ON VERY LARGE ONE SIDED IDEALS OF A RING

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(received December 16, 1965)

1. Introduction. If R is a ring, a right (left) ideal of R is said to be large if it has non-zero intersection with each non-zero right (left) ideal of R [8]. If S is a set, let $|S|$ be the cardinal number of S. We say a right (left) ideal I of a ring R is very large if $|R/I| < \aleph_0$. If a is an element of a ring R such that $(a)^r = \{r \in R \mid ar = 0\}$ is very large then we say a is very singular. The set of all very singular elements of a ring R is a two sided ideal of R. If R is a prime ring, then 0 is the only very singular element of R and a very large right (left) ideal of R is indeed large provided that R is not finite. In case R is a simple ring, every non-zero right (left) ideal of R is very large if and only if either R is finite or R is a division ring. If R is a prime ring with 1 such that the characteristic of R is zero, then R is a right order in a simple ring with minimum condition on one-sided ideals if every large right ideal of R is very large. In case R is a primitive ring with 1 such that the characteristic of R is zero, then R is a simple ring with minimum condition on one-sided ideals if and only if every large right ideal of R is very large.

2. If R is a ring, let R_r^{Δ} be the right singular ideal of R and let

$$Z(R) = \{a \in R \mid (a)^r \text{ is very large}\} .$$

PROPOSITION 2.1 If I and J are very large right (left) ideals of a ring R then $I \cap J$ is a very large right (left) ideal of R.

Proof. Since $|R/I| < \aleph_0$ and $|R/J| < \aleph_0$, $|R/I \cap J| < \aleph_0$ by Poincaré's theorem [6: p. 40, Exercise 3].

Canad. Math. Bull. vol. 9, no. 2, 1966

https://doi.org/10.4153/CMB-1966-025-9 Published online by Cambridge University Press

PROPOSITION 2.2 If I is a very large right ideal of a ring R then, for any $x \in R$, the set $(I:x) = \{r \in R \mid xr \in I\}$ is a very large right ideal of R.

Proof. Define a mapping f by $f(r) = xr + I$ for all $r \in R$. Then f is an R-homomorphism from the (right) R-module R onto the R-module $(xR + I)/I$. Since the kernel of f is $(I:x)$, $R/(I:x) \cong (xR + I)/I$. Hence

$$|R/(I:x)| = |(xR + I)/I| \leq |R/I| < \aleph_0.$$

THEOREM 2.3. If R is a ring, $Z(R)$ is a two sided ideal of R.

Proof. If $x, y \in Z(R)$ then $(x-y)^r \supseteq (x)^r \cap (y)^r$. Hence $|R/(x-y)^r| \leq |R/(x)^r \cap (y)^r|$ by [5: Theorem 1.5.3, p.12]. Since $|R/(x)^r \cap (y)^r| < \aleph_0$ by Proposition 2.1, $x - y \in Z(R)$. If $r \in R$, $x \in Z(R)$ then $(rx)^r \supseteq (x)^r$. Hence $rx \in Z(R)$. Now consider $(xr)^r$. By Proposition 2.2, $((x)^r : r)$ is very large. Thus $xr \in Z(R)$, since $(xr)^r \supseteq ((x)^r : r)$.

THEOREM 2.4. If R is a ring such that $Z(R) = 0$, then a very large right ideal of R is large.

Proof. Suppose there exists a very large right ideal I of R such that I is not large. Then there exists a non-zero right ideal J of R such that $I \cap J = 0$. Define a mapping f from the R-module J onto the R-module $(J + I)/I$ by $f(j) = j + I$ for all $j \in J$. Since $I \cap J = 0$, f is an isomorphism. Hence $|J| = |(J + I)/I| \leq |R/I| < \aleph_0$. Let $j \in J$ such that $jR \neq 0$ (if $jR = 0$ for all $j \in J$ then $J \subseteq Z(R)$). Then $jR \cong R/(j)^r$. Thus $|R/(j)^r| = |jR| \leq |J| < \aleph_0$ and $0 \neq j \in Z(R)$. This is impossible.

