)_+ = k$. Let $(b_0, b_1, b_2) \in (D_1)_+ \cap (D_2)_+$, where $b_i \in L$. Then there exist $a_i \in L$, $0 \leqslant i \leqslant 2$, such that $(b_0, b_1, b_2) = (\theta^{-1}a_0\theta, \theta^{-1}a_1, a_2\theta)$. We therefore have $\theta^{-1}a_1 = b_1$, so that $a_1 \neq 0$ would imply that $\theta \in L$, a contradiction. Thus $a_1 = 0$. Similarly $a_2 = 0$. Further, $\theta^{-1}a_0\,\theta = b_0 \in \theta^{-1}L\,\theta \cap L = k$. This proves that $(D_1)_+ \cap (D_2)_+ = k$. □

Let J be a Tits first construction Jordan division algebra over k. By (4.3), there exist cyclic division algebras D_1, D_2 of degree 3 over k such that $(D_1)_+, (D_2)_+$ are Jordan subalgebras of J with $(D_1)_+ \cap (D_2)_+ = k$. Then (cf. 1.8) $J = J(D_1e_1, \mu_1) = J(D_2e_2, \mu_2)$ for some $e_i \in J$ and $\mu_i\colon \mathcal{N}(D_ie_i) \simeq k$, isomorphisms. By (4.1), there exists, for each $i \geqslant 1$, a pair $(P_i^1, \widetilde{\mu_i^1})$, where P_i^1 is a nonfree rank 1 projective $D_1[X, Y]$-module and $\widetilde{\mu_i^1}$ a trivialization of its reduced norm and a polynomial $f_i \in k[X]$ such that the following conditions are satisfied:

(1) The polynomials f_i and f_j are coprime for $i \neq j$ and $(P_i^1)_{f_i}$ is free.
(2) The reduction of $(P_i^1, \widetilde{\mu_i^1})$ modulo Y is $(D_1e_1, \mu_1) \otimes k[X]$.

Similarly, for every $i \geqslant 1$, there exist, by (4.1), pairs $(P_i^2, \widetilde{\mu_i^2})$, P_i^2 a nonfree rank 1 projective $D_2[X, Y]$-module with a trivialization $\widetilde{\mu_i^2}$ of its reduced norm and polynomial $g_i \in k[X]$ satisfying

(1) The polynomials g_i and g_j are coprime for $i \neq j$, the polynomials f_i and g_j are coprime for all i, j and $(P_i^2)_{g_i}$ is free.
(2) The reduction of $(P_i^2, \widetilde{\mu_i^2})$ modulo Y is $(D_2e_2, \mu_2) \otimes k[X]$.

Let P be a rank 1 nonfree projective $D[X, Y]$-module and $\widetilde{\mu}\colon \mathcal{N}(P) \simeq k[X, Y]$ a trivialization of the reduced norm. The pair $(P, \widetilde{\mu})$ is a principal $SL_1(D)$-bundle over $\mathbb{A}_k^2$ which admits an extension $(\widetilde{P}, \widetilde{\mu})$ to $\mathbb{P}_k^2$ (cf. [PST], 4.5). The bundle $\widetilde{P}$ is simply an extension to $\mathbb{P}_k^2$ of the $D[X, Y]$-module P. Then $\mathcal{J} = \mathcal{J}(\widetilde{P}, \widetilde{\mu})$ is a Jordan algebra bundle over $\mathbb{P}_k^2$ which restricts on $\mathbb{A}_k^2$ to $J(P, \widetilde{\mu})$. Since the extension of $J(P, \widetilde{\mu})$ to $\mathbb{P}_k^2$ is unique, we have the following

PROPOSITION 4.4. *The Jordan algebra bundle $\mathcal{J} = J(P, \widetilde{\mu})$ admits a unique extension $\widetilde{\mathcal{J}}$ to $X = \mathbb{P}_k^2$ whose underlying vector bundle is given by*

$$\widetilde{\mathcal{J}} = (D \otimes \mathcal{O}_X) \oplus \widetilde{P} \oplus \widetilde{P}^{(*)},$$

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where $\widetilde{P}$ denote the extension of the $D[X, Y]$-module P to $\mathbb{P}^2_k$. □

Let $(P^1_i, \widetilde{\mu^1_i})$, $(P^2_i, \widetilde{\mu^2_i})$ be nonfree projective $D_1[X, Y]$ and $D_2[X, Y]$ modules respectively with trivializations of their reduced norms constructed above. Let

$$J^1_i = J(P^1_i, \widetilde{\mu^1_i}), \qquad J^2_i = J(P^2_i, \widetilde{\mu^2_i}).$$

