Glasgow Mathematical Journal (2024), **66**, pp. 382–439
doi:10.1017/S001708952400003X

Glasgow Mathematical Journal

RESEARCH ARTICLE

Models and integral differentials of hyperelliptic curves

Simone Muselli

University of Bristol, Bristol, UK
Email: simone@muselli.it

Received: 14 February 2022; **Revised:** 10 December 2023; **Accepted:** 5 January 2024; **First published online:** 18 March 2024

Keywords: Hyperelliptic curves; models of curves; dualising sheaf

2020 Mathematics Subject Classification: *Primary* - 11G20; *Secondary* - 14H45, 14F10

Abstract

Let $C : y^2 = f(x)$ be a hyperelliptic curve of genus $g \geq 1$, defined over a complete discretely valued field K, with ring of integers O_K. Under certain conditions on C, mild when residue characteristic is not 2, we explicitly construct the minimal regular model with normal crossings $\mathcal{C}/O_K$ of C. In the same setting we determine a basis of integral differentials of C, that is an O_K-basis for the global sections of the relative dualising sheaf $\omega_{\mathcal{C}/O_K}$.

1. Introduction

The purpose of this paper is to construct regular models of hyperelliptic curves and to describe a basis of integral differentials attached to them. Moreover, we want these constructions explicit and easy to compute.

1.1. Overview

To describe the arithmetic of curves over global fields, for example in the study of the Birch & Swinnerton-Dyer conjecture, it is essential to understand regular models and integral differentials over all primes, including those with very bad reduction. Constructing regular models of curves over discrete valuation rings is not an easy problem, even in the hyperelliptic curve case. In fact, there is no practical algorithm able to determine a model, unless the genus of the curve is 1 or we have some tameness or nondegeneracy hypothesis.

One possible approach to tackle this problem is giving a full classification of possible regular models in a fixed genus, as done by the Kodaira–Néron [7, 19] and Namikawa–Ueno [10, 18] classifications for curves of genera 1 and 2, respectively. However, this strategy seems impractical in general, since the number of models grows fast with the genus. Recently, new approaches based on clusters [14], Newton polytopes [1], and MacLane valuations [21], have been developed (see Section 1.5 for more detail).

On one side, clusters define nice and clear invariants from which one can extract information on the local arithmetic of hyperelliptic curves. Such invariants turn out to be particularly useful from a Galois theoretical point of view. However, for describing regular models, restrictions on the reduction type of the curve and on the residue characteristic of its base field [5, 14] need to be imposed. On the other side, Newton polytopes and MacLane valuations have a potential to solve the problem in general, but the respective constructions are more algorithmic and so do not give the result in closed form. Furthermore, they often depend on the chosen equation rather than on the curve itself.

In this paper, we present a new approach that preserves both positive aspects from the above and provides a link between the two sides. We describe a model from simple invariants defined from what we call *rational cluster picture* (Definition 1.10). This object modifies the theory in [14] and appears to be more suitable for our purpose (see Section 1.3). In fact, the rational cluster picture also carries intrinsic

© The Author(s), 2024. Published by Cambridge University Press on behalf of Glasgow Mathematical Journal Trust. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited.

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

connections with the other presented approaches, as it is closely related to Newton polygons and to degree 1 MacLane valuations (see [3]). When these valuations are enough to describe a regular model we say that the curve has an *almost rational cluster picture* (Definition 1.1; see also Corollary 3.29, Proposition 3.31). It turns out that the approach even works in residue characteristic 2, under an extra assumption that the curve is *y-regular* (Definition 1.4). Our main result is:

Let K be a complete[1] *discretely valued field with* $\mathrm{char}(K) \neq 2$, *and let K^{nr} be its maximal unramified extension. Let C/K be a hyperelliptic curve, having an almost rational cluster picture over K^{nr}. If the residue characteristic of K is* 2, *assume that $C_{K^{nr}}$* is *y-regular. Then via the rational cluster picture we determine:*

(i) *the minimal regular model with normal crossings $\mathcal{C}^{\min}$*,
(ii) *a basis of integral differentials of C.*

This result applies to a wide class of curves, covering wild cases and base fields with even residue characteristic. For example, if $g = 2$, then 107 out of 120 Namikawa-Ueno types [18] arise from hyperelliptic curves satisfying the conditions of our theorem. In addition, the author believes it has a potential to solve the problem in general. Heuristically speaking, the rational clusters invariants are expected to extend to general MacLane valuations. This approach could eventually lead to a full characterisation of minimal models with normal crossings of hyperelliptic curves (over any discretely valued field).

