AN AXIOMATIC LINE GEOMETRY

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In their classic treatment (**5**) Veblen and Young build n-dimensional projective geometry from points and lines. Naturally, each line becomes identified with the set of points with which it is incident, and many treatments build from points alone, postulating the existence of certain distinguished subsets of the set of points. From either point of view, some labour is required, even in the two-dimensional case, to establish duality; hence a considerable interest attaches to self-dual systems of axioms; cf. (**2**; **3**).

But all the self-dual axiom systems known to me build from all of the flats, i.e., from the points, lines, planes, . . . , and hyperplanes, postulating an imposing edifice of undefined objects right at the start. When the dimension of the geometry is odd, there are self-dual flats, and the others can be regarded as sets of these. To obtain a duality which is intrinsic we need only state the axioms in terms of a single undefined set and binary relations on that set, defining additional flats in an appropriate way.

This paper carries out in detail, for three dimensions, the project just suggested. Ultimately, a point will be defined as the set of lines which our intuition perceives as going through it, and a plane will be the set of lines which lie in it.

There is given a set $L = \{a, b, c, \ldots\}$ of undefined elements called "lines" together with a reflexive and symmetric binary relation T over L.

We write this relation in the form $a \mathrm{T} b$, which we read as "a is transverse to b" or "a and b are transverse". If a is not transverse to b, we say "a is skew to b" or "a and b are skew". Obviously, the pairs of skew lines form a relation S on L which is the complement of T in $L \times L$. Thus any two lines a and b satisfy either $a \mathrm{T} b$ or $a \mathrm{S} b$, but not both.

The letters Q and R shall always denote subsets of L.

If $a \mathrm{T} b$ for every $a \in Q$ and every $b \in R$, we write $Q \mathrm{T} R$; in particular, $a \mathrm{T} Q$ means that $a \mathrm{T} b$ for all $b \in Q$. Then a is called a transversal of Q. The set of all transversals of Q is given a symbol in the following definition.

Definition. $\mathrm{T}Q = \{a\colon a \mathrm{T} Q\}$.

We shall frequently speak of the set T^2Q of lines which are transverse to every transversal of Q. Thus

$$\mathrm{T}^2Q = \mathrm{T}(\mathrm{T}\,Q) = \{a\colon a \mathrm{T}\ (\mathrm{T}Q)\}.$$

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The notation $R \mathrm{T}^2 Q$ means $R \subset \mathrm{T}^2 Q$. Then every line of R is transverse to every transversal of Q.

For our geometry to have any structure, it is necessary that L and T satisfy certain conditions, the axioms of our geometry.

Axiom T1. (Reflexivity of T): *For every line* a, $a \mathrm{T} a$.

Axiom T2. (Symmetry of T): $a \mathrm{T} b$ *implies* $b \mathrm{T} a$.

Axiom T3. L *contains at least two lines.*

Axiom T4. *For every two lines* a *and* b, $\mathrm{T}\{a, b\}$ *contains at least two skew lines.*

Axiom T5. *If* $a \neq b$, $c \mathrm{S} c'$, *and* $\{a, b\} \mathrm{T} \{a, b, c, c'\}$, *then* $\mathrm{T}\{a, b, c, c'\}$ *contains a third line.*

Axiom T6. *If* $\{a, b\} \mathrm{T} \{a, b, c, c', c''\}$, $a \neq b$, $c \mathrm{S} c'$, *and* $c' \mathrm{S} c''$, *then* $c'' \mathrm{T} c$.

Axiom T7. *Let* $a \mathrm{T} b$, $b \mathrm{T} c$, $c \mathrm{T} d$, $a \mathrm{S} c$, *and* $b \mathrm{S} d$; *then* $\mathrm{T}\{a, b, c, d\}$ *contains exactly two lines and they are skew.*

The reader can easily verify intuitionally that the foregoing propositions are valid in real projective geometry of three dimensions. They are, in fact, valid in general projective geometry of three dimensions.

A long chain of definitions and theorems leads up to the definitions of points and planes.

The following obvious proposition does not involve the axioms; it simply facilitates calculations with the operator T.

PROPOSITION 1. *Let* Q *and* R *be sets of lines; then* $Q \subset R$ *implies* $\mathrm{T}R \subset \mathrm{T}Q$ *and hence* $\mathrm{T}^2Q \subset \mathrm{T}^2R$.

PROPOSITION 2. *Let* $a \mathrm{S} a'$ *and* $b \mathrm{T} \{a, a'\}$; *then* $\mathrm{T}\{a, a'\}$ *contains a line skew to* b.

Proof. Using Axiom T4, let $\{c, c'\} \mathrm{T} \{a, a'\}$ and $c \mathrm{S} c'$. If $c \mathrm{S} b$ or $c' \mathrm{S} b$, there is nothing to prove. Otherwise, $b \mathrm{T} \{a, c, a', c'\}$ and $a \mathrm{T} c$, $c \mathrm{T} a'$, $a' \mathrm{T} c'$, $a \mathrm{S} a'$, $c \mathrm{S} c'$. Hence, by Axiom T7, $\mathrm{T}\{a, c, a', c'\}$ contains exactly two lines, one of them b, the other skew to b.

Definition. A *skew quadrilateral* is a set of four lines a, a', b, b' such that

$$\{a, a'\} \mathrm{T} \{b, b'\}, \qquad a \mathrm{S} a', b \mathrm{S} b'. \tag{1}$$

The symbol for the quadrilateral is $\{a, a'; b, b'\}$, and whenever this symbol appears, it is to be understood that (1) holds.

Definition. A *tetrahedron*

$$\{a, a'; b, b'; c, c'\} \tag{2}$$

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consists of three pairs $\{a, a'\}$, $\{b, b'\}$, $\{c, c'\}$ of skew lines such that any two of the pairs are transverse. Thus (2) is equivalent to, e.g.,

$$\{a, a'\} \mathrm{T} \{b, b'; c, c'\}, \qquad a \mathrm{S} a'.$$

Remark. By Axiom T7, any skew quadrilateral can be completed to a tetrahedron in one and only one way. This is part of the following proposition.