THEOREM 2.5. If R is a semi-prime ring, then $R_r^\Delta \cap Z(R) = 0$.

Proof. Let $x \in R_r^\Delta \cap Z(R)$ such that $x \neq 0$. Then $|R/(x)^r| < \aleph_0$. Since R is a semi-prime ring, $0 < |R/(x)^r|$.

https://doi.org/10.4153/CMB-1966-025-9 Published online by Cambridge University Press

Since $xR \cong R/(x)^r$ and $0 < |xR| < \aleph_0$, there must exist a minimal right ideal I of R such that $I \subseteq xR \subseteq R_r^\Delta \cap Z(R)$. Hence if $i \in I$ then $(i)^r \cap I \neq 0$ and $iI = 0$. This is impossible, since $I^2 \neq 0$.

THEOREM 2.6. If R is a prime ring which is not finite then $Z(R) = 0$.

Proof. If $x \in Z(R)$ such that $x \neq 0$, then $|xR| = |R/(x)^r| < \aleph_0$. Hence R is a primitive ring with a minimal right ideal which is finite. Thus by [7, Theorem 3, p. 33] R is a finite ring. This is a contradiction.

COROLLARY 2.7. If R is a prime ring which is not finite, then a very large right ideal of R is large.

Proof. This is a consequence of Theorem 2.6 and Theorem 2.4.

COROLLARY 2.8. If R is a prime ring which is not finite, and is such that every non-zero right ideal of R is very large then R is a right Ore domain.

Proof. By hypothesis and by Corollary 2.7, if $a \in R$, $a \neq 0$, such that $(a)^r \neq 0$, then $a \in Z(R)$. This is impossible by Theorem 2.6. Thus R is a right Ore domain.

3. It is well known that if R is a ring with 1 such that every right (left) ideal of R is a direct summand of R, then R is a semi-simple ring with the minimum conditions on one sided ideals (See [2, Theorem 4.2, p. 11]). For a later reference, we will state the following trivial improvement of the above fact.

LEMMA 3.1*. If R is a ring with 1 such that each maximal right (left) ideal of R is not large, then R is a semi-simple ring with the minimum condition on one sided ideals.

Proof. Let F be the right socle of R. If $1 \notin F$, then

* The author proposed this lemma as a problem in the Canad. Math. Bull. vol. 8, No. 1, 1965.

https://doi.org/10.4153/CMB-1966-025-9 Published online by Cambridge University Press

by Zorn's Lemma there exists a maximal right ideal, say I of R, such that $I \supseteq F$. Since I is not large, there is a non-zero right ideal J of R such that $I \cap J = 0$. $I \oplus J = R$, since I is maximal. Hence J is a minimal right ideal of R which is not contained in F. This is impossible. Thus $1 \in F$ and $F = R$. From [1, Theorem 11, p.61], the assertion follows.

LEMMA 3.2. If every large right ideal of a simple ring R with 1 is very large, then R satisfies the minimum conditions on one-sided ideals.

Proof. If R is finite then clearly the assertion is true. Suppose R is not finite. Let I be a maximal right ideal of R. If I is large then $|R/I| < \aleph_0$. Hence there exists a finite number, n, of elements $\{x_i\}_{i=1}^{n}$ such that $R = \bigcup_{i=1}^{n} \{x_i + I\}$. By Proposition 2.2 and Corollary 2.7, $(I:x_i)$ is a large right ideal of R for each $i = 1, 2, \ldots, n$. Hence $K = \bigcap_{i=1}^{n} (I:x_i)$ is a non-zero right ideal of R. Since $R = \bigcup_{i=1}^{n} \{x_i + I\}$, $RK \subseteq I$. Since R is simple, this implies that $I = R$. This is impossible. Thus every maximal right ideal of R is not large. Now by Lemma 3.1, the assertion is true.

COROLLARY 3.3. If R is a simple ring, then R is finite or a division ring if and only if every non-zero right (left) ideal of R is very large.

