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A STUDY OF UNIFORM ONE-SIDED IDEALS IN SIMPLE RINGS

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Abstract. Using a variation on the concept of a CS module, we describe exactly when a simple ring is isomorphic to a ring of matrices over a Bézout domain. Our techniques are then applied to characterise simple rings which are right and left Goldie, right and left semihereditary.

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1. Introduction. In what follows, rings are always associative with identity, and modules are unitary modules. For a module M and a positive integer n, M^n denotes the direct sum of n copies of M.

Recall that a module M is called a *CS module* or an *extending module* if every complement (closed) submodule is a direct summand, or equivalently, if every submodule of M is essential in a direct summand. A ring R is called a *right CS ring* if the right R-module R_R is CS. Since their introduction almost thirty years ago by Chatters and Hajarnavis in [**3**], CS modules and rings have been studied extensively (in particular by Japanese algebraists), feature prominently in Mohamed and Müller's [**16**], and are the focus of the text [**9**] by Dung et al.

Recently, Hanada, Kuratomi and Oshiro in [**11**] have investigated CS modules of three different types, listed there as (A), (B) and (C). While their type (A) modules are just unmodified CS modules, those of types (B) and (C) are defined using decompositions of the module and involve the so-called "internal exchange property". However, for the ring $\mathbb{Z}$ of integers, although the $\mathbb{Z}$-module $\mathbb{Z}^n$ is CS for any $n \in \mathbb{N}$, $\mathbb{Z}^n$ is not CS under either types (B) or (C) of [**11**]. On the other hand, each uniform direct summand of $\mathbb{Z}^n$ is isomorphic to $\mathbb{Z}$ as a $\mathbb{Z}$-module. Motivated by this we give the following definition.

Definition. Given $n \in \mathbb{N}$, a uniform R-module U is called an $n-\mathrm{CS}^+$ module if U^n is CS and each uniform direct summand of U^n is isomorphic to U (as R-modules).

The $^+$ here is meant to indicate that this notion is slightly stronger than the usual CS condition. In section 2 we will assume that our CS^+ condition holds for some uniform right ideal of a simple ring R and use this to describe the structure of R using Bézout domains. Section 3 will deal with conditions that imply when a simple ring is semihereditary.

2. Simple rings that are full matrix rings over Bézout domains. Recall that a (not-necessarily commutative) integral domain D is called a *right* (*left*)*Bézout domain* if every finitely generated right (left) ideal of D is principal. A domain is called *Bézout* if it is both right and left Bézout. Basic properties of these domains are developed by Cohn in his text [**5**]. In particular, [**5**, Proposition 1.7] shows that a right Bézout domain is necessarily right Ore.

Although, by definition, the only nonzero ideal of a simple ring R is R itself, the one-sided ideal structure of R and the associated categories of left and right R-modules may be quite complex. This is evidenced in the texts of Cozzens and Faith [**8**], Goodearl [**10**], and McConnell and Robson [**15**]. Our purpose here is to show that, under an $n-\mathrm{CS}^+$ assumption, the structure of a simple ring can be described in terms of Bézout domains. We first give a simple but effective lemma, prompted by the proof of Hart's [**12**, Theorem 1].

LEMMA 2.1. *Let A be any nonzero right ideal of a simple ring R. Then, for some $n \in \mathbb{N}$, there are elements $x_1, x_2, \ldots, x_n \in R$ such that $R_R = x_1A + x_2A + \cdots + x_nA$. As a consequence, R_R is isomorphic to a direct summand of A^n.*

Proof. Since R is simple and RA is a nonzero two-sided ideal of R, we have $R = RA$. Thus, for some $n \in \mathbb{N}$, there are elements $x_i \in R$, $a_i \in A$ for $1 \le i \le n$ giving $1 = x_1a_1 + x_2a_2 + \cdots + x_na_n$. It now quickly follows that $R = x_1A + x_2A + \cdots + x_nA$.

Furthermore, we have an epimorphism $\varphi : A^n \to R$ defined by setting $\varphi : (b_1, b_2, \ldots, b_n) \mapsto x_1b_1 + x_2b_2 + \cdots + x_nb_n$ for all $b_1, b_2, \ldots, b_n \in A$. The projectivity of R_R splits φ and so R_R is isomorphic to a direct summand of A^n, as claimed. $\square$

We now prove our main result. It shows that the presence of a uniform right ideal with $2-\mathrm{CS}^+$ in a simple ring has a strong influence on the structure of the ring.