PROPOSITION 3. *Any two lines and a pair of skew transversals can be completed to a tetrahedron.*

Proof. Let $a \neq b$, $\{c, c'\} \mathrm{T} \{a, b\}$, and $c \mathrm{S} c'$. By the preceding remark, we may assume that $a \mathrm{T} b$. Let $a' \mathrm{T} \{c, c'\}$, $a' \mathrm{S} a$ (cf. Proposition 2); then $\{a, a'; c, c'\}$. If $a' \mathrm{T} b$, then $b \mathrm{T} \{a, a'; c, c'\}$, and, letting b' be the other transversal of $\{a, a'; c, c'\}$, we have $\{a, a'; b, b'; c, c'\}$. From now on let $a' \mathrm{S} b$.

By the above remark, we can construct $\{a, a'; b', b''; c, c'\}$. Axiom T6 and $\{b, b', b'', a, c\} \mathrm{T} \{a, c\}$ imply $b \mathrm{T} b''$, say. The relation $b \mathrm{T} b'$ would imply $b \mathrm{T} \{b', b''; c, c'\}$; $b \in \{a, a'\}$. Hence $b \mathrm{S} b'$. We now have $\{b, b'; c, c'\}$ which can be completed to a tetrahedron by two skew lines, one of which is a. Denote the other by a''. Then $\{a, a''; b, b'; c, c'\}$.

Remark. When $a \mathrm{T} b$, the completion is not unique (cf. Theorem 8).

PROPOSITION 4. $\mathrm{T}\{a, a'; b, b'; c, c'\} = \emptyset$.

Proof.

$$\mathrm{T}\{a, a'; b, b'; c, c'\} \subset \mathrm{T}\{a, a'; b, b'\} \cap \mathrm{T}\{a, a'; c, c'\} \quad \text{(cf. Proposition 1)}$$
$$= \{c, c'\} \cap \{b, b'\} = \emptyset \quad \text{(cf. Axiom T7)}.$$

We shall frequently mention sets of (linearly) dependent lines.

Definition (**1**, p. 205). Given a set Q of one, two, or three lines, a line a is called *dependent* on Q if $a \mathrm{T}^2 Q$, i.e., if a is transverse to every transversal of Q. Thus, dependence on Q does not depend on any ordering of Q.

THEOREM 5. $\mathrm{T}^2 a = a$.

Proof. Obviously, $a \mathrm{T} a$ and hence $\mathrm{T}^2 a \subset \mathrm{T} a$. Let $b \neq a$, If $b \mathrm{S} a$, then $b \notin \mathrm{T}^2 a$. If $b \mathrm{T} a$, use Axiom T4 and Proposition 3 to construct $\{a, a'; b, b'; c, c'\}$. Then $b \notin \mathrm{T}^2 a$ for all $b \neq a$.

THEOREM 6. *If* $a \mathrm{S} a'$, *then* $\mathrm{T}^2\{a, a'\} = \{a, a'\}$.

Proof. Use Axiom T4 and Proposition 3 to construct $\{a, a'; b, b'; c, c'\}$. Then

$$\mathrm{T}^2\{a, a'\} \subset \mathrm{T}\{b, b'; c, c'\} = \{a, a'\} \subset \mathrm{T}^2\{a, a'\}.$$

Theorem 6 disposes of the lines dependent on a pair of skew lines. The

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lines dependent upon a pair of transverse lines form a configuration which is fundamental to our geometry.

Definition. A pencil (of lines) is the set of lines dependent on a pair of transverse lines.

The following theorem is a characterization of a pencil.

THEOREM 7. *If* $a \mathrm{T} b$, $a \neq b$, *then*

$$\mathrm{T}^2\{a, b\} = \mathrm{T}\{a, b, c, c'\}$$

for every choice of c, c' *such that* $\{c, c'\} \mathrm{T} \{a, b\}$, $c \mathrm{S} c'$.

Proof. Proposition 1 and $\{a, b, c, c'\} \subset \mathrm{T}\{a, b\}$ imply

$$\mathrm{T}^2\{a, b\} \subset \mathrm{T}\{a, b, c, c'\}.$$

Conversely, let $d \mathrm{T} \{a, b, c, c'\}$ and let $d' \mathrm{T} \{a, b\}$. We wish to prove $d \mathrm{T} d'$, i.e., $d \mathrm{T}^2 \{a, b\}$.

Assume that $d \mathrm{S} d'$. Neither d nor d' coincides with a, b, c, or c'. Use Proposition 3 to complete $\{a, b, d, d'\}$ to a tetrahedron $\{a, a'; b, b'; d, d'\}$. Since $\{a', c, c'\} \mathrm{T} \{b, d\}$ and $c \mathrm{S} c'$, Axiom T6 implies $a' \mathrm{T} c$ or $a' \mathrm{T} c'$. Symmetrically, $\{b', c, c'\} \mathrm{T} \{a, d\}$ yields $b' \mathrm{T} c$ or $b' \mathrm{T} c'$. Finally, $\{d', c, c'\} \mathrm{T} \{a, b\}$ yields $d' \mathrm{T} c$ or $d' \mathrm{T} c'$. Thus, one of the two lines c and c' must be a transversal of at least two of the lines a', b', d'. Since $\{a, a'; b, b'; d, d'\}$ is a tetrahedron and neither c nor c' belongs to $\{a, a'; b, b', d, d'\}$, this is impossible.

THEOREM 8. *Every pencil has at least three lines.*

Proof. By Theorem 7, this theorem is simply a paraphrase of Axiom T5.

PROPOSITION 9. *If* $c \mathrm{T}^2 \{a, b\}$, *then* $\mathrm{T}\{a, b, c\} = \mathrm{T}\{a, b\}$.

Proof. Proposition 1 yields at once $\mathrm{T}\{a, b, c\} \subset \mathrm{T}\{a, b\}$. On the other hand, if $d \mathrm{T} \{a, b\}$, then $c \mathrm{T}^2 \{a, b\}$ implies $c \mathrm{T} d$. Thus $d \mathrm{T} \{a, b, c\}$, and $T\{a, b\} \subset \mathrm{T}\{a, b, c\}$.

COROLLARY. *If* $c \mathrm{T}^2 \{a, b\}$, *then* $\mathrm{T}^2\{a, b, c\} = \mathrm{T}^2\{a, b\}$.

PROPOSITION 10. *If* $c \mathrm{T}^2 \{a, b\}$, $c \neq a$, *then* $\mathrm{T}\{a, c\} = \mathrm{T}\{a, b\}$.

Proof. If $a \mathrm{S} b$, then $c \mathrm{T}^2 \{a, b\}$, $c \neq a$ imply, by Theorem 6, $c = b$, and the theorem is trivial.

