Student Problems

Students up to the age of 19 are invited to send solutions to either or both of the following problems to Tuya Sa, SCH.1.17, Schofield Building, Loughborough University, Loughborough, LE11 3TU. Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive solutions for either problem. It is not necessary to submit solutions to both. Solutions should arrive by 20th May 2023 and will be published in the July 2023 edition.

The Mathematical Association and the *Gazette* comply fully with the provisions of the 2018 GDPR legislation. Submissions must be accompanied by the SPC permission form which is available on the Mathematical Association website

https:www.m-a.org.uk/the-mathematical-gazette

Note that if permission is not given, a pupil ***may still participate and will be eligible for a prize in the same way as others****.*

Problem 2023.1 **(Christopher Starr)**

A shot-putter releases a shot with speed V from a height h at an angle A to the horizontal. Find an expression for the value of the angle that gives the greatest possible horizontal distance.

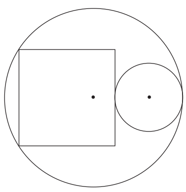
Problem 2023.2 **(Geoffrey Strickland)**

Given that a and b are strictly positive integers such that $a^3 - 3b^2 = 1$ and $x = 2a - 3b$, $y = 2b - a$, prove that $x^2 - 3y^2 = 1$, $0 < x < a$ and $0 \leqslant y < b$.

Solutions to 2022.5 and 2022.6

Both problems were solved by Soham Bhadra (Patha Bhavan, India), Emily Waren (Bishop Wordsworth's Grammar School), and Aparup Roy (Zoom International School, India). Problem 2022.5 was also solved by Hakan Knightly (The Sixth Form College Farnborough). Problem 2022.6 was also solved by Tom Rapley (The Sixth Form College Farnborough), Aryon Bhandari (Peter Symonds College), and Isabella Topley, Arya Golkari and Honsh Powl (King's College London Mathematics School). It was hard to choose the prizewinners as all students give impressive solutions.

Problem 2022.5 **(Paul Stephenson)**



A square and circle fit a circle of unit radius symmetrically as shown. Is it possible to adjust their sizes so that their combined area is under half of that unit circle?

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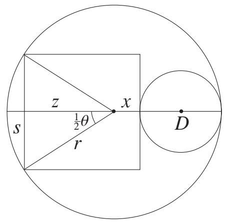
Solution by Hakan Knightley



D : The diameter of smaller circle
s: Half the side-length of the square
r: The radius of the larger circle
z: The perpendicular distance from the centre of the circle to the side of the square that is not touching the smaller circle.

x: The perpendicular displacement from the centre of the larger circle to the side of the square that is touching the smaller circle. If the diameter of the smaller circle is equal to or larger than the radius of the lager circle, the value of x is negative.

θ: The angle between the two radii of the larger circle that pass through its centre, and two distinct vertices of the square that touch the circumference of the larger circle, and are not diagonals of each other.

Use the Sine Rule and the fact that angles in a triangle add to π radians, to find s, z, x and D in terms of the radius of the larger circle and θ.

$$\frac{r}{\sin\frac{1}{2}\pi} = \frac{s}{\sin\frac{1}{2}\theta} = \frac{z}{\sin\left(\frac{1}{2}\pi - \frac{1}{2}\theta\right)}$$

$$\text{so } s = r\sin\tfrac{1}{2}\theta,\ z = r\sin\left(\tfrac{1}{2}\pi - \tfrac{1}{2}\theta\right),\ x = 2r\sin\tfrac{1}{2}\theta - r\sin\left(\tfrac{1}{2}\pi - \tfrac{1}{2}\theta\right)$$

$$D = r - 2r\sin\tfrac{1}{2}\theta + r\sin\left(\tfrac{1}{2}\pi - \tfrac{1}{2}\theta\right).$$

Use the formulas for the area of a square and the area of a circle to find the area of the three shapes in terms of the radius of the larger circle and θ.

Area of square = $\left(2r\sin\frac{1}{2}\theta\right)^2$; Area of small circle = $\frac{1}{4}\left(r - 2r\sin\frac{1}{2}\theta + r\sin\left(\frac{1}{2}\pi - \frac{1}{2}\theta\right)\right)^2\pi$;
Area of the outside circle = $r^2\pi$.

θ must be smaller than or equal to $\frac{1}{2}\pi$, or the square will not fit inside the larger circle. It must also be positive. This gives $0 < \theta \leqslant \frac{1}{2}\pi$.