THEOREM 2.2. *Let R be a simple ring. Then the following conditions are equivalent.*

(i) *R contains a uniform right ideal V such that V_R is $2-\mathrm{CS}^+$.*

(ii) *R contains a uniform left ideal T such that ${}_RT$ is $2-\mathrm{CS}^+$.*

(iii) *R is isomorphic to the $k \times k$ matrix ring over a Bézout domain D for some $k \in \mathbb{N}$.*

Proof. (i)$\Rightarrow$(iii). First note that, since it has a uniform right ideal V, R has finite right uniform dimension, by Hart's [**12**, Theorem 2].

We will show that V is $n-\mathrm{CS}^+$ for any positive integer $n \ge 2$. We prove this in two steps. To start with we show that every closed uniform submodule of V^n is a direct summand, in other words that V^n satisfies the so-called $(1-C_1)$ condition or is *uniform-extending* (see [**9**]). Then we show that each closed submodule of V^n is a direct summand, i.e. that V^n is CS.

Step 1. We claim that, for any $n \ge 2$, each uniform closed submodule of V^n is a direct summand that is isomorphic to V. This is true for $n = 2$ by our assumption (i). Suppose it is true for a fixed $n \ge 2$, i.e., each closed uniform submodule of V^n is a direct summand of V^n which is isomorphic to V. Now let W be a uniform closed submodule of $V^{n+1} = V^n \oplus V$. If $W \cap V^n \ne 0$, then by the nonsingularity of V^{n+1}, it follows that $W \subseteq V^n$. Then, by our induction hypothesis, W is a direct summand of V^n isomorphic to V, as required. On the other hand, suppose that $W \cap V^n = 0$. In this case, if W intersects the $(n+1)$th direct summand V nontrivially, then $W = V$ and we are done. If, instead, $W \cap V = 0$ we set $M = V \oplus W$. Then, by modularity, we have $M = V \oplus N$ where $N = M \cap V^n$. It follows that N is a uniform submodule of

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V^n and so, by the induction hypothesis, N is essential in a direct summand N^* of V^n where $N^* \cong V$. Let $M^* = V \oplus N^*$. Then M^* is a direct summand of V^{n+1}. Also W is a submodule of M^* and so, since it is a closed submodule of V^{n+1}, W is a closed submodule of M^*. On the other hand, $M^* \cong V^2$ and so every uniform direct summand of M^* is isomorphic to V. Then, since W is a closed submodule of M^*, it is a direct summand of M^* and is isomorphic to V. But M^* is a direct summand of V^{n+1}, and hence so also is W. This establishes our claim.

Step 2. It is not difficult to see that, for any direct summand H of V^n, every uniform direct summand of H is isomorphic to V. Let A be any closed nonzero submodule of V^n. Then A contains a closed uniform submodule U_1 of V^n (cf. [**3**, Proposition 2.2]). By step 1, $V^n = U_1 \oplus V_1$. By modularity, $A = U_1 \oplus A_1$ where $A_1 = A \cap V_1$. As a closed submodule of A, A_1 is a closed submodule of V_1 (again see [**3**, Proposition 2.2]). If $A_1 \neq 0$, A_1 again contains a closed uniform submodule U_2 of V_1 that is clearly a direct summand of V_1. Continuing in this way we finally see that A is a direct summand of V^n.

By Lemma 2.1, $R_R = \sum_{j=1}^{n} x_j V$ for some $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in R$, and R_R is isomorphic to a direct summand of V^n. Hence R_R is CS and each of its uniform direct summands is isomorphic to V. Noting that, by [**12**, Theorem 2], R has finite right uniform dimension, we then have the following information:

(a) V_R is projective and principal and
(b) $R_R \cong V^k$ where $k =$ u-dim(R_R), the right uniform dimension of R.