If $a = b$, then $c \mathrm{T}^2\{a, b\}$ implies the contradiction, $c = a$. Hence, we may suppose that $a \mathrm{T} b$, $a \neq b$, and $b \neq c$. First we show that $b \mathrm{T}^2 \{a, c\}$. Let $b' \mathrm{T} \{a, c\}$ and suppose, contrary to what we wish to prove, that $b' \mathrm{S} b$. Use Proposition 3 to construct $\{a, a'; b, b'; c, c'\}$. Then $c' \mathrm{T} \{a, b\}$. Since $c \mathrm{T}^2 \{a, b\}$, this would imply $c \mathrm{T} c'$, a contradiction. Hence $b \mathrm{T} b'$, and $b \mathrm{T}^2 \{a, c\}$. By Proposition 9, $c \mathrm{T}^2 \{a, b\}$ and $b \mathrm{T}^2 \{a, c\}$ imply

$$T\{a, b\} = T\{a, b, c\} = T\{a, c\}.$$

COROLLARY. *Suppose that* $a \neq c$. *Then* $c\,T^2\{a, b\}$ *if and only if* $T^2\{a, b\} = T^2\{a, c\}$.

THEOREM 11. *A pencil is determined by any two of its lines.*

Proof. Let $a\,T\,b$, $a \neq b$, $\{c, d\}\,T^2\,\{a, b\}$, and $c \neq d$. Choose the notation so that $a \neq c$. Using Proposition 10 and its Corollary twice, we obtain

$$T^2\{a, b\} = T^2\{a, c\} = T^2\{c, a\} = T^2\{c, d\}.$$

COROLLARY. *Two distinct pencils have at most one line in common.*

PROPOSITION 12. *For each pencil* R *there are pencils* Q, Q' *such that* $R \cap Q$ *is a line,* $R \cap Q'$ *is void.*

Proof. Let $R = T^2\{a, b\} = T\{a, b, c, c'\}$; cf. Theorem 7. Use Proposition 3 to construct $\{a, a'; b, b'; c, c'\}$, and let $Q = T^2\{a, c\}$, $Q' = T^2\{a', b'\}$. Then $a \in R \cap Q$ while $R \neq Q$. Furthermore, by Proposition 4,

$$R \cap Q' = T\{a, b, c, c'\} \cap T\{a', b', c, c'\} = T\{a, a'; b, b'; c, c'\} = \emptyset.$$

The next theorems deal with lines dependent on three lines which are independent in the following sense: no one depends on the other two. By Theorem 11, three mutually transverse lines are independent if and only if they do not all lie in one pencil.

Definition. A *bundle* is the set of lines dependent on three independent and mutually transverse lines.

Later on we shall be able to partition the bundles into two disjoint classes. The bundles of one class will be called *point bundles* or, simply, *points*, while the bundles of the other class will be called *plane bundles* or, simply, *planes*. The next theorem is a characterization of a bundle.

THEOREM 13. *If* a, b, *and* c *are independent and mutually transverse, then*

$$T^2\{a, b, c\} = T\{a, b, c\}.$$

Proof. By hypothesis, $\{a, b, c\} \subset T\{a, b, c\}$, and therefore, by Proposition 1, $T^2\{a, b, c\} \subset T\{a, b, c\}$. To prove the opposite inclusion, let $\{d, d'\}\,T\,\{a, b, c\}$ and suppose, contrary to what we wish to prove, that $d\,S\,d'$. Then by Theorem 7, $T\{a, b, d, d'\} = T^2\{a, b\}$, and, since $c\,T\,\{a, b, d, d'\}$, it follows that $c\,T^2\,\{a, b\}$ and a, b, c are not independent. Therefore $d\,T\,d'$.

Because of Theorem 13, bundles may be designated by $T\{a, b, c\}$ rather than by $T^2\{a, b, c\}$.

THEOREM 14. *A bundle is determined by any three of its independent lines.*

Proof. Let a, b, and c be independent and mutually transverse. Let $d\,T\{a, b, c\}$, $d \notin T^2\{b, c\}$. Since $\{d, b, c\} \subset T\{a, b, c\}$, Theorems 13 and 1 yield

$$\mathrm{T}\{a, b, c\} = \mathrm{T}^2\{a, b, c\} \subset \mathrm{T}\{d, b, c\} = \mathrm{T}^2\{d, b, c\} \subset \mathrm{T}^3\{a, b, c\} = \mathrm{T}\{a, b, c\}.$$

Hence $\mathrm{T}\{d, b, c\} = \mathrm{T}\{a, b, c\}$. Now let d, e, and f be three independent lines of $\mathrm{T}\{a, b, c\}$. By Proposition 10, and its Corollary, $\mathrm{T}^2\{a, c\} \neq \mathrm{T}^2\{b, c\}$. Hence by Theorem 11 and its Corollary, $T^2\{a, c\} \cap \mathrm{T}^2\{b, c\} = c$;

$$\mathrm{T}^2\{a, c\} \cap \mathrm{T}^2\{b, c\} \cap \mathrm{T}^2\{a, b\} = \emptyset.$$

Hence, without loss of generality, we may assume that $d \notin \mathrm{T}^2\{b, c\}$. Then, as we proved above, $\mathrm{T}\{a, b, c\} = \mathrm{T}\{d, b, c\}$. Since $e \neq d$, the above argument shows that $e \notin \mathrm{T}^2\{d, b\} \cap \mathrm{T}^2\{d, c\}$, say, $e \notin \mathrm{T}^2\{d, c\}$, so that

$$\mathrm{T}\{b, c, d\} = \mathrm{T}\{e, d, c\}.$$

Finally, $\mathrm{T}\{c, d, e\} = \mathrm{T}\{f, d, e\}$, since $f \notin \mathrm{T}^2\{d, e\}$. Combining these results yields $\mathrm{T}\{a, b, c\} = \mathrm{T}\{d, e, f\}$.

Proposition 15. *Any two lines of a bundle determine a pencil contained in the bundle.*

Proof. Let a, b, and c be independent and mutually transverse. Let $\{d, e\}\ \mathrm{T}\ \{a, b, c\}$, $d \neq e$. Then, since $T\{a, b, c\} = \mathrm{T}^2\{a, b, c\}$, we have $d\,\mathrm{T}\,e$, and $\mathrm{T}^2\{d, e\}$ is a pencil. Furthermore, $\{d, e\} \subset \mathrm{T}\{a, b, c\}$ together with Theorems 1 and 13 yield $\mathrm{T}^2\{d, e\} \subset \mathrm{T}^3\{a, b, c\} = \mathrm{T}\{a, b, c\}$.

