ON THE CRITERION OF STASHEFF

C. S. HOO

1. Introduction. In (**9**), Stasheff showed that a loop space ΩX is homotopy-commutative if and only if the map $e\nabla: \Sigma\Omega X \vee \Sigma\Omega X \rightarrow X$ may be extended to $\Sigma\Omega X \times \Sigma\Omega X$, where ∇ is the folding map and $e: \Sigma\Omega X \rightarrow X$ is the map whose adjoint is the identity map of ΩX. In (**5**), Ganea, Hilton, and Peterson showed that this criterion does not dualize. Our aim in this paper is to give a reformulation of Stasheff's criterion, which is equivalent to it, but in a form which does dualize. In the course of the paper, we shall discuss why Stasheff's criterion, in its original form, does not dualize. In (**5**), the authors have, of course, given an explicit counterexample to the dual of Stasheff's criterion in its original form.

We work in the category of spaces with base point and having the homotopy type of countable CW-complexes. All maps and homotopies are with respect to base points. For simplicity, we shall frequently use the same symbol for a map and its homotopy class. Given spaces X and Y, we denote the set of homotopy classes of maps from X to Y by $[X, Y]$. We also have an isomorphism

$$\tau: [\Sigma X, Y] \rightarrow [X, \Omega Y]$$

taking each map to its adjoint. For any space X, the maps $e: \Sigma\Omega X \rightarrow X$, $e': X \rightarrow \Omega\Sigma X$ shall be those given by $\tau(e) = 1_{\Omega X}$, $e' = \tau(1_{\Sigma X})$.

2. Before stating our results, we shall recall some homotopy operations which we shall be using. In (**1**), Arkowitz defined a generalized Whitehead product $[\ ,\ ]: [\Sigma A, X] \times [\Sigma B, X] \rightarrow [\Sigma(A \wedge B), X]$. Dually, he also defined a product, called the dual product, $[\ ,\ ]': [X, \Omega A] \times [X, \Omega B] \rightarrow [X, \Omega(A \flat B)]$, where $A \flat B$ is the flat product. Now, suppose that X is an H-space. Then in (**2**), Arkowitz defined a generalized Samelson product

$$\langle\ ,\ \rangle: [A, X] \times [B, X] \rightarrow [A \wedge B, X].$$

Dually, if X is an H'-space, in (**3**) he defined a flat product

$$\langle\ ,\ \rangle': [X, A] \times [X, B] \rightarrow [X, A \flat B].$$

These operations are related in the following way: If α is an element of $[\Sigma A, X]$ and β is an element of $[\Sigma B, X]$, then $\tau\langle\tau^{-1}(\alpha), \tau^{-1}(\beta)\rangle' = [\alpha, \beta]'$.

We now state our results.

THEOREM 1. *ΩX is homotopy-commutative if and only if $[e, e] = 0$, where $e: \Sigma\Omega X \rightarrow X$.*

Received October 6, 1967. This research was supported by NRC Grant A-3026.

https://doi.org/10.4153/CJM-1969-024-6 Published online by Cambridge University Press

THEOREM 2. *ΣX is homotopy-commutative if and only if $[e', e']' = 0$, where $e'\colon X \to \Omega\Sigma X$.*

The proofs of these theorems are exact duals, and we shall give the proofs informally. We consider Theorem 1. We have the element $\langle 1_{\Omega X}, 1_{\Omega X}\rangle$ of $[\Omega X \wedge \Omega X, \Omega X]$. Let $q\colon \Omega X \times \Omega X \to \Omega X \wedge \Omega X$ be the projection. Then $q^{\#}\langle 1_{\Omega X}, 1_{\Omega X}\rangle = [c]$, where $c\colon \Omega X \times \Omega X \to \Omega X$ is the basic commutator. Since $q^{\#}$ is a monomorphism (see **2**), it follows that ΩX is homotopy-commutative if and only if $\langle 1_{\Omega X}, 1_{\Omega X}\rangle = 0$. Since $\langle 1_{\Omega X}, 1_{\Omega X}\rangle = \langle \tau(e), \tau(e)\rangle = \tau[e, e]$, and since τ is an isomorphism, this proves Theorem 1. The proof of Theorem 2 is exactly dual.