1.2. Main results

We will now present (a simplified version of) the main results of this paper. We will then illustrate them with an explicit example in Section 1.4.

Let K be a complete discretely valued field of residue characteristic p, with normalised discrete valuation v and ring of integers O_K. We require $\mathrm{char}(K)$ to be not 2, but we allow $p = 2$ and $p = 0$. In this subsection we will assume for simplicity that $K = K^{nr}$. Extend the valuation v to an algebraic closure $\bar{K}$ of K. Let C/K be a hyperelliptic curve, that is a geometrically connected smooth projective curve, double cover of $\mathbb{P}^1_K$. Let g be the genus of C. Assume $g \geq 1$. Fix a Weierstrass equation

$$C : y^2 = f(x).$$

Let $\mathfrak{R}$ be the set of roots of f in $\bar{K}$. Thus

$$f(x) = c_f \prod_{r\in\mathfrak{R}} (x - r).$$

For any $r, r' \in \mathfrak{R}$, with $r \neq r'$, denote by $\mathcal{D}_{r,r'}$ the smallest v-adic disc containing r and r'.

Definition 1.1 (Definition 3.26). *We say that C has an almost rational cluster picture if for any roots $r, r' \in \mathfrak{R}$ with $r \neq r'$, either*

(a) *$\mathcal{D}_{r,r'} \cap K \neq \varnothing$, or*
(b) *$p > 0$ and $|\mathcal{D}_{r,r'} \cap \mathfrak{R}| \leq |v(r - w)|_p$ for some $w \in K$,*

where $|\cdot|_p$ denotes the canonical p-adic absolute value on $\mathbb{Q}$.

Definition 1.2. *A rational cluster is a non-empty subset $\mathfrak{s} \subset \mathfrak{R}$ of the form $\mathcal{D} \cap \mathfrak{R}$, where $\mathcal{D}$ is a v-adic disc $\mathcal{D} = \{x \in \bar{K} \mid v(x - w) \geq \rho\}$ for some $w \in K$ and $\rho \in \mathbb{Q}$. We denote by Σ_K the set of rational clusters.*

In the following definition we introduce most of the notation and quantities, associated with rational clusters, needed in order to state our main theorems.

[1] The assumption on the completeness of K is not restrictive since regular models do not change under completion of the base field.

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

Definition 1.3. *For any $\mathfrak{s} \in \Sigma_K$ we say:*

$\mathfrak{s}$ *proper,*	$\lvert\mathfrak{s}\rvert > 1$
$\mathfrak{s}'$ *is a child of* $\mathfrak{s}$,	$\mathfrak{s}' \in \Sigma_K$ *and* $\mathfrak{s}' \subsetneq \mathfrak{s}$ *is a maximal subcluster*
$\mathfrak{s}$ *minimal,*	$\mathfrak{s}$ *has no proper children*
$\mathfrak{s}$ *übereven,*	$\mathfrak{s} = \bigcup_{\mathfrak{s}' \text{ child of } \mathfrak{s}} \mathfrak{s}'$ *and* $\lvert\mathfrak{s}'\rvert$ *even for all children* $\mathfrak{s}'$ *of* $\mathfrak{s}$

Moreover, we write $\mathfrak{s}' < \mathfrak{s}$, or $\mathfrak{s} = P(\mathfrak{s}')$, for a child $\mathfrak{s}' \in \Sigma_K$ of $\mathfrak{s}$, and $r \wedge \mathfrak{s}$ for the smallest rational cluster containing the root $r \in \mathfrak{R}$ and $\mathfrak{s}$.

Let Σ_K be the set of proper rational clusters. For any $\mathfrak{s} \in \Sigma_K$, define its radius

$$\rho_{\mathfrak{s}} = \max_{w \in K} \min_{r \in \mathfrak{s}} v(r - w)$$

and the following quantities:

