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CHARACTERIZATIONS OF QUASI-METRIZABLE BITOPOLOGICAL SPACES

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Abstract

In this paper we prove that a pairwise Hausdorff bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is quasi-metrizable if and only if for each point $x \in X$ and for $i, j = 1, 2$, $i \neq j$, one can assign $\mathscr{T}_i$ nbd bases $\{S(n, i;\ x) \mid n = 1, 2, \ldots\}$ such that (i) $y \notin S(n-1, i;\ x)$ implies $S(n, i;\ x) \cap S(n, j;\ y) = \phi$, (ii) $y \in S(n, i;\ x)$ implies $S(n, i;\ y) \subset S(n-1, i;\ x)$. We derive two further results from this.

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The concept of quasi-metric spaces was first introduced by Wilson [11]. The fact that a quasi-metric gives rise to a conjugate quasi-metric was noticed by Kelly [1], thus leading to the study of bitopological spaces. Since then one of the main problems in this area has been to find necessary and sufficient conditions for quasi-metrization. This problem was considered by Kelly [1] Patty [5], Lane [2], Reilly [6], Salbany [9] and later by Pareek [4] and Romaguera [7, 8].

The related notion of quasi-uniform spaces and their properties have been discussed in great detail in Murdheswar and Naimpally [3] and Stoltenberg [10]. In the proof of Theorem 1 we make use of the quasi-uniform analogue of the metrization theorem of Alexandroff and Urysohn, namely, a pairwise Hausdorff quasi-uniform space $(X, \mathscr{V}, \mathscr{V}^{-1})$ is quasi-metrizable if and only if $\mathscr{V}$ has a countable base. From Theorem 1 we derive Theorems 2 and 3 as corollaries. It must be noted that Theorem 2 has been proved by Pareek [4].

We write nbd for neighbourhood. If A is a subset of X and $\mathscr{T}_i$ is a topology on X, then $\mathscr{T}_i$ cl A($\mathscr{T}_i$ int A) is the closure (interior) of A in the space $(X, \mathscr{T}_i)$.

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The letters m, n, n_i, m_j, n_j represent positive integers. The letters i, j always take the values $i, j = 1, 2$; $i \neq j$. $S(n, i;\ x)$ represents a $\mathscr{T}_i$ nbd of x where n is a positive integer.

1. THEOREM. *A pairwise Hausdorff bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is quasi-metrizable if and only if for each point $x \in X$ one can assign $\mathscr{T}_i$ neighbourhood bases $\{S(n, i;\ x) \mid n = 1, 2, \dots\}$ such that*

(i) $y \notin S(n-1, i;\ x)$ *implies* $S(n, i;\ x) \cap S(n, j;\ y) = \varnothing$,

(ii) $y \in S(n, i;\ x)$ *implies* $S(n, i;\ y) \subset S(n-1, i;\ x)$ $(i, j = 1, 2;\ i \neq j)$.

PROOF. To prove that the conditions are sufficient, we show first that $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise regular. If $S(n, i;\ x) \cap S(n, j;\ y) = \varnothing$, then $S(n, i;\ x) \subset X - S(n, j;\ y)$ so that $\mathscr{T}_j \operatorname{cl} S(n, i;\ X) \subset X - \mathscr{T}_j \operatorname{int} S(n, j;\ y)$. Thus if $y \notin S(n-1, i;\ x)$, then $y \notin \mathscr{T}_j \operatorname{cl} S(n, i;\ x)$ so that

$$x \in S(n, i;\ x) \subset \mathscr{T}_j \operatorname{cl} S(n, i;\ x) \subset S(n-1, i;\ x).$$