We now show that Theorem 1 is equivalent to Stasheff's criterion. Let $i_1, i_2\colon \Sigma\Omega X \to \Sigma\Omega X \vee \Sigma\Omega X$ be the inclusions in the first and second coordinates, respectively. Then we have an element $[i_1, i_2]$ of

$$[\Sigma(\Omega X \wedge \Omega X), \Sigma\Omega X \vee \Sigma\Omega X].$$

It is easily seen that $[e, e] = (\nabla(e \vee e)_{\#}[i_1, i_2]) = (e\nabla)_{\#}[i_1, i_2]$. Let k represent $[i_1, i_2]$. We can consider k to be a co-fibration, if necessary, by replacing the situation by a homotopically equivalent situation, using standard constructions of homotopy theory. The co-fibre is then the space

$$(\Sigma\Omega X \vee \Sigma\Omega X) \bigcup_k C\Sigma(\Omega X \wedge \Omega X).$$

The exact sequence of the co-fibration k now shows that $[e, e] = 0$ if and only if $e\nabla$ extends to

$$(\Sigma\Omega X \vee \Sigma\Omega X) \bigcup_k C\Sigma(\Omega X \wedge \Omega X).$$

Since this last space is homotopically equivalent to $\Sigma\Omega X \times \Omega\Sigma X$ (see **1**, Corollary 4.3), we have established the equivalence between Theorem 1 and Stasheff's criterion.

Let us now discuss Theorem 2 briefly. Let $\pi_1, \pi_2\colon \Omega\Sigma X \times \Omega\Sigma X \to \Omega\Sigma X$ be the projections. Then we have that

$$[e', e']' = ((e' \times e')\Delta)^{\#}[\pi_1, \pi_2]' = (\Delta e')^{\#}[\pi_1, \pi_2]'.$$

Let k_1 represent $[\pi_1, \pi_2]'$. Making it a fibration using standard constructions, we have as fibre the space

$$(\Omega\Sigma X \times \Omega\Sigma X) \bigcap_{k_1} P\Omega(\Sigma X \flat \Sigma X),$$

where, if $f\colon X \to Y$ is a map, then

$$X \bigcap_f PY = \{(x, l) \in X \times Y^I \text{ such that } f(x) = l(0), l(1) = *\}.$$

Then, of course, the exact sequence of the fibration now tells us that $[e', e']' = 0$ if and only if $\Delta e'$ can be compressed into

$$(\Omega\Sigma X \times \Omega\Sigma X) \bigcap_{k_1} P\Omega(\Sigma X \flat \Sigma X).$$

However, we do not know whether or not this last space is homotopically equivalent to $\Omega\Sigma X \vee \Omega\Sigma X$. The counterexample given in (**6**) would indicate that this would not be true in general. We hope that our discussion has provided some insight into the problem.

It is now fairly easy to see how the above theorems may be generalized. In fact, let us make some conventions about our homotopy operations. Let us write $[\alpha_1, \alpha_2, \ldots, \alpha_n]$ for $[[\alpha_1, \alpha_2, \ldots, \alpha_{n-1}], \alpha_n]$ whenever these operations are defined, and write $[\alpha] = \alpha$. Similarly for the other operations. Given a sequence of spaces $\{X_i\}_{i=1}^n$, let us define

$$\bigwedge_{i=1}^{n} X_i = \left(\bigwedge_{i=1}^{n-1} X_i\right) \wedge X_n \quad \text{and} \quad q_n\colon \prod_{i=1}^{n} X_i \to \bigwedge_{i=1}^{n} X_i$$

by $q_n = q(q_{n-1} \times 1)$, where q is the projection of a cartesian product of two spaces onto its smashed product. Then, according to (**4**, Lemma 4.1),

$$q_n^{\#}\colon \left[\bigwedge_{i=1}^{n} X_i, G\right] \to \left[\prod_{i=1}^{n} X_i, G\right]$$

is a monomorphism for any H-space G. Dually, we define

$$\bigvee_{i=1}^{n} X_i = \left(\bigvee_{i=1}^{n-1} X_i\right) \vee X_n \quad \text{and} \quad \flat_{i=1}^{n} X_i = \left(\flat_{i=1}^{n-1} X_i\right) \flat X_n.$$