Now let W be a finitely generated uniform right ideal of R, say $W = w_1R + w_2R + \cdots + w_mR$ for some $w_i \in W$, for $i = 1, 2, \ldots, m$. Then, substituting $R = \sum_{j=1}^{n} x_j V$ (from above), we get

$$W = \sum_{i=1}^{m} w_i R = \sum_{i=1}^{m} w_i \left(\sum_{j=1}^{n} x_j V \right) = \sum_{i=1}^{m} \sum_{j=1}^{n} w_i x_j V$$

and so there are elements z_k in D, for $k = 1, 2, \ldots, mn$, such that $W = z_1V + z_2V + \cdots + z_{mn}V$. Then, as in the proof of Lemma 2.1, we see that there is an epimorphism ψ of V^{mn} onto W. Since V is $mn - \mathrm{CS}^+$, W is isomorphic to a uniform direct summand of V^{mn}. Thus $W \cong V$ and so, in particular, W_R is projective and principal.

On the other hand, from (b) we see that $R \cong \mathrm{End}_R(V^k) \cong M_k(D)$ where $D = \mathrm{End}_R(V)$. It follows that every finitely generated uniform right ideal of D is a principal right ideal. Moreover, by [**12**, Theorem 1] D is a right Ore domain. Thus D is a right Bézout domain. In particular, R is a right Goldie ring (see also [**12**]).

Finally we look at the left side of R. For this we consider the uniform dimension k of R_R.

Case 1. $k = 1$. In this case (i) implies that R^n is a CS right R-module for any $n \in \mathbb{N}$. Hence, by [**9**, Corollary 12.9], R is a right and left Ore domain. As above we see that R is a right Bézout domain. This and [**5**, Proposition 1.9] show that every finitely generated left ideal of R is free. But R is a left Ore domain, hence every finitely generated left ideal of R is principal. Thus R is a left Bézout domain.

Case 2. $k \geq 2$. By Huynh et al. [**13**], R is then left CS. We have seen that every uniform direct summand of R_R is isomorphic to V. But every uniform direct summand of R is generated by a primitive idempotent. Therefore, for any pair of primitive idempotents $e, f \in R$, $eR \cong fR$ and so, by Anderson and Fuller [**1**, Proposition 17.18], we have $Re \cong Rf$. As ${}_RR$ is CS, ${}_RR$ is then a direct sum of isomorphic uniform left ideals. Moreover, if U is a uniform direct summand of ${}_RR$, then $U = Rg$ for some primitive idempotent $g \in R$. But $gR \cong eR$ for any primitive idempotent $e \in R$, so $Rg \cong Re$. This implies that all uniform direct summands of ${}_RR$ are isomorphic to each other. Now, arguing as we did for the right side of R, we see that R is isomorphic to the full matrix ring over a left Bézout domain $\mathrm{End}_R({}_RT)$ for some uniform direct summand T of ${}_RR$.

It is clear that if $e \in R$ is an idempotent, then eR is uniform if and only if Re is uniform. Moreover, $\mathrm{End}_R(V) \cong \mathrm{End}_R(eR) \cong \mathrm{End}_R(Re) \cong eRe$ (see e.g. [**1**, Proposition 4.15]). But as we saw above, eRe is a right and left Bézout domain. This proves (iii).

It is clear that (iii) $\Rightarrow$ (i) and so (ii) $\Leftrightarrow$ (iii) follows by the symmetry of (iii). $\square$

Corollary 2.3. *Let R be a simple ring with ACC on principal right and principal left ideals. Then the following conditions are equivalent.*

(i) *R contains a uniform right ideal V such that V_R is $2-\mathrm{CS}^+$.*
(ii) *R contains a uniform left ideal T such that ${}_RT$ is $2-\mathrm{CS}^+$.*
(iii) *R is isomorphic to the $k \times k$ matrix ring over a principal right and principal left ideal domain D for some $k \in \mathbb{N}$.*

Proof. By Theorem 2.2, we only need to show that if (i) holds then the domain D of Theorem 2.2 is left and right noetherian. By the proof of the Theorem, we have $R = V^k$ for some $k \in \mathbb{N}$, where V is the given uniform right ideal of R, and, if R has also ACC on principal right ideals, each submodule of V_R is principal. Hence R is right noetherian. Similarly R is also left noetherian. Consequently, D is both left and right noetherian as required. $\square$

Remark 2.4. The question arises as to whether Corollary 2.3 remains true if we assume the ACC only for principal right ideals. However, the answer to this is negative because Cohn and Schofield give an example in [**7**] (see also Cohn [**6**]) of a simple Bézout domain D which is a right principal ideal domain but not left principal. Obviously, each full matrix ring $R = M_n(D)$ over D is a ring satisfying the conditions of Theorem 2.2. In this case R is right noetherian, but not left noetherian. Hence our rings in Theorem 2.2 are generally not noetherian. More interestingly, this ring R is an example showing that although condition (i) of Theorem 2.2 is left-right symmetric, this symmetry is not a perfect one in the sense that while every finitely generated uniform right ideal of R is principal, there are uniform left ideals of R that are not principal. The following corollary gives more information on this.