Proposition 16. *If A is a bundle and d is a line not in A, then $Q = A \cap \mathrm{T}d$ is a pencil.*

Proof. Let $A = \mathrm{T}\{a, b, c\}$, where a, b, and c are independent and mutually transverse. Then $Q = \mathrm{T}\{a, b, c, d\}$. Because $d \notin A$, we may suppose that $d\,\mathrm{S}\,a$. Let $e\,\mathrm{T}\,\{a, d\}$. If $e \notin A$, suppose (to be definite) that $e\,\mathrm{S}\,b$; then d, e, a, and b satisfy the hypothesis of Axiom T7 and there are skew transversals f and f' of $\{d, e, a, b\}$. Because $\{c, f, f'\}\ \mathrm{T}\ \{a, b\}$, Axiom T6 implies $c\,\mathrm{T}\,f$, say. Thus $f\,\mathrm{T}\,\{a, b, c, d\}$. In other words, Q is never void. We next wish to construct a second line in Q. Since $Q \subset A$, we may assume, by renaming f, that $c \in Q$. If $a\,\mathrm{S}\,d$, use Proposition 2 to construct $\{a, d; c, c'\}$ and Proposition 3 to construct $\{a, d; c, c'; e, e'\}$. By Axiom T6, the relation $\{b, e, e'\}\ \mathrm{T}\ \{a, c\}$ implies $b\,\mathrm{T}\,e$, say. Thus $e\,\mathrm{T}\,\{a, b, c, d\}$, i.e., $e \in Q$.

Since $e'\,\mathrm{T}\,\{a, c\}$ with $e\,\mathrm{S}\,e'$, we have $e \notin \mathrm{T}^2\{a, c\}$ although $e\,\mathrm{T}^2\,\{a, b, c\}$. Hence, by Theorem 14, $A = \mathrm{T}\{a, e, c\}$. Finally, Theorem 7 yields

$$Q = A \cap (\mathrm{T}d) = \mathrm{T}\{a, e, c, d\} = \mathrm{T}\{e, c, a, d\} = \mathrm{T}^2\{e, c\}.$$

Proposition 17. *Two pencils in the same bundle have a line in common.*

Remark. Of course, by the corollary of Theorem 11, two pencils with more than one line in common coincide.

Proof. Let $\mathrm{T}^2\{a, b\}$ and $\mathrm{T}^2\{c, d\}$ be pencils in the bundle A. We may assume that $c \notin \mathrm{T}^2\{a, b\}$, in particular, $A = \mathrm{T}\{a, b, c\}$; cf. Theorems 13 and 14. By

Axiom T4, a and b have a pair of skew transversals, and, if both were transverse to c, they would both be in $\mathrm{T}\{a, b, c\}$ and, hence not skew. Thus there exists a line c' transverse to $\{a, b\}$ but skew to c. Symmetrically, we may assume that $b \notin \mathrm{T}^2\{c, d\}$ and construct a line $b' \notin \mathrm{T}\{c, d\}$ such that $b \,\mathrm{S}\, b'$. Applying Axiom T7 to the four lines b', c, b, and c', we see that $\mathrm{T}\{b', c, b, c'\}$ is a pair of skew lines e, e', say. By Axiom T6, we can choose the notation so that $a \,\mathrm{T}\, e$. Thus $e \,\mathrm{T}\, \{a, b, c, c'\} = \mathrm{T}^2\{a, b\}$; cf. Theorem 7. Furthermore, because $\{d, e\} \subset A$, we have $d \,\mathrm{T}\, e$. Hence $e \,\mathrm{T}\, \{c, d, b, b'\} = \mathrm{T}^2\{c, d\}$ as well.

PROPOSITION 18. *If Q is a pencil and d is not a transversal of Q, then there is one and only one line in Q transverse to d.*

Proof. We first prove that $Q \cap \mathrm{T}d$ contains at most one line. Suppose that $\{a, b\} \subset Q \cap \mathrm{T}d$, $a \neq b$. Then $Q = \mathrm{T}^2\{a, b\}$, by Theorem 11; and, by Proposition 1, $d \,\mathrm{T}\, \{a, b\}$ implies $\mathrm{T}^2\{a, b\} \subset \mathrm{T}d$. That is, $d \,\mathrm{T}\, Q$, contrary to hypothesis. To prove that $Q \cap \mathrm{T}d$ has at least one line, let

$$Q = \mathrm{T}^2\{a, b\} = \mathrm{T}\{a, b, c, c'\}, \qquad c \,\mathrm{S}\, c'.$$

Then $A = \mathrm{T}\{a, b, c\}$ is a bundle containing the pencils Q and $\{e\colon e \in A, e \,\mathrm{T}\, d\}$; cf. Proposition 16. Proposition 17 implies the existence of a line in $Q \cap \mathrm{T}d$.

PROPOSITION 19. *Each pencil is contained in exactly two bundles.*

Proof. Let $Q = \mathrm{T}^2\{a, b\} = \mathrm{T}\{a, b, c, c'\}$, where $c \,\mathrm{S}\, c'$; cf. Theorem 7. Then $c \notin Q$ and $c' \notin Q$, so that $\mathrm{T}\{a, b, c\}$ and $\mathrm{T}\{a, b, c'\}$ are both bundles. They are distinct, and both contain Q. Conversely, let $\mathrm{T}\{a, b, d\}$ denote a bundle containing Q. Then $\{c, c', d\} \,\mathrm{T}\, \{a, b\}$; by Axiom T6, $c \,\mathrm{T}\, d$, say, and

$$\mathrm{T}\{a, b, d\} = \mathrm{T}\{a, b, c\};$$

cf. Theorem 14.

PROPOSITION 20. *If A and A' are distinct bundles each containing a pencil Q, then $Q = A \cap A'$.*

Proof. Let $Q = \mathrm{T}^2\{a, b\}$. Obviously, $Q \subset A \cap A'$. If $A \cap A'$ were to contain a line c not in Q, then, by Theorem 14, $A = \mathrm{T}\{a, b, c\} = A'$, contradicting $A \neq A'$.

We now interrupt the discussion of pencils and bundles to prove two theorems characterizing sets of lines dependent on three lines which are not mutually transverse. They will not be used in the sequel.