Let $i\colon A \flat B \to A \vee B$ by the inclusion for general spaces A, B. Then we define

$$i_n\colon \flat_{i=1}^{n} X_i \to \bigvee_{i=1}^{n} X_i$$

by $i_n = (i_{n-1} \vee 1)i$. According to (**3**, Proposition 5.1),

$$i_{n\#}\colon \left[X, \flat_{i=1}^{n} X_i\right] \to \left[X, \bigvee_{i=1}^{n} X\right]$$

is a monomorphism if X is an H'-space and the X_i are 1-connected. Let $\Delta\colon X \to \prod_{i=1}^n X$, be the generalized diagonal map, and $\nabla\colon \bigvee_{i=1}^n X \to X$ the generalized folding map. For each $n \geqq 1$, and each j such that $1 \leqq j \leqq n$, let $i_j\colon \Sigma\Omega X \to \bigvee_{i=1}^n \Sigma\Omega X$ be the inclusion in the jth copy. Then, if $e\colon \Sigma\Omega X \to X$ is our usual map, it is easily checked that $[e, e, \ldots, e] = (e\nabla)_{\#}[i_1, i_2, \ldots, i_n]$. Now, $\tau[e, e, \ldots, e] = \langle 1_{\Omega X}, \ldots, 1_{\Omega X}\rangle$ and $q_n^{\#}\langle 1_{\Omega X}, \ldots, 1_{\Omega X}\rangle = c_n$, the commutator of weight n of ΩX. Putting the facts together, we have the following generalization.

THEOREM 3. *Let $e\colon \Sigma\Omega X \to X$ be such that $\tau(e) = 1_{\Omega X}$. Then the following statements are equivalent:*

(a) nil $\Omega X \leqq n$;

(b) *the $(n+1)$-fold generalized Whitehead product $[e, e, \ldots, e] = 0$;*

(c) *$e\nabla\colon \bigvee_{i=1}^{n+1} \Sigma\Omega X \to X$ extends to*

$$\left(\bigvee_{i=1}^{n+1} \Sigma\Omega X\right) \underset{[i_1, \ldots, i_{n+1}]}{\cup} C\Sigma\left(\bigwedge_{i=1}^{n+1} \Omega X\right).$$

https://doi.org/10.4153/CJM-1969-024-6 Published online by Cambridge University Press

Dually, we have the following theorem.

THEOREM 4. *Let X be 0-connected, and let $e'\colon X \to \Omega\Sigma X$. Then the following statements are equivalent:*

(a) conil $\Sigma X \leqq n$;

(b) *the $(n+1)$-fold generalized dual product $[e', e', \ldots, e']' = 0$;*

(c) *$\Delta e'\colon X \to \prod_{i=1}^{n+1} \Omega\Sigma X$ can be compressed into*

$$\left(\prod_{i=1}^{n+1} \Omega\Sigma X\right) \underset{[\pi_1, \ldots, \pi_{n+1}]'}{\cap} P\Omega\left(\flat_{i=1}^{n+1} \Sigma X\right),$$

where $\pi_j\colon \prod_{i=1}^{n+1}\Omega\Sigma X \to \Omega\Sigma X$ is the projection onto the jth coordinate.