Furthermore the space is pairwise normal. Indeed, if A and B are $\mathscr{T}_1$ closed and $\mathscr{T}_2$ closed subsets (of X) respectively such that $A \cap B = \varnothing$ and $y \in B$, then there exists a positive integer $n(y)$ such that $A \cap \mathscr{T}_2 \operatorname{cl} S(n(y), 1;\ y) = \varnothing$. Since $x \notin S(n(y), 1;\ y)$ for each $x \in A$, $S(n(y)+1, 1;\ y) \cap S(n(y)+1, 2;\ x) = \varnothing$ for all $x \in A$. If $Q_{n(y)} = \bigcup\{\mathscr{T}_2 \operatorname{int} S(n(y)+1, 2;\ x) \mid x \in A\}$, then $Q_{n(y)} \supset A$ and $Q_{n(y)} \cap \mathscr{T}_2 \operatorname{cl} S(n(y)+1, 1;\ y) = \varnothing$. If we write $\bigcup\{\mathscr{T}_1 \operatorname{int} S(n(y)+1, 1;\ y) \mid n(y) = k\} = W(k, 1)$, then $Q_k \cap \mathscr{T}_2 \operatorname{cl} W(k, 1) = \varnothing$ so that we get a $\mathscr{T}_1$ open covering $\{W(k, 1) \mid k = 1, 2, \dots\}$ of B such that $A \cap \mathscr{T}_2 \operatorname{cl} W(k, 1) = \varnothing$ for each k. Similarly we can form a $\mathscr{T}_2$ open covering $\{W(k, 2) \mid k = 1, 2 \dots\}$ of A such that $B \cap \mathscr{T}_1 \operatorname{cl} W(k, 2) = \varnothing$ for each k. Then a standard argument produces disjoint sets $W_1 \in \mathscr{T}_1$ and $W_2 \in \mathscr{T}_2$ such that $W_1 \supset B$ and $W_2 \supset A$.

Let $\mathscr{K}(m, i) = \{\mathscr{T}_i \operatorname{int} S(m, i;\ y) \mid y \in X\}$. Let $S(x, \mathscr{K}(m, i)) = \bigcup\{\mathscr{T}_i \operatorname{int} S(m, i;\ y) \mid x \in \mathscr{T}_j \operatorname{int} S(m, j;\ y)\}$. Let $\mathscr{B}(i;\ x) = \{S(x, \mathscr{K}(m, i)) \mid m = 1, 2 \dots\}$. We claim $\mathscr{B}(i;\ x)$ is a $\mathscr{T}_i$ local base at x. If x is fixed initially and $U(i;\ x)$ are arbitrary $\mathscr{T}_i$ nbds of x then there exists n_i such that $x \in S(n_i - 1, i;\ x) \subset U(i;\ x)$. Consider $m = \max(n_1 + 1, n_2 + 1)$. Then clearly $S(m, i;\ x) \subset S(n_i, i;\ x)$. In order to avoid confusion, let us now prove specifically $\mathscr{B}(2;\ x)$ is a $\mathscr{T}_2$ local base at x. Let y be such that $x \in \mathscr{T}_1 \operatorname{int} S(m, 1;\ y)$. Then $S(m, 1;\ y) \cap S(m, 2;\ x) \neq \varnothing$ so that $y \in S(m-1, 2;\ x) \subset S(n_2, 2;\ x)$. Hence $S(n_2, 2;\ y) \subset S(n_2 - 1, 2;\ x)$. Since $m = \max\ (n_1 + 1, n_2 + 1)$, $\mathscr{T}_2 \operatorname{int} S(m, 2;\ y) \subset S(n_2, 2;\ y) \subset S(n_2 - 1, 2;\ x) \subset U(2;\ x)$. Thus $\mathscr{B}(2;\ x)$ is a $\mathscr{T}_2$ local base at x.

If $x \in \mathscr{T}_1 \operatorname{int} S(n+2, 1;\ y)$, then $S(n+2, 1;\ y) \cap S(n+2, 2;\ x) \neq \varnothing$ so that by (i) $y \in S(n+1, 2;\ x)$. Hence by (ii) $S(n+1, 2;\ y) \subset S(n, 2;\ x)$ so that $\bigcup\{\mathscr{T}_2 \operatorname{int} S(n+2, 2;\ y) \mid x \in \mathscr{T}_1 \operatorname{int} S(n+1, 1;\ y)\} \subset \mathscr{T}_2 \operatorname{int} S(n, 2;\ x)$. If we define $\mathscr{L}(m, i) = \{S(x, \mathscr{K}(m, i)) \mid x \in X\}$, then $\mathscr{L}(n+2, i) < \mathscr{K}(n, i)$ for all

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$n = 1, 2, 3 \dots$. If we write $V(m, i) = \bigcup\{\mathscr{T}_j \text{ int } S(m, j;\ y) \times \mathscr{T}_i \text{ int } S(m, i;\ y) \mid y \in X\}$, then $(x, y) \in V(m+2, i) \circ V(m+2, i)$ implies, for some $z \in X$ that $(x, z) \in V(m+2, i)$ and $(z, y) \in V(m+2, i)$.