Proof. It suffices to prove that every non-zero right ideal of R is R in case R is not finite. However, this follows from the proof of Lemma 3.2.

THEOREM 3.4. Let R be a prime ring with 1 such that the characteristic of R is zero. Then R is a right order in a simple ring with the minimum condition on one-sided ideals if every large right ideal of R is very large.

Proof. Since the characteristic of R is zero, R is not finite. Hence by hypothesis and Theorem 2.5, $R_r^{\Delta} = 0$. By [8, Theorem 3] and [9, Theorem 2.7], the maximal right quotient ring $\hat{R}$ of R is a prime ring which is regular (von Neumann). Let $\hat{I}$ be a large right ideal of $\hat{R}$. Then

$I = \hat{I} \cap R$ is a large right ideal of R. Hence I is very large. Thus there exists a positive integer n such that $nR \subseteq I$. Let $a = n1$. If $a = 0$ then the characteristic of R is zero. Hence $a \neq 0$. If $(a)^r \neq 0$ then for any $t \in (a)^r$, $r \in \hat{R}$, $art = n(rt) = rnt = r0 = 0$. Hence $(a)^r$ would be a non-zero two-sided ideal of a prime ring $\hat{R}$. This is impossible. Thus $(a)^r = 0$. Since $\hat{R}$ is regular, there exists $x \in \hat{R}$ such that $axa = a$ and $a(xa - 1) = 0$. Since $(a)^r = 0$, this implies that $xa = 1$. Now $xa = xn1 = nx = ax$. Thus $1 = ax \in \hat{I}$ and $\hat{I} = \hat{R}$. Thus, by Lemma 3.1, $\hat{R}$ is a simple ring with minimum conditions on one-sided ideals. By [4, Proposition 5.6] and [9, Theorem 4.2], R is a right order in a simple ring with minimum conditions on one-sided ideals.

THEOREM 3.5. If R is a primitive ring with 1 such that the characteristic of R is zero, then R is a simple ring with minimum condition on one-sided ideals if and only if every large right ideal of R is very large.

Proof. If R is a simple ring with the minimum condition on one sided ideals, then R is a direct sum of a finite number of minimal right ideals. Hence any large right ideal of R must be R itself. Conversely, suppose every large right ideal of R is very large and R is not finite. Let M be a faithful simple R-module. If $m \in M$, $m \neq 0$, then $(m)^r = \{r \in R \mid mr = 0\}$ is not large. Otherwise, $|R/(m)^r| < \aleph_0$ and $mR = M$ would be a finite set. In this case R is a finite ring by [7, Theorem 3, p. 33]. If $(m)^r$ is not large then there is a non-zero right ideal I of R such that $(m)^r \cap I = 0$, and $(m)^r \oplus I = R$ since $(m)^r$ is a maximal right ideal of R. Hence I must be a minimal right ideal of R. By Theorem 3.4, the maximal right quotient ring $\hat{R}$ of R is a simple ring with minimum conditions on right ideals. Let $\hat{I}$ be a minimal injective hull of the R-module I which is contained in $\hat{R}$. Then $\hat{I}$ is a right ideal of $\hat{R}$ and $\hat{I}$ is a minimal right ideal of $\hat{R}$ by [10, Lemma 2.2]. Now by [3, Theorem 2], $Hom_{\hat{R}}(\hat{I}, \hat{I})$ is a right quotient ring of $Hom_R(I, I)$. Since $Hom_R(I, I)$ is a division ring, $Hom_{\hat{R}}(\hat{I}, \hat{I}) = Hom_R(I, I)$. Since the dimension of $Hom_{\hat{R}}(\hat{I}, \hat{I})$ - space $\hat{I}$ is finite so is the dimension of

https://doi.org/10.4153/CMB-1966-025-9 Published online by Cambridge University Press

$Hom_R(I, I)$ - space I. Thus R is a simple ring with minimum conditions on one sided ideals.

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https://doi.org/10.4153/CMB-1966-025-9 Published online by Cambridge University Press