We now consider how we may reformulate (**9**, Theorem 1.9). Let (X, ϕ, μ) be an H-space. We assume that H-spaces are homotopy associative, have right homotopy units and right homotopy inverses, that is, they are G-spaces in the terminology of (**7**). From the multiplication $\phi\colon X \times X \to X$, the Hopf construction (see, for example, **8** for definition) yields a map

$$J(\phi)\colon \Sigma(X \wedge X) \to \Sigma X.$$

Consider the co-fibration

$$\Sigma(X \wedge X) \xrightarrow{J(\phi)} \Sigma X \xrightarrow{l} \Sigma X \underset{J(\phi)}{\cup} C\Sigma(X \wedge X).$$

Let us write $P(\phi)$ for the co-fibre. Then (**9**, Theorem 1.9) states that (X, ϕ, μ) is homotopy-commutative if and only if $l\nabla\colon \Sigma X \vee \Sigma X \to P(\phi)$ extends to $\Sigma X \times \Sigma X$. We shall show how this may be proved in terms of the generalized Whitehead product. Let $i_1, i_2\colon \Sigma X \to \Sigma X \vee \Sigma X$ be the inclusions in the appropriate coordinates. Then, as above, we see that $l\nabla$ extends to $\Sigma X \times \Sigma X$ if and only if $(l\nabla)_\#[i_1, i_2] = 0$. In order to see how the dual can be formulated, we observe, of course, that the extension is to the space

$$(\Sigma X \vee \Sigma X) \underset{[i_1, i_2]}{\cup} C\Sigma(X \wedge X).$$

We now observe that $(l\nabla)_\#[i_1, i_2] = (\nabla(l \vee l))_\#[i_1, i_2] = [l, l]$. Now,

$$\tau[l, l] = \langle \tau(l), \tau(l) \rangle.$$

Let $q\colon X \times X \to X \wedge X$ be the projection. Then

$$q^\#\langle \tau(l), \tau(l)\rangle = c(\tau(l)\pi_1 \times \tau(l)\pi_2)\Delta = c(\tau(l) \times \tau(l)),$$

where c is the basic commutator of the loop space $\Omega P(\phi)$, $\pi_1, \pi_2\colon X \times X \to X$ are the projections, and Δ is the diagonal map. Let us denote the basic commutator of (X, ϕ, μ) by c also. Since $\tau(l)$ is an H-map, as can be easily proved (cf. also **9**, Proposition 3.5), we have that $c(\tau(l) \times \tau(l)) \simeq \tau(l)c$. Since $q^\#$ is a monomorphism and τ is an isomorphism, it follows that $l\nabla$ extends to $\Sigma X \times \Sigma X$ if and only if $[l, l] = 0$, and this last statement holds if and only if $\tau(l)c = 0$.

Since by (**9**, Lemma 4.2) there exists a map $v\colon \Omega P(\phi) \to X$ such that $v\tau(l) \simeq 1_X$, we clearly have that $l\nabla$ extends to $\Sigma X \times \Sigma X$ if and only if $[l, l] = 0$, and this is true if and only if $c = 0$. This proves (**9**, Theorem 1.9) and also yields the following reformulation which is equivalent to it.

THEOREM 5. *Let (X, ϕ, μ) be an H-space. Then (X, ϕ, μ) is homotopy-commutative if and only if $[l, l] = 0$, where $l\colon \Sigma X \to P(\phi)$ is the inclusion.*

We remark, of course, that in order to use Stasheff's result of the existence of a map $v\colon \Omega P(\phi) \to X$ such that $v\tau(l) \simeq 1_X$, it is necessary that right translation in our H-spaces be a homotopy equivalence. This is always true under our assumptions.

Using the methods above, we can generalize Theorem 5 to the following theorem.

THEOREM 6. *Let (X, ϕ, μ) be an H-space. Let $l\colon \Sigma X \to P(\phi)$ be the inclusion. Then the following statements are equivalent:*

(a) $\operatorname{nil}(X, \phi, \mu) \leqq n$;

(b) *the $(n + 1)$-fold generalized Whitehead product $[l, l, \ldots, l] = 0$;*

(c) $l\nabla\colon \vee_{i=1}^{n+1}\Sigma X \to P(\phi)$ *extends to*

$$\left(\bigvee_{i=1}^{n+1} \Sigma X\right) \bigcup_{[i_1, \ldots, i_{n+1}]} C\Sigma\left(\bigwedge_{i=1}^{n+1} X\right),$$

where $i_j\colon \Sigma X \to \vee_{i=1}^{n+1}\Sigma X$ is the embedding in the jth coordinate.