Then $\{J^j_i, j = 1, 2, i \geqslant 1\}$, is a family of Jordan algebras over $k[X, Y]$ with the property that $J^j_i = J \otimes k[X]$ modulo Y and

$$J^1_i \otimes k[X]_{f_i}[Y] \simeq J \otimes k[X]_{f_i}[Y], \qquad J^2_i \otimes k[X]_{g_i}[Y] \simeq J \otimes k[X]_{g_i}[Y]$$

with $(g_i, g_j) = 1 = (f_i, f_j)$, $i \neq j$ and $(f_i, g_j) = 1$ for all i, j.

PROPOSITION 4.5. *The Jordan algebras J^1_i (resp J^2_i) are mutually nonisomorphic.*

Proof. Suppose that $J^1_i \simeq J^1_j$, for some $i \neq j$. Since J^1_i and J^1_j are extended after inverting f_i and f_j respectively and $(f_i, f_j) = 1$, by ([BCW]), J^1_i is extended from $J \otimes k[X]$. Then the extension $\widetilde{J^1_i}$ of J^1_i to $\mathbb{P}^2_k$, by uniqueness of extension, is isomorphic to $\pi^* J$, $\pi : \mathbb{P}^2_k \to \operatorname{Spec} k$ denoting the structure morphism. Then

$$(D_1 \otimes \mathcal{O}_{\mathbb{P}^2_k}) \oplus \widetilde{P^1_i} \oplus \widetilde{P^1_i}^{(*)} \simeq \pi^* J$$

as vector bundles on $\mathbb{P}^2_k$. While $\pi^* J$ is a trivial vector bundle, $\widetilde{P^1_i}$ is an indecomposable vector bundle by (3.2). This is a contradiction. □

Let

$$\pi^1_i : (P^1_i, \widetilde{\mu^1_i}) \otimes k[X]_{f_i}[Y] \simeq (D_1 e_1, \mu_1) \otimes k[X]_{f_i}[Y]$$

and

$$\pi^2_i : (P^2_i, \widetilde{\mu^2_i}) \otimes k[X]_{g_i}[Y] \simeq (D_2 e_2, \mu_2) \otimes k[X]_{g_i}[Y]$$

be isomorphisms such that $\overline{\pi^j_i} =$ identity, for $j = 1, 2$. We then have induced isomorphisms

$$J(\pi^1_i) : J^1_i \otimes k[X]_{f_i}[Y] \simeq J \otimes k[X]_{f_i}[Y],$$

$$J(\pi^2_i) : J^2_i \otimes k[X]_{g_i}[Y] \simeq J \otimes k[X]_{g_i}[Y],$$

with $\overline{J(\pi^j_i)} =$ identity, for $j = 1, 2$. Let J_i be the Jordan algebra obtained by patching J^1_i on $k[X]_{g_i}[Y]$ and J^2_i on $k[X]_{f_i}[Y]$ over $k[X]_{f_i g_i}[Y]$ by $\phi_i = J(\pi^2_i)^{-1}$

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$J(\pi_i^1)$. Then, since $\overline{J_i^j} = J$ modulo Y and $\overline{J(\pi_i^j)} =$ identity, $\overline{\phi}_i =$ identity and $\overline{J_i} = J \otimes k[X]$ modulo Y. By the very construction,

$$J_i \otimes k[X]_{f_i g_i}[Y] \simeq J \otimes k[X]_{f_i g_i}[Y]$$

and the polynomials $r_i = f_i g_i$ are mutually coprime. We now show that the algebras J_i are mutually nonisomorphic. Suppose that $J_i \simeq J_j$ for $i \neq j$. Then both $(J_i)_{r_i}$ and $(J_i)_{r_j}$ are extended from J. Since $(r_i, r_j) = 1$, $J_i \simeq J \otimes k[X, Y]$. Restricting J_i to $k[X]_{g_i}[Y]$ we get $J_i^1 \otimes k[X]_{g_i}[Y]$ and $J_i^1 \otimes k[X]_{f_i}[Y]$ are extended. Since $(f_i, g_i) = 1$, J_i^1 is extended from J. This contradicts (4.5). We record this as

PROPOSITION 4.6. *The Jordan algebras J_i on $\mathbb{A}_k^2$ have the following properties:*