$b_{\mathfrak{s}}$	*denominator of* $\rho_{\mathfrak{s}}$
$\epsilon_{\mathfrak{s}}$	$v(c_f) + \sum_{r \in \mathfrak{R}} \rho_{r \wedge \mathfrak{s}}$
$D_{\mathfrak{s}}$	1 *if* $b_{\mathfrak{s}}\epsilon_{\mathfrak{s}}$ *odd,* 2 *if* $b_{\mathfrak{s}}\epsilon_{\mathfrak{s}}$ *even*
$m_{\mathfrak{s}}$	$(3 - D_{\mathfrak{s}})b_{\mathfrak{s}}$
$p_{\mathfrak{s}}$	1 *if* $\lvert\mathfrak{s}\rvert$ *is odd,* 2 *if* $\lvert\mathfrak{s}\rvert$ *is even*
$s_{\mathfrak{s}}$	$\frac{1}{2}(\lvert\mathfrak{s}\rvert\rho_{\mathfrak{s}} + p_{\mathfrak{s}}\rho_{\mathfrak{s}} - \epsilon_{\mathfrak{s}})$
$\gamma_{\mathfrak{s}}$	2 *if* $\lvert\mathfrak{s}\rvert$ *is even and* $\epsilon_{\mathfrak{s}} - \lvert\mathfrak{s}\rvert\rho_{\mathfrak{s}}$ *is odd,* 1 *otherwise*
$p^0_{\mathfrak{s}}$	1 *if* $\mathfrak{s}$ *is minimal and* $\mathfrak{s} \cap K \neq \varnothing$, 2 *otherwise*
$s^0_{\mathfrak{s}}$	$-\epsilon_{\mathfrak{s}}/2 + \rho_{\mathfrak{s}}$
$\gamma^0_{\mathfrak{s}}$	*2 if* $p^0_{\mathfrak{s}} = 2$ *and* $\epsilon_{\mathfrak{s}}$ *is odd, 1 otherwise*

Definition 1.4 (Definition 4.10). *We say that the hyperelliptic curve C is y-regular if either $p \neq 2$ or $D_{\mathfrak{s}} = 1$ for any $\mathfrak{s} \in \Sigma_K$.*

Definition 1.5. *Let $\mathfrak{s} \in \Sigma_K$ and let $c \in \{0, \ldots, b_{\mathfrak{s}} - 1\}$ such that $c\rho_{\mathfrak{s}} - \frac{1}{b_{\mathfrak{s}}} \in \mathbb{Z}$. Define*

$$\tilde{\mathfrak{s}} = \{\mathfrak{s}' \in \Sigma_K \cup \{\varnothing\} \mid \mathfrak{s}' < \mathfrak{s} \text{ and } \tfrac{\lvert\mathfrak{s}'\rvert}{b_{\mathfrak{s}}} - c\epsilon_{\mathfrak{s}} \notin 2\mathbb{Z}\},$$

where $\varnothing < \mathfrak{s}$ if $\mathfrak{s}$ is minimal and $p^0_{\mathfrak{s}} = 2$.

The genus $g(\mathfrak{s})$ of a rational cluster $\mathfrak{s} \in \Sigma_K$ is defined as follows:

- *If $D_{\mathfrak{s}} = 1$, then $g(\mathfrak{s}) = 0$.*
- *If $D_{\mathfrak{s}} = 2$, then $2g(\mathfrak{s}) + 1$ or $2g(\mathfrak{s}) + 2$ equals* $\dfrac{\lvert\mathfrak{s}\rvert - \sum_{\mathfrak{s}' < \mathfrak{s}} \lvert\mathfrak{s}'\rvert}{b_{\mathfrak{s}}} + \lvert\tilde{\mathfrak{s}}\rvert.$

Notation 1.6. *Let $\alpha \in \mathbb{Z}_+$, $a, b \in \mathbb{Q}$, with $a > b$, and fix $\frac{n_i}{d_i} \in \mathbb{Q}$ so that*

$$\alpha a = \frac{n_0}{d_0} > \frac{n_1}{d_1} > \ldots > \frac{n_r}{d_r} > \frac{n_{r+1}}{d_{r+1}} = \alpha b, \quad \text{with} \quad \begin{vmatrix} n_i & n_{i+1} \\ d_i & d_{i+1} \end{vmatrix} = 1,$$

and r minimal. We write $\mathbb{P}^1(\alpha, a, b)$ for a chain of $\mathbb{P}^1$s (Notation 4.16) of length r and multiplicities $\alpha d_i, \ldots, \alpha d_r$. Denote by $\mathbb{P}^1(\alpha, a)$ the chain $\mathbb{P}^1(\alpha, a, \lfloor \alpha a - 1 \rfloor / \alpha)$.