Dually, let (X, ϕ', μ') be an H'-space. From the co-multiplication

$$\phi'\colon X \to X \vee X,$$

the co-Hopf construction yields a map $H(\phi')\colon \Omega X \to \Omega(X \flat X)$. We consider the fibration

$$\Omega X \bigcap_{H(\phi')} P\Omega(X \flat X) \xrightarrow{l'} \Omega X \xrightarrow{H(\phi')} \Omega(X \flat X).$$

Let us write $P(\phi')$ for the fibre. Then we can easily show that the dual product $[l', l']' = 0$ if and only if $\Delta l'\colon P(\phi') \to \Omega X \times \Omega X$ can be compressed into

$$(\Omega X \times \Omega X) \bigcap_{[\pi_1, \pi_2]'} P\Omega(X \flat X),$$

where $\pi_1, \pi_2\colon \Omega X \times \Omega X \to \Omega X$ are the projections onto the coordinates. It is easily shown that $\tau^{-1}(l')\colon \Sigma P(\phi') \to X$ is an H'-map; see (**5**, Proposition 3.1). However, we do not know whether or not $\tau^{-1}(l')$ has a right homotopy inverse. Because of this lack of knowledge, the methods above give us only the following result. The converse would hold if such a right homotopy inverse exists.

THEOREM 7. *Let (X, ϕ', μ') be an H'-space. Let $l'\colon P(\phi') \to \Omega X$ be the projection. If (X, ϕ', μ') is homotopy commutative, then $[l', l']' = 0$, or equivalently, $\Delta l'\colon P(\phi') \to \Omega X \times \Omega X$ can be compressed into*

$$(\Omega X \times \Omega X) \bigcap_{[\pi_1, \pi_2]'} P\Omega(X \flat X).$$

https://doi.org/10.4153/CJM-1969-024-6 Published online by Cambridge University Press

More generally, if X is 0*-connected, and if* $\operatorname{conil}(X, \phi', \mu') \leqq n$, *then the* $(n+1)$*-fold dual product* $[l', l', \ldots, l']' = 0$, *or equivalently, the map*

$$\Delta l' \colon P(\phi') \to \prod_{i=1}^{n+1} \Omega X$$

can be compressed into

$$\left(\prod_{i=1}^{n+1} \Omega X\right) \underset{[\pi_1, \ldots, \pi_{n+1}]'}{\cap} P\Omega\left(\flat_{i=1}^{n+1} X\right),$$

where $\pi_j \colon \prod_{i=1}^{n+1} \Omega X \to \Omega X$ *is the projection onto the jth coordinate.*

References

1. M. Arkowitz, *The generalized Whitehead product,* Pacific J. Math. *12* (1962), 7–23.
2. ——— *Homotopy products for H-spaces,* Michigan Math. J. *10* (1963), 1–9.
3. ——— *Commutators and cup products,* Illinois J. Math. *8* (1964), 571–581.
4. I. Berstein and T. Ganea, *Homotopical nilpotency,* Illinois J. Math. *5* (1961), 99–130.
5. I. Berstein and P. J. Hilton, *On suspensions and comultiplications,* Topology *2* (1963), 73–82.
6. T. Ganea, P. J. Hilton, and F. P. Peterson, *On the homotopy-commutativity of loop-spaces and suspensions,* Topology *1* (1962), 133–141.
7. P. J. Hilton, *Homotopy theory and duality* (Gordon and Breach, New York, 1965).
8. C. S. Hoo, *A note on a theorem of Ganea, Hilton and Peterson,* Proc. Amer. Math. Soc. *19* (1968), 909–911.
9. J. Stasheff, *On homotopy-abelian H-spaces,* Proc. Cambridge Philos. Soc. *57* (1961), 734–745.

University of Alberta,
Edmonton, Alberta