AN INEQUALITY FOR A PAIR OF POLYNOMIALS THAT ARE RELATIVELY PRIME

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(received 28 February 1964)

To T. M. Cherry

Let $f(x)$ and $g(x)$ be two polynomials with arbitrary complex coefficients that are relatively prime. Hence the maximum

$$m(x) = \max(|f(x)|, |g(x)|)$$

is positive for all complex x. Since $m(x)$ is continuous and tends to infinity with $|x|$, the quantity

$$E(f, g) = \min_{x} m(x)$$

is therefore also positive.

In the theory of transcendental numbers one often requires a good positive estimate for $E(f, g)$. The usual method for obtaining such an estimate is as follows. If $R(f, g)$ denotes the resultant of f and g, then identically in x

$$f(x)F(x)+g(x)G(x) = R(f, g)$$

where $F(x)$ and $G(x)$ are two polynomials that can be defined explicitly in terms of determinants. It follows that

$$m(x) \geqq |R(f, g)|/\{|F(x)|+|G(x)|\},$$

and hence it suffices to give an upper estimate for $|F(x)|+|G(x)|$. For this purpose one may assume that $|x|$ is not too large; for when $|x|$ is large, $m(x)$ trivially cannot be small. (See e.g. A. O. Gelfond, Transcendentnye i algebraitcheskie tchisla, Moskva 1952, pp. 181—2.)

In the present note I shall apply a different and better method that is due to N. Feldman. It has the additional advantage of leading to a best-possible result.

1. Let, in explicit form,

$$f(x) = a_0(x-\alpha_1)\cdots(x-\alpha_m), \quad g(x) = b_0(x-\beta_1)\cdots(x-\beta_n),$$

where $a_0 \neq 0$ and $b_0 \neq 0$, and where $\alpha_h \neq \beta_k$ for all h and k. Put, for any given complex number x,

$$\alpha = \min_{1 \leq h \leq m} |x-\alpha_h| \quad \text{and} \quad \beta = \min_{1 \leq k \leq n} |x-\beta_k|,$$

and denote by r and s two suffixes for which

$$\alpha = |x-\alpha_r| \quad \text{and} \quad \beta = |x-\beta_s|.$$

Then at least one of the two numbers α and β is positive.

Assume, first, that

$$0 < \alpha \leqq \beta$$

and number the zeros of $g(x)$ such that, say,

$$|\alpha_r-\beta_k| \begin{cases} < 2\alpha & \text{if} \quad k = 1, 2, \cdots, N, \\ \geqq 2\alpha & \text{if} \quad k = N+1, N+2, \cdots, n; \end{cases}$$

here N is a certain integer satisfying $0 \leqq N \leqq n$.

If $k = 1, 2, \cdots, N$, then

$$(1) \qquad |x-\beta_k| \geqq \beta \geqq \alpha > |\alpha_r-\beta_k|/2.$$

If, however, $k = N+1, N+2, \cdots, n$, then

$$|\alpha_r-\beta_k| \geqq 2\alpha = 2|x-\alpha_r|$$

and therefore

$$(2) \qquad |x-\beta_k| = |(x-\alpha_r)+(\alpha_r-\beta_k)| \geqq |\alpha_r-\beta_k|-|x-\alpha_r| \geqq |\alpha_r-\beta_k|/2.$$

On combining the inequalities (1) and (2) it follows that

$$|g(x)| = |b_0 \prod_{k=1}^{n} (x-\beta_k)| \geqq 2^{-n}|b_0 \prod_{k=1}^{n} (\alpha_r-\beta_k)|.$$

It is obvious that this formula remains true also when

$$\alpha = 0,$$

and hence we have proved that

$$|g(x)| \geqq 2^{-n}|g(\alpha_r)| \quad \text{if} \quad 0 \leqq \alpha \leqq \beta.$$

In exactly the same way it follows that

$$|f(x)| \geqq 2^{-m}|f(\beta_s)| \quad \text{if} \quad 0 \leqq \beta \leqq \alpha.$$

These two inequalities together imply the following result.

THEOREM 1. *Let $f(x)$ and $g(x)$ have the degrees m and n and the zeros $\alpha_1, \cdots, \alpha_m$ and $\beta_1, \cdots, \beta_n$, respectively. Then*

$$E(f, g) \geqq \min_{\substack{1 \leq h \leq m \\ 1 \leq k \leq n}} (2^{-m}|f(\beta_k)|, 2^{-n}|g(\alpha_h)|).$$

https://doi.org/10.1017/S1446788700025246 Published online by Cambridge University Press

This result is best possible because the assertion holds with equality in the special case when f and g are the two polynomials

$$(3) \qquad f(x) = (x-1)^m \quad \text{and} \quad g(x) = (x+1)^n.$$

2. Theorem 1 gives a lower bound for $E(f, g)$ in terms of the zeros of f and g. It is now not difficult to replace this estimate by one that involves instead only the coefficients of these two polynomials.

Let in explicit form

$$f(x) = a_0x^m+a_1x^{m-1}+\cdots+a_m, \quad g(x) = b_0x^n+b_1x^{n-1}+\cdots+b_n.$$

Further denote by

$$L(f) = |a_0|+|a_1|+\cdots+|a_m|, \quad L(g) = |b_0|+|b_1|+\cdots+|b_n|$$

the lengths of the two polynomials. By a theorem of R. Güting [1],

$$|f(\beta_k)| \geqq |R(f, g)|/L(f)^{n-1}L(g)^m, \quad |g(\alpha_h)| \geqq |R(f, g)|/L(f)^nL(g)^{m-1}$$

for all suffixes h and k. Hence, by Theorem 1,

$$E(f, g) \geqq |R(f, g)|L(f)^{-n}L(g)^{-m} \min \{2^{-m}L(f), 2^{-n}L(g)\}.$$

For the applications, the most important case is that of polynomials with integral coefficients. The resultant $R(f, g) \neq 0$ is then also an integer and hence its absolute value is not less than 1. Therefore, in this particular case,

$$E(f, g) \geqq L(f)^{-n}L(g)^{-m} \min \{2^{-m}L(f), 2^{-n}L(g)\}.$$

While this formula is very simple, it is, however, no longer best possible.

Canberra, 19 February, 1964.

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[1] Approximation of algebraic numbers by algebraic numbers, Michigan Math. J. 8 (1961), 149—159.

https://doi.org/10.1017/S1446788700025246 Published online by Cambridge University Press