(1) $\overline{J_i} = J \otimes k[X]$ *modulo* Y.
(2) *There are mutually coprime polynomials $r_i \in k[X]$ such that $J_i \otimes k[X]_{r_i}[Y] \simeq J \otimes k[X]_{r_i}[Y]$.*
(3) *The algebras J_i are nonextended and mutually nonisomorphic.* □

5. Nontrivial Jordan Algebra Bundles on $\mathbb{A}_k^2$ via Tits Second Construction

Let K be a quadratic extension of k. Let (D, σ) be a central division algebra of degree 3 over K with an involution σ of second kind over K/k. Let $u \in D^*$ be such that $\sigma(u) = u$ and $Nrd(u) = \mu\sigma(\mu)$ for some $\mu \in K^*$. In ([R], 4.9), it is shown that there exists a rank 1 projective $D[X, Y]$-module P with a nonsingular Hermitian form $h: P \times P \to D[X, Y]$ and a trivialization $\widetilde{\mu}: \mathrm{disc}(h) \to (K[X, Y], \langle 1 \rangle)$ with the following properties:

(1) The reduction of $(P, h, \widetilde{\mu})$ modulo Y is isomorphic to $(D, \langle u \rangle, \mu)$, where $\langle u \rangle$ denotes the rank one Hermitian form given by $a \mapsto au\sigma(a)$ and μ is treated as a trivialization of the discriminant of $\langle u \rangle$.
(2) There exists $f \in k[X]$ such that $(P, h, \widetilde{\mu}) \otimes k[X]_f [Y] \simeq (D, \langle u \rangle, \mu) \otimes k[X]_f [Y]$.
(3) The principal $\mathrm{SU}(D, \sigma)$-bundle on $\mathbb{A}_k^2$ associated to $(P, h, \widetilde{\mu})$ admits no reduction of the structure group to any proper connected reductive subgroup of $\mathrm{SU}(D, \sigma)$. In particular, $(P, h, \widetilde{\mu})$ is not extended from $(D, \langle u \rangle, \mu)$.

In (1), we may further assume, through a twist argument ([PST], 6.1), that $(P, h, \widetilde{\mu})$ reduces modulo Y to $(De, u_e, \mu_e) \otimes k[X]$ where De is the free module of rank 1 over D with a basis element e, u_e the Hermitian form on De given by $u_e(xe, ye) = xu\sigma(y)$ and $\mu_e(\mathcal{N}(e)) = \mu$. In (2), we may further assume that $f(0) \neq 0$ in view of the following

LEMMA 5.1. *For $\eta \in k^*$, let $(P, h, \widetilde{\mu})_\eta$ be the pull back of $(P, h, \widetilde{\mu})$ under the automorphism $D[X, Y] \xrightarrow{\phi_\eta} D[X, Y]$, given by $X \mapsto X - \eta$, $Y \mapsto Y$ and $\phi_\eta | D =$*

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identity. *Then* $(P, h, \widetilde{\mu})_\eta \otimes k[X]_{f_\eta}[Y]$ *is extended from* $(D, \langle u\rangle, \mu)$ *where* $f_\eta(X) = f(X - \eta)$.

Proof. We need only to check that the fibre of $(P, h, \widetilde{\mu})_\eta$ at $(0, 0)$ is isomorphic to $(D, \langle u\rangle, \mu)$. In fact the fibre of $(P, h, \widetilde{\mu})_\eta$ at $(0, 0)$ is precisely the fibre of $(P, h, \widetilde{\mu})$ at $(\eta, 0)$. Since $(P, h, \widetilde{\mu})$ is stably extended from D, the fibre of $(P, h)_\eta$ at $(\eta, 0)$ is isomorphic to the fibre of $(P, h, \widetilde{\mu})$ at $(0, 0)$, which is $(D, \langle u\rangle, \mu)$. □

LEMMA 5.2. *Let* $J = J(D, \sigma, u, \mu)$ *be a Tits second construction Jordan division algebra, where* D *is a central simple algebra of degree 3 over a quadratic extension* K *of* k, *with an involution* σ *of second kind over* K/k. *Assume that* J *is not a Tits first construction. Then there exist central simple algebras* (D^1, σ^1), (D^2, σ^2) *of degree 3 over a quadratic extension* F/k *with involutions of second kind over* F/k, *such that* $(D^1)^+$, $(D^2)^+$ *are Jordan subalgebras of* J *with* $(D^1)^+ \cap (D^2)^+ = k$.