The following theorem describes the special fibre of a regular model of C with strict normal crossings.[2] It follows from a more general result constructing a proper flat model of C unconditionally

[2] In this paper a ‘normal crossings’ divisor is not a ‘strict normal crossings’ divisor in general (see e.g. [9, Remark 9.1.7]).

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

(Theorem 4.18). For the special fibre $\mathcal{C}_s^{\min}$ of the minimal regular model with normal crossings, the reader can refer to Theorem 4.23, where we also describe a defining equation for all components of $\mathcal{C}_s^{\min}$ and discuss the Galois action (for general K). Finally, note that all these models are constructed in Section 5 by giving an explicit open affine cover (see Sections 5.1–5.3).

Theorem 1.7 (Regular SNC model). *Suppose C is y-regular and has almost rational cluster picture. Then we can explicitly construct a regular model with strict normal crossings $\mathcal{C}/O_K$ of C (Sections 5.1–5.3). Its special fibre $\mathcal{C}_s/k$ is given as follows.*

(1) *Every $\mathfrak{s} \in \Sigma_K$ gives a 1-dimensional closed subscheme $\Gamma_{\mathfrak{s}}$ of multiplicity $m_{\mathfrak{s}}$. If $\mathfrak{s}$ is übereven and $\epsilon_{\mathfrak{s}}$ is even, then $\Gamma_{\mathfrak{s}}$ is the disjoint union of $\Gamma_{\mathfrak{s}}^- \simeq \mathbb{P}^1$ and $\Gamma_{\mathfrak{s}}^+ \simeq \mathbb{P}^1$, otherwise $\Gamma_{\mathfrak{s}}$ is a smooth geometrically integral curve of genus $g(\mathfrak{s})$ (write $\Gamma_{\mathfrak{s}}^- = \Gamma_{\mathfrak{s}}^+ = \Gamma_{\mathfrak{s}}$ in this case).*
(2) *Every $\mathfrak{s} \in \Sigma_K$ with $D_{\mathfrak{s}} = 1$ gives $(|\mathfrak{s}| - \sum_{\mathfrak{s}' \in \Sigma_K, \mathfrak{s}' < \mathfrak{s}} |\mathfrak{s}'| + p_{\mathfrak{s}}^0 - 2)/b_{\mathfrak{s}}$ open-ended $\mathbb{P}^1$s of multiplicity $b_{\mathfrak{s}}$ from $\Gamma_{\mathfrak{s}}$.*
(3) *Finally, for any $\mathfrak{s} \in \Sigma_K$ draw the following chains of $\mathbb{P}^1$s:*

Conditions	*Chain*	*From*	*To*
$\mathfrak{s}$ *minimal*	$\mathbb{P}^1(\gamma_{\mathfrak{s}}^0, -s_{\mathfrak{s}}^0)$	$\Gamma_{\mathfrak{s}}^-$	*open-ended*
$\mathfrak{s}$ *minimal,* $p_{\mathfrak{s}}^0/\gamma_{\mathfrak{s}}^0 = 2$	$\mathbb{P}^1(\gamma_{\mathfrak{s}}^0, -s_{\mathfrak{s}}^0)$	$\Gamma_{\mathfrak{s}}^+$	*open-ended*
$\mathfrak{s} \neq \mathfrak{R}$	$\mathbb{P}^1(\gamma_{\mathfrak{s}}, s_{\mathfrak{s}}, s_{\mathfrak{s}} - p_{\mathfrak{s}} \cdot \frac{\rho_{\mathfrak{s}} - \rho_{P(\mathfrak{s})}}{2})$	$\Gamma_{\mathfrak{s}}^-$	$\Gamma_{P(\mathfrak{s})}^-$
$\mathfrak{s} \neq \mathfrak{R}$, $p_{\mathfrak{s}}/\gamma_{\mathfrak{s}} = 2$	$\mathbb{P}^1(\gamma_{\mathfrak{s}}, s_{\mathfrak{s}}, s_{\mathfrak{s}} - p_{\mathfrak{s}} \cdot \frac{\rho_{\mathfrak{s}} - \rho_{P(\mathfrak{s})}}{2})$	$\Gamma_{\mathfrak{s}}^+$	$\Gamma_{P(\mathfrak{s})}^+$
$\mathfrak{s} = \mathfrak{R}$	$\mathbb{P}^1(\gamma_{\mathfrak{s}}, s_{\mathfrak{s}})$	$\Gamma_{\mathfrak{s}}^-$	*open-ended*
$\mathfrak{s} = \mathfrak{R}$, $p_{\mathfrak{s}}/\gamma_{\mathfrak{s}} = 2$	$\mathbb{P}^1(\gamma_{\mathfrak{s}}, s_{\mathfrak{s}})$	$\Gamma_{\mathfrak{s}}^+$	*open-ended*