Add the area of the small circle and the area of the square to make an expression for the total area of the shapes inside the larger circle, and define it as function δ of θ: $\delta(\theta) = \frac{1}{4}\left(r - 2r\sin\frac{1}{2}\theta + r\sin\left(\frac{1}{2}\pi - \frac{1}{2}\theta\right)\right)^2\pi + \left(2r\sin\frac{1}{2}\theta\right)^2$.

Verify if it is possible for the area of the square and the smaller circle to be less than half the area of the unit circle:

$$\tfrac{1}{4}\left(r - 2r\sin\tfrac{1}{2}\theta + r\sin\left(\tfrac{1}{2}\pi - \tfrac{1}{2}\theta\right)\right)^2\pi + \left(2r\sin\tfrac{1}{2}\theta\right)^2 < \tfrac{1}{2}r^2\pi$$

r is arbitrary, so we can use $r = 1$ and the verification will still be valid, leaving us an inequality in terms of θ to be solved. Verify

$$\tfrac{1}{4}\left(1 - 2\sin\tfrac{1}{2}\theta + \sin\left(\tfrac{1}{2}\pi - \tfrac{1}{2}\theta\right)\right)^2\pi + \left(2\sin\tfrac{1}{2}\theta\right)^2 < \tfrac{1}{2}\pi.$$

Find a minimum pair in $y = \delta(\theta)$ by differentiating and solving $\delta'(\theta) = 0$

$$\delta'(\theta) = 4\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta - \tfrac{\pi}{4}\left(-2\sin\tfrac{1}{2}\theta + \cos\tfrac{1}{2}\theta + 1\right)\left(2\sin\tfrac{1}{2}\theta + 2\cos\tfrac{1}{2}\theta\right) = 0$$

$$\text{so } \theta = -2.59,\ 0.97715,\ 3.4489.$$

Since $\theta = 0.97715$ is the minimum value, we get $\delta(0.97715) = 1.58152$. But since $1.58152 > \frac{1}{2}\pi$, this is not possible.

***Problem 2022.6* (Paul Stephenson)**

The primes p and q $(p > q)$ are greater than 3. They are respectively the hypotenuse and a side of a Pythagorean triangle. Show that 12 divides the other side, r.

Solution by Thomas Rapley

By the Pythagorean Theorem $r^2 = p^2 - q^2$. Since p is a prime number greater than 3, it is not divisible by 3, so we deduce that p is congruent to one of $\{1, 2\}$ modulo 3. Note that $a \equiv b \pmod{c}$ implies $a^2 \equiv b^2 \pmod{c}$. Hence, p^2 is congruent to one of $\{1^2, 2^2\} = \{1, 4\}$ modulo 3. Thus, p^2 is congruent to 1 modulo 3.

Similarly, q^2 is congruent to 1 modulo 3. Thus

$$r^2 \equiv p^2 - q^2 \equiv 1 - 1 \equiv 0 \pmod{3}.$$

This means that 3 is a prime factor of r^2, and 3 is also a prime factor of r, so $3 \mid r$. Since p is a prime number greater than 3, it is not 2, so it is not divisible by 2, and we deduce that p is congruent to one of $\{1, 3, 5, 7\}$ modulo 8. Hence, p^2 is congruent to one of $\left\{1^2, 3^2, 5^2, 7^2\right\}$ modulo 8. Thus, p^2 is congruent to 1 modulo 8.

Similarly, q^2 is congruent to 1 modulo 8. Thus

$$r^2 \equiv p^2 - q^2 \equiv 1 - 1 \equiv 0 \pmod{8}.$$

This means that r^2 is divisible by 8, so the power of 2 in the prime factorization of r^2 is at least 3. However, the power of each prime factor of r^2 must be even in its factorization, so the power of 2 in the prime factorization of r^2 is at least 4. Suppose that the power of 2 in the prime factorization of r is α. Then the power of 2 in the prime factorization of r^2 is 2α

$$2\alpha \geqslant 4 \Rightarrow \alpha \geqslant 2.$$

Thus the power of 2 in the prime factorization of r^2 is at least 2, meaning that r is divisible by 4.

So $3 \mid r$ and $4 \mid r$. Since $\gcd(3, 4) = 1$, we deduce that $(3 \times 4) \mid r$, and so r is divisible by 12.

Prize Winners

The first prize of £25 is awarded to Emily Waren. The second prize of £20 is awarded to Hakan Knightley.

TUYA SA

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