Corollary 2.5. *Let R be a simple ring with a uniform right ideal V as in* (i) *of Theorem 2.2, so that R is isomorphic to the full matrix ring $M_k(D)$ for some Bézout domain D and some $k \in \mathbb{N}$. If D is a principal right ideal domain the following conditions are equivalent.*

(i) *D is a principal left ideal domain.*
(ii) *Each uniform left ideal of D is finitely generated.*
(iii) *Each maximal left ideal of D is finitely generated.*
(iv) *Each countably generated left ideal of D is projective.*
(v) *D satisfies the left restricted minimum condition.*

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Proof. This is a consequence of a result of Camillo and Cozzens, namely [**2**, Theorem 5]. (For (iii) $\Rightarrow$ (i), note that every simple left R-module is finitely presented and so, by [**2**, Lemma 1], embeds in the left R-module $Q \oplus (Q/D)$ where Q is the left and right classical quotient ring of D; it then follows from [**2**, Theorem 2] that D is left noetherian.) □

Now suppose that R is a ring of Theorem 2.2, that all uniform right ideals of R are principal, but there is a uniform left ideal U of R which is not finitely generated. By the proof of Theorem 2.2, there exists a principal uniform left ideal V of R with $R \cong M_n(\mathrm{End}(_R V))$ for some $n \in \mathbb{N}$. The left analogue of Lemma 2.1 also gives $R = Ux_1 + Ux_2 + \cdots + Ux_m$ for some $m \in \mathbb{N}$ and $x_i \in R$. By [**12**, Lemma 1], there is an idempotent $e \in M_m(\mathrm{End}(_R U))$ such that $R \cong eM_m(\mathrm{End}(_R U))e$. On the other hand, this isomorphism does not in general give a Morita equivalence between R and $\mathrm{End}(_R U)$ since being a simple ring is a Morita invariant (see, e.g., [**1**, Corollary 21.12]) but, by Hart [**12**,, Lemma 4 and Theorem 5], $\mathrm{End}(_R U)$ is not simple. We wonder what relationship exists between the two domains $\mathrm{End}(_R V)$ and $\mathrm{End}(_R U)$.

3. Simple rings with a finitely Σ-CS uniform one-sided ideal. Recall that a module M is said to be *finitely* Σ-*CS* if M^n is CS for any $n \in \mathbb{N}$. If the right R-module R_R is finitely Σ-CS, then R is called a *right finitely* Σ-*CS ring* and left finitely Σ-CS rings are defined similarly.

THEOREM 3.1. *For a simple ring R the following conditions are equivalent.*
(i) *R contains a finitely Σ-CS uniform right ideal V.*
(ii) *R contains a finitely Σ-CS uniform left ideal T.*
(iii) *R is left and right Goldie, left and right finitely Σ-CS.*
(iv) *R is left and right Goldie, left and right semihereditary.*

Proof. (i) $\Rightarrow$ (iii). By [**12**], R is right Goldie and, by Lemma 2.1, there is a positive integer m such that R_R is a direct summand of V^m. Hence, for any positive integer k, R^k is a direct summand of $(V^m)^k = V^{mk}$. This proves that $(R^k)_R$ is CS for any $k \in \mathbb{N}$, i.e., R_R is finitely Σ-CS.

We consider the case $k > 1$. As R^k is CS, $M_k(R)$ is CS by [**9**, 12.8]. Since $M_k(R)$ is a simple right Goldie ring with uniform dimension at least 2, it follows that $M_k(R)$ is left CS and left Goldie by [**13**]. Then, again by [**9**, 12.8], $_R(R^k)$ is CS, proving that $_RR$ is finitely Σ-CS. Of course, R is left Goldie.