Proof. Let $J = D^+ \oplus D$. Then J is the descent of $J(D, \mu) = D \oplus D \oplus D$ over K under the descent map $\psi_u(x, y, z) = (\sigma(x), \sigma(z)u^{-1}, u\sigma(y))$ (cf. [M-1], proof of Theorem 7). Let M_1 denote the subalgebra of $J(D, \mu)$ generated over K by $(u, 0, 0)$ and $(0, 1, 0)$. Since $\psi_u(u, 0, 0) = (u, 0, 0)$ and $\psi_u(0, 1, 0) = (0, 0, u)$, it is easily verified that ψ_u stabilizes M_1. Since J is a Jordan division algebra, $u \notin k^*$ and M_1 is a nine-dimensional subalgebra of J. Since J is not a Tits first construction, by ([J], Lemma 2, p.420), M_1 descends to a subalgebra $M^1 = (D^1)^+$ of J for a central division algebra (D^1, σ^1) of degree 3 over a quadratic extension F of k, with an involution of second kind over F/k. Choose an element $v \in D$ with $vu\sigma(v) \notin K(u)$, $v\sigma(v) = 1$ and $Nrd(v) = 1$. Then $\widetilde{\phi_v}\colon J \to J$ given by $\widetilde{\phi_v}(a, b) = (v^{-1}av, v^{-1}b)$, is an automorphism of J. Let $M^2 = \widetilde{\phi_v}(M^1)$. Since M^1 and M^2 are isomorphic as Jordan algebras, $M^2 = (D^2)^+$ for some degree 3 division algebra (D^2, σ^2) with an involution of second kind over F/k. The map $\widetilde{\phi_v} \otimes 1\colon J_S \to J_S$ transports to the automorphism $\phi_v\colon D \oplus D \oplus D \to D \oplus D \oplus D$ given by $\phi_v(a_0, a_1, a_2) = (v^{-1}a_0v, v^{-1}a_1, a_2v)$.

We have $M_2 = M^2 \otimes_k K = \phi_v(M_1)$. We prove that $M^1 \cap M^2 = k$. For this, it is sufficient to prove that $M_E^1 \cap M_E^2 = E$ in J_E for some finite extension E of k. Let $E = K(\sqrt{d})$, where d is the discriminant of the minimal polynomial of u over K. Then $E(u)/E$ is cyclic and $M_{KE}^2 = \phi_v(M_{KE}^1)$. The proof of (4.3) gives $M_{KE}^1 \cap M_{KE}^2 = KE$, noting that $v^{-1}KE(u)v \neq KE(u)$. Thus $M^1 \cap M^2 = k$, proving the lemma. □

Let J be a Tits second construction Jordan algebra which is not a Tits' first construction. By the above lemma, we may write (cf. 2.5)

$$J = J(D^1e_1, u_{e_1}, \mu_{e_1}) = J(D^2e_2, u_{e_2}, \mu_{e_2})$$

with $(D^1)^+ \cap (D^2)^+ = k$.

By ([R], 4.9), there exist $(P_1^i, h_1^i, \widetilde{\mu_1^i})$, $(P_2^i, h_2^i, \widetilde{\mu_2^i})$, rank 1, nontrivial Hermitian spaces over $(D^1[X, Y], \sigma^1)$ and $(D^2[X, Y], \sigma^2)$ respectively and $f_i, g_i \in k[X]$ with the following properties:

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(1) $(P_1^i, h_1^i, \widetilde{\mu_1^i})$ modulo Y reduces to $(D^1e_1, u_{e_1}, \mu_{e_1})$, $(P_2^i, h_2^i, \widetilde{\mu_2^i})$ modulo Y reduces to $(D^2e_2, u_{e_2}, \mu_{e_2})$.
(2) $(P_1^i, h_1^i, \widetilde{\mu_1^i}) \otimes k[X]_{f_i}[Y]$ is isomorphic to $(D^1e_1, u_{e_1}, \mu_{e_1}) \otimes k[X]_{f_i}[Y]$, $(P_2^i, h_2^i, \widetilde{\mu_2^i}) \otimes k[X]_{g_i}[Y]$ is isomorphic to $(D^2e_2, u_{e_2}, \mu_{e_2}) \otimes k[X]_{g_i}[Y]$ with $(f_i, f_j) = (g_i, g_j) = 1$, $i \neq j$ and $(f_i, g_j) = 1$ for all i, j.
(3) The bundles (P_1^i, h_1^i) and (P_2^i, h_2^i) are not extended from D^1 and D^2 respectively.