THEOREM 21. *If a, a', and a'' are mutually skew, then they are independent, and any two lines of $\mathrm{T}\{a, a', a''\}$ or of $\mathrm{T}^2\{a, a', a''\}$ are skew.*

Proof. The independence follows from Theorem 6. Also, by Axiom T6, no two lines of $\mathrm{T}\{a, a', a''\}$ can be transverse. If we can show that $\mathrm{T}\{a, a', a''\}$ contains at least three lines, Axiom T6 also implies that no two lines of

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$T^2\{a, a', a''\}$ are transverse. Let $d \mathrm{T} \{a, a'\}$. By Proposition 19 there is a bundle A' containing $T^2\{a', d\}$, and, since $a \mathrm{S} a'$, we have $a \notin A'$. By Proposition 16, $\{g: g \in A', g \mathrm{T} a\}$ is a pencil Q. Theorem 8 assures that Q has three distinct lines d_1, d_2, and d_3, say. By Theorem 11,

$$Q = T^2\{d_i, d_j\}, \qquad 1 \leqq i < j \leqq 3.$$

Since $a' \mathrm{T} Q$, $a \mathrm{T}^2 Q$ would imply $a \mathrm{T} a'$. Therefore a is independent of Q; in particular, if $i \neq j$, then a, d_i, d_j, are independent. Since, e.g., $d_3 \mathrm{T} \{a, d_1, d_2\}$, Theorem 14 shows that the three bundles $T\{a, d_i, d_j\}$ are identical. Let $A = T\{a, d_i, d_j\}$. For $k = 1, 2, 3$, let $B_k \neq A$ be the other bundle containing $T^2\{a, d_k\}$; cf. Proposition 19. If, for $k \neq i$, $d_i \in B_k$, Theorem 14 would imply that $B_k = T\{a, d_i, d_k\} = A$. Hence $d_i \in B_k$ if and only if $i = k$; and the B_k are distinct.

By Proposition 17, the pencils

$$B_k \cap \mathrm{T}a' \quad \text{and} \quad B_k \cap \mathrm{T}a''$$

have a line b_k in common. Thus $b_k \mathrm{T} \{a, a', a''\}$. It remains to show that the b_k are distinct. Suppose, for example, that $b_1 = b_2$. Then $b_1 \in B_1 \cap B_2$, and this would imply that $b_1 \mathrm{T} \{a, d_1, d_2\}$, i.e., $b_1 \in A$. Hence,

$$T^2\{a, b_1\} \subset A \cap B_1 \cap B_2.$$

Then $A \neq B_1$, $A \neq B_2$ would, by Theorem 19, imply the contradiction $B_1 = B_2$. Hence $\{a, a', a''\}$ has three distinct transversals b_1, b_2, and b_3.

Corollary. *The two sets* $T\{a, a', a''\}$ *and* $T^2\{a, a', a''\}$ *are disjoint.*

Proof. By Theorem 21, any two lines of $T\{a, a', a''\}$ are skew. Hence no line of $T\{a, a', a''\}$ can belong to $T^2\{a, a', a''\}$.

Remark. The set $T\{a, a', a''\}$ is called a *regulus* and its elements, *generators*. The elements of $T^2\{a, a', a''\}$ are called *directrices* of $T\{a, a', a''\}$. By Proposition 1, $\{a, a', a''\} \mathrm{T} \{b, b', b''\}$ implies $T^2\{b, b', b''\} \subset T\{a, a', a''\}$, i.e., the directrices of $T\{b, b', b''\}$ are contained in another regulus of which $\{b, b', b''\}$ are generators. If every line of $T\{a, a', a''\}$ is transverse to every line of $T\{b, b', b''\}$, then we also have $T\{a, a', a''\} \subset T^2\{b, b', b''\}$, so that $T\{a, a', a''\} = T^2\{b, b', b''\}$ and the regulus $T\{a, a', a''\}$ is uniquely determined by any three of its generators. For a counterexample, see (**4**, p. 319).

Theorem 22. *Let* $a \mathrm{T} b$, $a \neq b$, $a' \mathrm{S} a$. *Then*

$$T^2\{a, b, a'\} = T^2\{a, b\} \cup T^2\{a', c\},$$

where c *is the line in* $T^2\{a, b\}$ *which is transverse to* a'; cf. Proposition 18.

Proof. We first show that $T^2\{a, b, a'\}$ is unaltered if b is replaced by any line of $T^2\{a, b\}$ other than a. Let $d \mathrm{T}^2 \{a, b\}$, $d \neq a$.

Let $e \mathrm{T} \{a, b, a'\}$; e could be, e.g., the line of Proposition 18. Then $e \mathrm{T} \{a, b\}$

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and $d\,\mathrm{T}^2\,\{a, b\}$ imply $d\,\mathrm{T}\,e$. Thus $e\,\mathrm{T}\,\{a, d, a'\}$, and $\mathrm{T}\{a, b, a'\} \subset \mathrm{T}\{a, d, a'\}$; hence $\mathrm{T}^2\{a, d, a'\} \subset \mathrm{T}^2\{a, b, a'\}$. But we also have $b \in \mathrm{T}^2\{a, d\}$, $b \neq a$, and, hence, symmetrically, $\mathrm{T}^2\{a, b, a'\} \subset \mathrm{T}^2\{a, d, a'\}$.

From now on we may suppose that $b\,\mathrm{T}\,a'$, and we have to prove that

$$\mathrm{T}^2\{a, b, a'\} = \mathrm{T}^2\{a, b\} \cup \mathrm{T}^2\{a', b\}.$$

The relations $\{a, b\} \subset \{a, b, a'\}$ and $\{a', b\} \subset \{a, b, a'\}$ imply that

$$\mathrm{T}^2\{a, b\} \cup \mathrm{T}^2\{a', b\} \subset \mathrm{T}^2\{a, b, a'\}.$$

Conversely, suppose that $d\,\mathrm{T}^2\,\{a, b, a'\}$. We have to show that

$$d \in \mathrm{T}^2\{a, b\} \cup \mathrm{T}^2\{a', b\}.$$

Let C and C' be the bundles containing $\mathrm{T}^2\{a, b\}$; cf. Proposition 19. By Proposition 16 there are lines $e \in C$ and $e' \in C'$ such that $e\,\mathrm{T}\,a'$, $e \neq b$, and $e'\,\mathrm{T}\,a'$, $e' \neq b$. Then $e\,\mathrm{T}^2\,\{a, b\}$ would imply $a\,\mathrm{T}^2\,\{e, b\}$, and, in particular, $a\,\mathrm{T}\,a'$; hence $C = \mathrm{T}\{a, b, e\}$. Similarly, $C' = \mathrm{T}\{a, b, e'\}$, and $C \neq C'$ implies $e\,\mathrm{S}\,e'$. By Theorem 7,

$$\mathrm{T}^2\{a, b\} = \mathrm{T}\{a, b, e, e'\} \quad \text{and} \quad \mathrm{T}^2\{a', b\} = \mathrm{T}\{a', b, e, e'\}.$$

We have $\{d, a, a'\}\,\mathrm{T}\,\{b, e\}$, $b\,\mathrm{T}\,e$, $a\,\mathrm{S}\,a'$, so that, by Axiom T6, $d\,\mathrm{T}\,a$, say. Hence $d \in \mathrm{T}\{a, b, e, e'\} = \mathrm{T}^2\{a, b\}$.