The implications (iii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii) are clear, and (iv) $\Leftrightarrow$ (iii) holds by [**9**, 12.18]. □

Notice that in the proof of Theorem 2.2, using the assumption that V is $2-\mathrm{CS}^+$, we were able to establish that V is $n-\mathrm{CS}^+$ for any $n \in \mathbb{N}$. However, assuming simply that V^2 is CS, we have been unable to show that even V^3 is CS. Therefore we pose the following question.

QUESTION 3.2 *Let R be a simple right Goldie ring. Is R necessarily right finitely Σ-CS if $(R^2)_R$ is CS?*

If $(R^2)_R$ is CS, then $M_2(R)$ is right CS (cf. [**9**, 12.8]). Then, if R is also a simple right Goldie ring, it follows from [**13**] that $M_2(R)$ is left Goldie and, consequently, so is R. Thus, for any $n \in \mathbb{N}$, $M_n(R)$ is a simple right and left Goldie ring. It now follows from [**9**, 12.6] that every right (left) closed ideal of $M_n(R)$ is a right (left) annihilator. Thus the ring $M_n(R)$ is right and left CS if and only if $M_n(R)$ is a Baer ring (cf. [**9**, 12.7]). (Here a ring is *Baer* if each of its right annihilators, or equivalently each of its left annihilators, is generated by an idempotent.)

Secondly, comparing with a result of Huynh et al. in [**14**], we see that any finitely generated right or left ideal of the ring R in Theorem 3.1 is CS. Moreover, the following holds.

COROLLARY 3.3. *Let R be a simple ring with a uniform right ideal. Then the following conditions are equivalent.*

(i) *R has a nonzero finitely Σ-CS right ideal A.*

(ii) *Every finitely generated right ideal of R is finitely Σ-CS.*

(iii) *Every finitely generated left ideal of R is finitely Σ-CS.*

(iv) *R is left and right Goldie, left and right semihereditary.*

Proof. R is right Goldie by [**12**, Theorem 2]. Thus Theorem 3.1 gives (i) $\Leftrightarrow$ (iv) and (ii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (ii). By [**9**, Corollary 12.18], R is right finitely Σ-CS. Let A be a finitely generated right ideal of R. Since R is right semihereditary, A_R is projective and so isomorphic to a summand of R^m for some $m \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$, A^k is isomorphic to a direct summand of $(R^m)^k = R^{mk}$. Hence A^k is CS for each $k \in \mathbb{N}$, giving (ii). Similarly we obtain (iv) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv). By (iii), ${}_RR$ in particular is finitely Σ-CS. Hence, by [**9**, Corollary 11.4], R is left semihereditary. Since R is right Goldie, R is also right semihereditary (cf. Small [**17**]). Now since ${}_R(R^2)$ is CS, the ring $M_2(R)$ is left CS (cf. [**9**, Lemma 12.8]). By [**13**, Theorem 1], $M_2(R)$ is then left Goldie and so R is left Goldie. This proves (iv). □

For a module M and a set I we let $M^{(I)}$ denote the direct sum of $|I|$ copies of M.

COROLLARY 3.4. *For a simple ring R the following conditions are equivalent.*

(i) *R contains a uniform right ideal V such that $V^{(\mathbb{N})}$ is CS.*

(ii) *R contains a uniform left ideal T such that $T^{(\mathbb{N})}$ is CS.*

(iii) *R is right and left artinian, i.e., $R \cong M_n(S)$ for some division ring S and some $n \in \mathbb{N}$.*

Proof. (i) $\Rightarrow$ (iii). By Lemma 2.1, R_R is isomorphic to a direct summand of V^m for some $m \in \mathbb{N}$. Hence $R^{(\mathbb{N})}$ is isomorphic to a direct summand of $V^{(\mathbb{N})}$ and so $R^{(\mathbb{N})}$ is also CS. Then, since R is a right Goldie ring, R is semisimple artinian by Clark and Wisbauer [**4**, Corollary 2.9]. This show that (iii) holds.

The implication (iii) $\Rightarrow$ (i) is clear, while (iii) $\Leftrightarrow$ (ii) follows from the symmetry of (iii). □

REMARK 3.5. Further to the results above, the authors have shown that if R is a simple ring in which every right ideal is the direct sum of quasi-continuous right ideals then either R is artinian or R is a non-selfinjective right Goldie ring in which each right ideal is a direct sum of uniform right ideals (and there are such non-artinian rings). Details will appear elsewhere.

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