We define two families J_1^i and J_2^i of Jordan algebras on $\mathbb{A}_k^2$ as follows

$$J_1^i = J(P_1^i, h_1^i, \widetilde{\mu_1^i}), \qquad J_2^i = J(P_2^i, h_2^i, \widetilde{\mu_2^i}).$$

Let

$$\pi_1^i\colon (P_1^i, h_1^i, \widetilde{\mu_1^i})_{f_i} \simeq (D^1e_1, u_{e_1}, \mu_{e_1}) \otimes k[X]_{f_i}[Y]$$

and

$$\pi_2^i\colon (P_2^i, h_2^i, \widetilde{\mu_2^i})_{g_i} \simeq (D^2e_2, u_{e_2}, \mu_{e_2}) \otimes k[X]_{g_i}[Y]$$

be isometries such that $\overline{\pi_j^i}$ = identity for $j = 1, 2$. Then these induce isomorphisms

$$J(\pi_1^i)\colon J_1^i \otimes k[X]_{f_i}[Y] \simeq J \otimes k[X]_{f_i}[Y],$$

$$J(\pi_2^i)\colon J_2^i \otimes k[X]_{g_i}[Y] \simeq J \otimes k[X]_{g_i}[Y],$$

which reduce to identity modulo Y.

PROPOSITION 5.3. *The Jordan algebras J_1^i and J_2^i over $k[X, Y]$ have the following properties:*

(1) *J_1^i and J_2^i reduce modulo Y to J.*
(2) *$J_1^i \otimes k[X]_{f_i}[Y]$ is extended from $J \otimes k[X]_{f_i}[Y]$ and $J_2^i \otimes k[X]_{g_i}[Y]$ is extended from $J \otimes k[X]_{g_i}[Y]$ with $(f_i, f_j) = (g_i, g_j) = 1$, $i \neq j$ and $(f_i, g_j) = 1$ for all i, j. In particular, J_1^i are mutually nonisomorphic and the same holds for J_2^i.*
(3) *$J_j^i \otimes_k F = J(P_j^i, \widetilde{\mu_j^i})$ for $j = 1, 2$, where the Jordan algebras $J(P_j^i, \widetilde{\mu_j^i})$ are those constructed in Section 4 and F is as in (5.2). (Here we use the identification $J(P, h, \mu) \otimes_R S = J(P, \mu)$ mentioned in the remark after (2.3)).*

Proof. Properties 1 and 2 follow from the corresponding properties for $(P_j^i, h_j^i, \widetilde{\mu_j^i})$. The fact that J_1^i are mutually nonisomorphic follows from ([BCW]) provided we show that J_1^i is not extended from J. Since $J_1^i \otimes K \simeq J(P_1^i, \widetilde{\mu_1^i})$ is not extended from $J(D^1e_1, \widetilde{\mu_{e_1}})$, P_1^i being nonfree (4.5), it follows that J_1^i is not extended.

Proof of the assertions for J_2^i is similar. The third property follows from the very construction of these algebras. □

Let J^i be the Jordan algebra over $k[X, Y]$ obtained by patching $(J_1^i)_{g_i}$ on $k[X]_{g_i}[Y]$, $(J_2^i)_{f_i}$ on $k[X]_{f_i}[Y]$ over $k[X]_{f_i g_i}[Y]$, by the isomorphism

$$\psi_i\colon J_1^i \otimes k[X]_{f_i g_i}[Y] \simeq J_2^i \otimes k[X]_{f_i g_i}[Y],$$

defined by $\psi_i = J(\pi_2^i)^{-1} J(\pi_1^i)$. Since J_j^i reduce modulo Y to J and $\overline{\psi}_i =$ identity, J^i reduce modulo Y to J. Further, by the very construction,

$$J^i \otimes k[X]_{f_i g_i}[Y] \simeq J \otimes k[X]_{f_i g_i}[Y]$$

and the polynomials $s_i = f_i g_i$ satisfy $(s_i, s_j) = 1$, $i \neq j$. Arguing as in the proof of (4.6), one shows that J^i are mutually nonisomorphic. We record this in the following

PROPOSITION 5.4. *The Jordan algebras J^i over $\mathbb{A}_k^2$ have the following properties:*