We now resume the investigation of bundles and pencils.

PROPOSITION 23. *The intersection of a bundle and a pencil is either void or a line or the pencil.*

Proof. When the intersection contains more than one line, Proposition 15 implies that the entire pencil is contained in the bundle.

PROPOSITION 24. *For each pencil Q there are bundles A, A', and A'' such that $A \cap Q = Q$, $A' \cap Q$ is a line, $A'' \cap Q$ is void.*

Proof. Let $Q = \mathrm{T}\{a, b, c, c'\}$ and use Proposition 3 to construct $\{a, a'; b, b'; c, c'\}$. Let $\mathrm{T}\{a, b, c\} = A$, $\mathrm{T}\{a', b, c\} = A'$; $\mathrm{T}\{a', b', c\} = A''$. Then $a, b \in A \cap Q$, so that, by Proposition 23, $A \cap Q = Q$. If $d \in A' \cap Q$, then $d\,\mathrm{T}\,\{a, a', c, c'\}$. By Axiom T7, $d \in \{b, b'\}$; and $b\,\mathrm{S}\,b'$ and $d\,\mathrm{T}\,b$ imply $d = b$. Finally, by Proposition 4, $A'' \cap Q \subset \mathrm{T}\{a, a'; b, b'; c, c'\} = \emptyset$.

PROPOSITION 25. *For each bundle A there are pencils Q, Q', and Q'' such that $Q \subset A$, $Q' \cap A$ is a line, $Q'' \cap A$ is void.*

Proof. Let $A = \mathrm{T}\{a, b, c\}$, $\{c, c'\}\,\mathrm{T}\,\{a, b\}$, $c\,\mathrm{S}\,c'$; cf. Proposition 19. Use Proposition 3 to construct $\{a, a'; b, b'; c, c'\}$. Let $Q = \mathrm{T}^2\{a, b\}$, $Q' = \mathrm{T}^2\{a', b\}$, $Q'' = \mathrm{T}^2\{a', b'\}$, and compare the proof of Proposition 24.

PROPOSITION 26. *The intersection of two distinct bundles is either a pencil, a line, or void.*

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Proof. If the bundles C and C' have two distinct lines a and b in common, Proposition 15 shows that they contain the pencil $\mathrm{T}^2\{a, b\}$. If $c \in C \cap C'$, $c \notin \mathrm{T}^2\{a, b\}$, then by Theorem 14, $C = \mathrm{T}\{a, b, c\} = C'$.

PROPOSITION 27. *For each bundle A, there are other bundles B, B', and B'' such that $A \cap B$ is a pencil, $A \cap B'$ is a line, and $A \cap B''$ is void.*

Proof. Let $A = \mathrm{T}\{a, b, c\}$, $\{c, c'\}\ \mathrm{T}\ \{a, b\}$, $c\,\mathrm{S}\,c'$. Use Proposition 3 to construct $\{a, a'; b, b'; c, c'\}$. Let $B = \mathrm{T}\{a', b, c\}$, $B' = \mathrm{T}\{a', b', c\}$, $B'' = \mathrm{T}\{a', b', c'\}$. Then $\mathrm{T}^2\{b, c\} \subset A \cap B$, and, by Proposition 26, $A \cap B = \mathrm{T}^2\{b, c\}$. Furthermore, $A \cap B' \subset \mathrm{T}\{a, a'; b, b'\} = \{c, c'\}$. Since $c \in A \cap B'$ and $c\,\mathrm{S}\,c'$, we have $c' \notin A \cap B'$. Thus $A \cap B' = c$.

Finally, by Proposition 4, $A \cap B'' = \mathrm{T}\{a, a'; b, b'; c, c'\}$ is void.

LEMMA 28. *If A, B, and C are distinct bundles, and if $A \cap B$ and $A \cap C$ are lines, then $B \cap C$ is a line.*

Proof. Case (i). $b = A \cap B \neq c = A \cap C$. Thus $b \notin C$, $c \notin B$; hence lines $b' \in C$ and $c' \in B$ exist such that $b\,\mathrm{S}\,b'$ and $c\,\mathrm{S}\,c'$. Since $b, c \in A$, we have $b\,\mathrm{T}\,c$. Then b', c, b, c' satisfy the hypothesis of Axiom T7. Let

$$\{a, a'\}\ \mathrm{T}\ \{b', c, b, c'\}, \qquad a\,\mathrm{S}\,a'.$$

Then $\mathrm{T}\{a, b, c\}$, $\mathrm{T}\{a', b, c\}$ are distinct bundles containing $\mathrm{T}^2\{b, c\}$. Let $A = \mathrm{T}\{a, b, c\}$; cf. Proposition 19. Since $a \neq b$, $a \notin B$, and we obtain, by the same proposition, $B = \mathrm{T}\{a', b, c'\}$; similarly, $C = \mathrm{T}\{a', b', c\}$. Then $a' \in B \cap C$. $B \cap C = \mathrm{T}\{a', b, b', c, c'\} \subset \mathrm{T}\{b, b'; c, c'\} = \{a, a'\}$ by Axiom T7. Since $a\,\mathrm{S}\,a'$, we obtain $B \cap C = a'$.