Indeed $x \in V(m+2, j)[z] \subset \mathscr{T}_j \text{ int } S(m, j;\ z)$ and $y \in V(m+2, i)[z] \subset \mathscr{T}_i \text{ int } S(m, i;\ z)$ so that $(x, y) \in V(m, i)$. Also notice that $(V(m, i))^{-1} = V(m, j)$. Thus the conditions are sufficient.

The necessity is proved as follows. Let p_1 be the quasi-metric that induces $\mathscr{T}_1$ and $\mathscr{T}_2$ be induced by its conjugate p_2. Let us write $S(n, i;\ x) = \{y \mid p_i(x, y) < (\frac{1}{2})^n\}$. If $x \notin S(n-1, i;\ x)$ and $S(n, i;\ x) \cap S(n, j;\ z) \neq \varnothing$, then there exists $y \in X$ such that $p_i(x, y) < (\frac{1}{2})^n$ and $p_j(z, y) < (\frac{1}{2})^n$. Hence $p_i(x, z) \leqslant p_i(x, y) + p_i(y, z) < (\frac{1}{2})^{n-1}$, a contradiction. Also, if $y \in S(n, i;\ x)$ and $z \in S(n, i;\ y)$, then $p_i(x, y) < (\frac{1}{2})^n$ and $p_i(y, z) < (\frac{1}{2})^n$ so that $p_i(x, z) < (\frac{1}{2})^{n-1}$ and hence $z \in S(n-1, i;\ x)$.

2. THEOREM. *A pairwise Hausdorff space* $(X, \mathscr{T}_1, \mathscr{T}_2)$ *is quasi-metrizable if and only if for each* $x \in X$ *one can assign* $\mathscr{T}_i$ *nbd bases* $\{S(n, i;\ x) \mid n = 1, 2, \dots\}$ *such that*

(i) $y \notin S(n-1, i;\ x)$ *implies* $S(n, i;\ x) \cap S(n, j;\ y) = \varnothing$,

(ii) $y \in S(n, i;\ x)$ *implies* $x \in S(n, j;\ y)$ $(i, j = 1, 2;\ i \neq j)$.

PROOF. We have to verify only condition (ii) of Theorem 1. Now

$$y \notin S(n-1, i; x)$$

implies $S(n, i;\ x) \cap S(n, j;\ y) = \varnothing$ so that if $z \in S(n, i;\ x)$, then $z \notin S(n, j; y)$. Thus $y \notin S(n, i;\ z)$. The necessity is obvious.

3. THEOREM. *A pairwise Hausdorff space* $(X, \mathscr{T}_1, \mathscr{T}_2)$ *is quasi-metrizable if and only if for each* $x \in X$ *one can assign* $\mathscr{T}_i$ *nbd bases* $\{S(n, i;\ x) \mid n = 1, 2, \dots\}$ *such that*

(i) $y \in S(n, i;\ x)$ *implies* $S(n, i;\ y) \subset S(n-1, i;\ x)$,

(ii) $y \in S(n, i;\ x)$ *implies* $x \in S(n, j;\ y)$ $(i, j = 1, 2;\ i \neq j)$.

PROOF. We only have to verify condition (i) of Theorem 1. If

$$S(n, i;\ x) \cap S(n, j;\ y) \neq \varnothing,$$

then there is a point $z \in S(n, i;\ x)$ and $z \in S(n, j;\ y)$ so that $S(n, i;\ z) \subset S(n-1, i;\ x)$ and $y \in S(n, i;\ z)$. Thus $y \in S(n-1, i;\ x)$.

The necessity is obvious.

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