(1) $\overline{J^i} = J \otimes k[X]$ *modulo* Y.
(2) *There exists* $\pi^i\colon J^i \otimes k[X]_{s_i}[Y] \simeq J \otimes k[X]_{s_i}[Y]$ *such that* $\overline{\pi^i} =$ identity, *for some* $s_i \in k[X]$ *with* $(s_i, s_j) = 1$ *for* $i \neq j$.
(3) J^i *are mutually nonisomorphic.*
(4) $J^i \otimes_k F = J_i$, *where* J_i *are the algebras constructed in Section 4.* □

6. F_4 Bundles with no Reduction of the Structure Group to any Proper Connected Reductive Subgroup

Let J be an exceptional Jordan division algebra over k. Let $G = \operatorname{Aut} J$. Then G is an anisotropic group of type F_4 over k. If J arises from a Tits first construction, then G_L is anisotropic for any extension L of degree coprime to 3. In fact, if $J = J(D, \mu)$ with $\mu \in k^*$ then G is isotropic if and only if J is split and this is so if and only if $\mu \in Nrd(D^*)$ ([J], Theorem 20, p. 416). In particular, if $\mu \notin Nrd(D^*)$, $\mu \notin Nrd(D_L^*)$, if $[L\colon k]$ is coprime to 3.

For the rest of the section we shall fix the following notation: if $\mathcal{G}$ is a simply connected group over $\overline{k}$, we say that a connected reductive group G over k is of *type* $\mathcal{G}$ if the simply connected cover of $[G_{\overline{k}}, G_{\overline{k}}]$ is a product of groups each isomorphic to $\mathcal{G}$. We say that a representation $\rho\colon G \to \mathrm{GL}(V)$ is faithful if the kernel of ρ is finite. We call a representation V of G of *type n* if it is a direct sum of irreducible representations each of which has dimension n.

PROPOSITION 6.1. *Let J be an exceptional Jordan division algebra arising from a Tits' first construction. Then the only possible proper connected reductive subgroups of* $G = \mathrm{Aut}(J)$ *over k are of type A_1, A_2 or D_4. Further, if a subgroup H*

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is of type A_1, *then the simply connected cover of* $[H, H]$ *is isomorphic to* $R_{L/k}H'$, *where* L *is a degree 3 extension of* k, H' *an absolutely almost simple group of type* A_1 *defined over* L *and* $R_{L/k}$ *denotes the Weil's restriction.*

Proof. Let H be a proper connected reductive subgroup of G. Then by Tits' classification of simply connected groups over k, it follows that the simply connected cover of $[H, H]$ is isomorphic to $\prod_{i=1}^{r} R_{L_i/k}H_i$ for some finite extensions L_i over k and absolutely almost simple groups H_i over L_i. Since the rank of G is 4, if H_i is not of type A_1, A_2 or D_4, then $[L_i : k] \leqslant 2$ and H_i becomes isotropic in an extension of degree 2^l for some l. Since G remains anisotropic over any finite extension of degree coprime to 3, it follows that each H_i is of type A_1, A_2 or D_4. Further, if H_i is of type A_1 for some i, then $[L_i : k] = 3$ and $r = 1$. □

PROPOSITION 6.2. *Let* J *be an exceptional Jordan division algebra arising from a Tits' first construction and* V *the space of trace zero elements of* J. *Then the action of any proper connected reductive subgroup of* $G = \mathrm{Aut}(J)$ *on* V *decomposes as* $V_1 \oplus V_2$, *with* $1 \leqslant \dim_k V_1 \leqslant 8$.

Proof. Let H be a proper connected reductive subgroup of G. Replacing H by $[H, H]$, we assume that H is semisimple. In view of ([PST], 7.5), under the action of H, V decomposes as $V_1 \oplus V_2$ with $V_i \neq 0$, $i = 1, 2$. If either of V_1 or V_2 is reducible for the action of H, then clearly there exists a nonzero H-stable subspace of V of dimension $\leqslant 8$. We therefore assume that V_1 and V_2 are irreducible representations of H and $\dim_k V_i \geqslant 9$, $i = 1, 2$. Without loss of generality we may assume that $9 \leqslant \dim_k V_1 \leqslant 13$.

Suppose that H is of type D_4. Since $H \subset G$ and rank of G is 4, H must be simple. Since G is anisotropic in any extension of degree 2^n, the simply connected cover of H is a trialitarian D_4 over k. The least dimension of a nontrivial irreducible representation over k of a trialitarian D_4 being 24, we get a contradiction.