Case (ii). $A \cap B = A \cap C = a$, say. Then $a = A \cap B \cap C \in B \cap C$. Suppose that $B \cap C \neq a$; then Proposition 26 implies that $Q = B \cap C$ is a pencil. By Theorem 11, there exists a b such that $Q = \mathrm{T}^2\{a, b\}$. By Proposition 16, the lines in A transverse to b form a pencil R. Let $c \in R$, $c \neq a$; thus $c \in A$. But, by hypothesis, $c \notin B$, $c \notin C$, and therefore

$$c \notin B \cap C = \mathrm{T}^2\{a, b\}.$$

Therefore, a, b, and c are independent, and $\mathrm{T}^2\{a, b\} \subset B \cap C \cap \mathrm{T}\{a, b, c\}$. Since $B \neq C$, Proposition 19 would yield $\mathrm{T}\{a, b, c\}$ is either B or C. But c is in neither B nor C.

LEMMA 29. *If A, B, and C are distinct bundles, and if $A \cap B$ and $A \cap C$ are pencils, then $B \cap C$ is a line.*

Proof. Let $A \cap B = R$, $A \cap C = Q$. Since A, B, and C are distinct, Proposition 19 excludes the possibility that $A \cap B \cap C = R$. Thus $R \neq Q$. As $R \subset A$, $Q \subset A$, it follows from Proposition 17 that $R \cap Q$ is a line a, say. Let $R = \mathrm{T}^2\{a, b\}$, $Q = \mathrm{T}^2\{a, c\}$. Then a, b, and c are independent in A so that $A = \mathrm{T}\{a, b, c\}$. Suppose that $d \in B \cap C$; then $d\ \mathrm{T}\ \{a, b, c\}$. Therefore $d \in A \cap B \cap C = (A \cap B) \cap (A \cap C) = R \cap Q = a$.

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LEMMA 30. *If A, B, and C are bundles with $B \neq C$, and if $A \cap B$ and $A \cap C$ are void, then $B \cap C$ is a line.*

Proof. Let $A = \mathrm{T}\{a, b, c\}$. Then $b, c \notin B$, and by Propositions 16 and 17, $(B \cap \mathrm{T}b) \cap (B \cap \mathrm{T}c)$ is a line a', say. Then $a' \mathrm{T} \{b, c\}$, $a' \notin A$; hence $a' \mathrm{S}\, a$. Let $A' = \mathrm{T}\{a', b, c\}$. Then $a' \in A' \cap B$. Since

$$A \cap A' = \mathrm{T}^2\{b, c\} \neq \emptyset,$$

we have $A' \neq B$. Also, since $A \cap A'$ is a pencil while $A \cap B$ is void, Lemma 29 shows that $A' \cap B$ cannot be a pencil; hence $A' \cap B = a'$; cf. Theorem 26. Symmetrically, $A' \cap C$ is a line. Therefore, by Lemma 28, $B \cap C$ is a line.

THEOREM 31. *Given three distinct bundles, the pairwise intersections are either:*

(a) *void, void, line; or*
(b) *line, line, line; or*
(c) *pencil, pencil, line; or*
(d) *void, pencil, line.*

Proof. If two of the intersections are both void, both lines, or both pencils, then Lemmas 30, 28, 29, respectively, lead to (a), (b), or (c). But if no two of the intersections are of the same type, then only (d) is possible.

We now divide the set of all bundles into two disjoint classes.

Definition. Two bundles A and B are said to be *equivalent*, $A \sim B$, if they coincide or have just one line in common.

PROPOSITION 32. *Equivalence is reflexive, symmetric, and transitive.*

Proof. It is obvious that equivalence is symmetric and reflexive. Its transitivity follows from Lemma 28.

THEOREM 33. *There are exactly two equivalence classes.*

Proof. Let A, B, and C be bundles, $A \nsim B$, $A \nsim C$. Thus $A \cap B$ is either void or a pencil, and so is $A \cap C$. Hence, by Theorem 31, either $B = C$ or $B \cap C$ is a line, i.e., $B \sim C$.

We now name the bundles of one of the equivalence classes "points", those of the other, "planes". We note that our axioms do not enable us to distinguish between the two classes. Thus, our designations are arbitrary, and our line geometry naturally allows dualities. Hereafter, points will be denoted by capital italics, and planes, by lower case Greek letters. Lines shall continue to be denoted by $a, b, \ldots, g$. The set of all points shall be denoted by P. The letters x, y, and z shall have a meaning to be given later.

PROPOSITION 34. *If the intersection of two bundles is a pencil, then one of them is a point and the other is a plane.*

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Proof. By definition, the two bundles are not equivalent.

In conclusion, we prove that line geometry is projective. Let us state explicitly what we mean by a three-dimensional projective geometry.

There is given an abstract set $P = \{A, B, C, \ldots\}$ of points and a class $\{a, b, c, d, \ldots\}$ of subsets of P called lines. A flat is a subset of P which contains with every two of its points every line containing them. The points, lines, and flats satisfy the following axioms.

Axiom P1. *Two distinct points are contained in exactly one line.*

Axiom P2. *If* $A \neq B$; $C \neq D$; $A, B \in a$; $C, D \in b$; $A, C \in c$; $B, D \in d$; *and* $a \cap b = \emptyset$, *then* $c \cap d = \emptyset$.

Axiom P3. *Every line contains at least three points.*

Axiom P4. *There is a set of four points but no set of less than four points such that any flat containing that set contains all the points.*

The fundamental relation in projective geometry is incidence of a point with a line. This notion is introduced into line geometry by the following definition.

Definition. If $a \in A$, we say the point A is (or lies) on the line a.

THEOREM 35. (Cf. Axiom P1.) *Two distinct points are on exactly one line.*

Proof. Two distinct points are equivalent.

THEOREM 36. (Cf. Axiom P2.) *If* $A \neq B$ *and* $C \neq D$, *then,*

$$(A \cap B)\,\mathrm{S}\,(C \cap D) \text{ implies } (A \cap C)\,\mathrm{S}\,(B \cap D).$$

Proof. $A = C$ would imply $(A \cap B)\,\mathrm{T}\,(C \cap D)$. Similarly, $B \neq D$. Let $A \cap B = a$, $C \cap D = a'$, $A \cap C = b$, $B \cap D = c$. Then $a\,\mathrm{T}\,c$, $c\,\mathrm{T}\,a'$, $a'\,\mathrm{T}\,b$, $b\,\mathrm{T}\,a$, and $a\,\mathrm{S}\,a'$. Suppose, contrary to what we wish to prove, that $b\,\mathrm{T}\,c$. Then b and c have the skew transversals a and a', so that a, b, and c are independent and mutually transverse. Since $c \notin A$, we have $A \neq \mathrm{T}\{a, b, c\}$, but both contain the pencil $\mathrm{T}^2\{a, b\}$. Hence, by Proposition 34, $\mathrm{T}\{a, b, c\}$ is a plane α, say. Similarly, $\mathrm{T}\{a', b, c\}$ is a plane α', say, and $a\,\mathrm{S}\,a'$ implies $\alpha \neq \alpha'$. By equivalence of planes, $\alpha \cap \alpha'$ is a line; but $\alpha \cap \alpha'$ contains $\mathrm{T}^2\{b, c\}$.