Suppose that H is of type A_2. Since the actions of H on V_1 and V_2 are nontrivial, there exist simple factors H_1 and H_2 (possibly $H_1 = H_2$) of $H_{\overline{k}}$ such that H_1 acts nontrivially on V_1 and H_2 acts non-trivially on V_2. If the dimension of V_1 is a prime, then H_1 acts irreducibly on $V_{1_{\overline{k}}}$ and similarly if the dimension of V_2 is a prime, then H_2 acts irreducibly on $V_{2_{\overline{k}}}$. Since H is of type A_2, H_1 and H_2 are of type A_2. It follows from the table of formulae for dimensions of irreducible representations of simple groups over an algebraically closed field ([OV], p. 300–305), that the dimension of V_i is not equal to 11, 13 or 17, for $i = 1, 2$. Since $\dim_k V_1 + \dim_k V_2 = \dim_k V$ and $9 \leqslant \dim_k V_1 \leqslant 13$, it follows that $\dim_k V_1 = 10$ or 12. Suppose that $\dim V_1 = 10$. Then $\dim_k V_2 = 16$. Since V_2 is irreducible, $V_2 \otimes \overline{k}$ is isotypical of some type. Looking at the dimensions of irreducible representations of A_2, we see that $V_2 \otimes \overline{k}$ must decompose as a sum of two 8-dimensional irreducible representations of H. Since there is a unique irreducible representation of dimension 8 which is rational (the adjoint representation), V_2 itself must have a decomposition $V_2 = V_2' \oplus V_2''$ with each of the summands of dimension 8 (cf. [PST], 7.1), leading to a contradiction. Therefore $\dim_k V_1 = 12$ and $\dim_k V_2 = 14$.

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Since there are no irreducible representations of dimension 2, 7 or 14 for A_2 over $\overline{k}$, V_2 can not be irreducible for H_2, leading to a contradiction once again.

Suppose that the simply connected cover of H is isomorphic to $R_{L/k}H'$ for some degree 3 extension L over k and a simple group H' over L. In this case, we have $H_{\overline{k}} = H_1H_2H_3$, an almost direct product of simple groups of type A_1. We note that any irreducible representation of H is of the form $W_1 \otimes W_2 \otimes W_3$ for some irreducible H_i-representation W_i, $1 \leqslant i \leqslant 3$. Since there are no absolutely simple subgroups of G over k of type A_1, we assume that H is simple over k. Since the action of H on V_1 is nontrivial, it is also faithful. Therefore the action of $H_{\overline{k}}$ on $V_{1_{\overline{k}}}$ is faithful. Suppose that $\dim_k V_1 = 11$ or 13. Then one of the three simple factors should act trivially on V_1, leading to a contradiction. If $\dim_k V_1 = 9$ then $\dim_k V_2 = 17$ and arguing as above with V_2, we again get a contradiction. Therefore $\dim_k V_1 = 10$ or 12. Let us consider the case of $\dim_k V_1 = 10$. Then, as above, the action of $H_{\overline{k}}$ on $V_{1_{\overline{k}}}$ is not irreducible. Then, under the action of $H_{\overline{k}}$, $V_{1_{\overline{k}}}$ decomposes as $W_1 \oplus W_2$ with $\dim_{\overline{k}} W_i = 5$, $i = 1, 2$ or $V_{1_{\overline{k}}} = W_1 \oplus \cdots \oplus W_5$ with $\dim_{\overline{k}} W_i = 2$, $1 \leqslant i \leqslant 5$. In either case, by looking at the Galois action, one concludes that 3 divides 2 or 5, which is absurd. Therefore the $\dim_k V_1 = 12$. In this case $\dim_k V_2 = 14$, which is also seen to impossible, arguing as above with V_2.

Now the proposition follows from (6.1). $\square$

THEOREM 6.3. *Let J_i be the Jordan algebras on $\mathbb{A}^2_k$ constructed in Section 4. Then the corresponding principal G-bundle P_{J_i} admits no reduction of the structure group to any proper connected reductive subgroup of G.*