THEOREM 37. (Cf. Axiom P3.) *For each line* a, *there are distinct points,* B, C, *and* D *such that* $B \cap C \cap D = a$.

Proof. Using Axioms T3 and T4 and Propositions 2 and 3, construct $\{a, a'; b, b'; c, c'\}$, choosing notation so that $\mathrm{T}\{a, b, c\}$ is a plane α, say; cf. Proposition 34. By Theorem 8, the pencil $\mathrm{T}^2\{b, c\} = \mathrm{T}\{b, c, a, a'\}$ has a third line d, say. If $b'\,\mathrm{T}\,d$, then $b' \in \mathrm{T}^2\{d, c\} = \mathrm{T}^2\{b, c\}$, and $b'\,\mathrm{T}\,b$. Hence $b'\,\mathrm{S}\,d$, and, symmetrically, $c'\,\mathrm{S}\,d$. Next, observe that

$$\alpha = \mathrm{T}\{a, b, c\} = \mathrm{T}\{a, d, c\} = \mathrm{T}\{a, b, d\}$$

and that α contains the pencils

$$\mathrm{T}^2\{a, b\}, \qquad \mathrm{T}^2\{a, c\}, \qquad \mathrm{T}^2\{a, d\}.$$

Let B, C, and D be the points containing $\mathrm{T}^2\{a, b\}$, $\mathrm{T}^2\{a, c\}$, and $\mathrm{T}^2\{a, d\}$, respectively. The relations $b' \mathrm{S}\, b$, $b' \mathrm{S}\, d$, $c' \mathrm{S}\, d$ imply that B, C, and D are distinct. Finally, $a \in B \cap C \cap D \subset B \cap C = a$, since $B \sim C$.

We have now established in line geometry the validity of all theorems which do not depend upon the finite dimension axiom, P4. We express the definition of a flat in the language of line geometry.

Definition. A *flat* is a set x of points such that $A, B \in x$, $A \neq B$, $A \cap B \cap G \neq \emptyset$ imply $G \in x$.

Definition. A "line of points" is the set $\bar{a} = \{C: a \in C\}$ determined by a given line a.

Definition. A "plane of points" is the set $\bar{\alpha} = \{C: C \cap \alpha \neq \emptyset\}$ determined by a given plane α.

PROPOSITION 38. *A point set is a flat if and only if it is one of the following*:
(i) *the void set* $\emptyset$,
(ii) *a point*,
(iii) *a line of points*,
(iv) *a plane of points*,
(v) *the set P of all points*.

Proof. Obviously, (i), (ii), and (v) are flats. Furthermore, if $\bar{a}$ is a line of points, then $G \in \bar{a}$ if and only if $a \in G$. Let $A, B \in \bar{a}$, $A \neq B$, $A \cap B \cap G \neq \emptyset$. Then $A \cap B = a$; therefore, $A \cap B \cap G = a$; $a \in G$. Hence $G \in \bar{a}$, and $\bar{a}$ is a flat. Also, if $\bar{\alpha}$ is a plane of points, then $G \in \bar{\alpha}$ if and only if $\alpha \cap G \neq \emptyset$. Let $A, B \in \bar{\alpha}$, $A \neq B$ and $A \cap B \cap G \neq \emptyset$; then $\alpha \cap A$, $\alpha \cap B$, being not void, are pencils. By Proposition 17, $b = (\alpha \cap A) \cap (\alpha \cap B)$ is a line, and hence $b = A \cap B \in \alpha$. Since $A \cap B \cap G \neq \emptyset$, we have $A \cap B \cap G = b$; $b \in G$; $\alpha \cap G \supset b \neq \emptyset$, and $G \in \bar{\alpha}$. Conversely, let x be a flat. If $x \neq \emptyset$, it contains a point A, say. If $x \neq A$, then x contains another point B, say. Putting $c = A \cap B$, we have that

$$\bar{c} = \{G: c \in G\} \subset x.$$

If $\bar{c}$ is a proper subset of x, then x contains a point C which does not contain c. Let $B \cap C = a$, $C \cap A = b$. Then a, b, and c are mutually transverse and, since $c \notin C$, independent. If $\alpha = \mathrm{T}\{a, b, c\}$, then $\alpha \cap C = \mathrm{T}^2\{a, b\}$, and, by Theorem 34, α is a plane. Let $G \in \bar{\alpha}$. Thus $\alpha \cap G$ and $\alpha \cap C$ are pencils in the same bundle, and, by Proposition 17, $\alpha \cap C \cap G$ is or contains a line d; $d \in C \cap G$. Furthermore, $d \mathrm{T}\, c$. Let E be the point containing $\mathrm{T}^2\{c, d\}$.

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Then $c \in E$ implies $E \in \bar{c} \subset x$, while $C \in x$, $C \notin \bar{c}$; hence $C \neq E$. Furthermore, $C \cap E \cap G \supset d \cap E = d \neq \emptyset$. Therefore, $G \in x$ and $\bar{\alpha} \subset x$. Finally, let $\bar{\alpha}$ be a proper subset of x. Thus there is a point $D \in x$ such that $D \cap \alpha = \emptyset$. Let G be any point in space and let $D \cap G = e$. By Proposition 16, $R = \alpha \cap \mathrm{T}e$ is a pencil. Let E denote the point containing R. Then $E \cap \alpha \neq \emptyset$ and $D \cap a = \emptyset$ imply that $D \neq E$, while $D \cap E \cap G = e \neq \emptyset$; hence $G \in x$. Thus $x = P$.

The following proposition is well known to be equivalent to Axiom P4.

THEOREM 39. *There is a linearly ordered set of five distinct flats, and no linearly ordered set of flats has more than five elements.*

Our demonstration that the geometry of the points of a line geometry defined by Axioms T1–T7 is projective is now complete. It may be mentioned that, by modifying their definitions, the flats of our geometry may also be interpreted as sets of lines.

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