Proof. By an abuse of notation, we say that a Jordan algebra bundle admits a reduction of the structure group if the corresponding principal G-bundle admits such a reduction. Suppose that J_i admits a reduction of the structure group to a proper connected reductive subgroup H of G. By (6.2), the action of H on the space of trace zero elements of J decomposes as $V_1 \oplus V_2$ with $1 \leqslant \dim_k V_1 \leqslant 8$. Then $H \hookrightarrow (\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)) \cap G$. The restriction of J_i to $k[X]_{g_i}[Y]$ also has a reduction of the structure group to H. Since $J_i \otimes k\,[X]_{g_i}[Y] \simeq J_i^1 \otimes k\,[X]_{g_i}[Y]$, the bundle J_i^1 over $k[X, Y]$ has the property that $J_i^1 \otimes k[X]_{g_i}[Y]$ has a reduction of the structure group to H. Further, $J_i^1 \otimes k[X]_{f_i}\,[Y] \simeq J \otimes k[X]_{f_i}[Y]$ with $(f_i, g_i) = 1$. Hence by ([PST], 4.7), J_i^1 over $k[X, Y]$ admits a reduction of the structure group to H. Let $\widetilde{J_i^1}$ denote the extension of J_i^1 to $\mathbb{P}^2_k$ as a H-bundle. Then by the uniqueness of extension of G-bundles from $\mathbb{A}^2_k$ to $\mathbb{P}^2_k$ ([PST], 4.6), $\widetilde{J_i^1}$ is an extension of the Jordan algebra J_i^1 on $k[X, Y]$ and hence the underlying vector bundle of $\widetilde{J_i^1}$ has a decomposition (4.4) $(D_1 \otimes \mathcal{O}_{\mathbb{P}^2_k}) \oplus \widetilde{P_1^i} \oplus \widetilde{P_1^i}^{(*)}$ with $\widetilde{P_1^i}$ indecomposable (3.1). Further, $\widetilde{P_1^i}^{(*)} = \mathrm{Hom}_{D_1 \otimes \mathcal{O}_{\mathbb{P}^2_k}}(\widetilde{P_1^i}, D_1 \otimes \mathcal{O}_{\mathbb{P}^2_k})$ is an extension of $P_1^{i\,(*)} = \mathrm{Hom}_{D_1[X,Y]}\,(P_1^i, D_1[X, Y])$ to $\mathbb{P}^2_k$. Once again, by the uniqueness of the extension, $\widetilde{P_1^i}^{(*)} \simeq \widetilde{P_1^{i\,(*)}}$ is indecomposable by (3.1). Thus the trace zero sub-

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bundle $(\widetilde{J_i^1})_0$ decomposes as $(D_1 \otimes \mathcal{O}_{\mathbb{P}_k^2})_0 \oplus \widetilde{P_1^i} \oplus \widetilde{P_1^i}^{(*)}$ with $(D_1 \otimes \mathcal{O}_{\mathbb{P}_k^2})_0 =$ trace zero sub-bundle of $D_1 \otimes \mathcal{O}_{\mathbb{P}_k^2}$, which is trivial as a vector bundle, $\widetilde{P_1^i}$ and $\widetilde{P_1^{i\,(*)}}$ being indecomposable as vector bundles. However, $(\widetilde{J_i^1})_0$, being a $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$-bundle, decomposes as $(\widetilde{J_i^1})_0 = \mathcal{E}_1 \oplus \mathcal{E}_2$ with $\mathcal{E}_i$ a $\mathrm{GL}(V_i)$-bundle, $i = 1, 2$. Any direct summand of $(\widetilde{J_i^1})_0$ of rank $\leqslant 8$ must necessarily be contained in $(D_1 \otimes \mathcal{O}_X)_0$. Since the fibre of $(\widetilde{J_i^1})_0$ at $(0, 0)$ decomposes as $V_1 \oplus V_2$, specializing at $(0, 0)$, we conclude that $V_1 \hookrightarrow (D_1)_0 =$ trace zero elements in D_1. Arguing in a similar way, by restricting J_i to $k[X]_{f_i}[Y]$, we conclude that $V_1 \hookrightarrow (D_2)_0$ so that $V_1 \hookrightarrow (D_1)_0 \cap (D_2)_0 \hookrightarrow J$. Since $(D_1)_+ \cap (D_2)_+ = k$, $(D_1)_0 \cap (D_2)_0 = 0$ in J, leading to a contradiction. This proves the theorem. $\square$

THEOREM 6.4. *Let J^i denote the Jordan algebra bundle on $\mathbb{A}_k^2$ constructed in Section 5. Then the principal G-bundle P_{J^i} admits no reduction of the structure group to any proper connected reductive subgroup of G.*

Proof. By (6.3), $J^i \otimes_k F = J_i$ admits no reduction of the structure group to any proper connected reductive subgroup of G, where F is as in (5.2). Hence J^i itself admits no such reduction. $\square$

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