Definition 1.8. *For any $\mathfrak{s} \in \Sigma_K$, an element $w_{\mathfrak{s}} \in K$ is called rational centre of $\mathfrak{s}$ if $\min_{r \in \mathfrak{s}} v(r - w_{\mathfrak{s}}) = \rho_{\mathfrak{s}}$.*

If $\mathfrak{s}' < \mathfrak{s}$ and $w_{\mathfrak{s}'}$ is a rational centre of $\mathfrak{s}'$, then $w_{\mathfrak{s}'}$ is also a rational centre of $\mathfrak{s}$. For any minimal rational cluster $\mathfrak{s}'$ fix a rational centre $w_{\mathfrak{s}'}$. For any $\mathfrak{s} \in \Sigma_K$ fix $w_{\mathfrak{s}} = w_{\mathfrak{s}'}$ for some minimal rational cluster $\mathfrak{s}' \subseteq \mathfrak{s}$.

The following result gives a basis of integral differentials when $K = K^{nr}$. In Theorem 6.4 we extend it to the case $K \neq K^{nr}$.

Theorem 1.9 (Theorem 6.3). *Suppose C is y-regular and has almost rational cluster picture. For $i = 0, \ldots, g-1$, inductively*

(i) *define* $e_i := \max_{\mathfrak{t} \in \Sigma_K} \left\{ \frac{\epsilon_{\mathfrak{t}}}{2} - \rho_{\mathfrak{t}} - \sum_{j=0}^{i-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{t}} \right\}$;

(ii) *let* $\Sigma_i = \left\{ \mathfrak{t} \in \Sigma_K \mid e_i = \frac{\epsilon_{\mathfrak{t}}}{2} - \rho_{\mathfrak{t}} - \sum_{j=0}^{i-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{t}} \right\}$;

(iii) *choose a maximal element $\mathfrak{s}_i$ of Σ_i freely.*

Then a basis of integral differentials is given by

$$\mu_i = \pi^{\lfloor e_i \rfloor} \prod_{j=0}^{i-1} (x - w_{\mathfrak{s}_j}) \frac{dx}{2y}, \qquad i = 0, \ldots, g-1.$$

Note that given e_i as in the previous theorem, the sum $\sum_{i=0}^{g-1} \lfloor e_i \rfloor$ is the quantity, often denoted by $v(\omega^\circ/\omega)$, appearing in the period in the Birch and Swinnerton-Dyer conjecture (for more details see [4], [25, §1.3]).

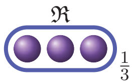
https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

1.3. Rational cluster picture

In this subsection we define the rational cluster picture and compare it with the *classical* cluster picture defined in [14]. We will show, via a simple example, in which sense the new object we introduce appears to be more suitable for the study of regular models.

Definition 1.10 (Definition 3.9). *Let K and C as before. The rational cluster picture of C is the collection of its rational clusters Σ_K together with their radii.*

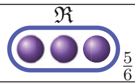
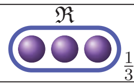
Example 1.11. *Let p be any prime number and set $K = \mathbb{Q}_p^{nr}$. Let $E_p/\mathbb{Q}_p^{nr}$ given by $y^2 = x^3 - p$. Then E_p is an elliptic curve with Kodaira-Néron reduction type II. Therefore, the minimal regular model (with normal crossings) of E_p does not depend on p. This is in accordance with the fact that the rational cluster picture of E_p is the same for all p. Indeed, the set of roots of the polynomial $x^3 - p$ is $\mathfrak{R} = \{\sqrt[3]{p}, \zeta_3\sqrt[3]{p}, \zeta_3^2\sqrt[3]{p}\}$, where ζ_3 is a primitive 3rd of unity. Hence the rational cluster picture of E_p is*



for any p,

where we denoted with bullet points the roots in $\mathfrak{R}$, with a surrounding oval the only rational cluster $\mathfrak{R}$, and with the subscript the radius $\rho_{\mathfrak{R}}$ of $\mathfrak{R}$.

A different behaviour is observed when we consider the cluster picture [14, Definition 1.26] of E_p, collection of its clusters together with their depths. The cluster picture of E_p is

$p = 2$	$p = 3$	$p > 3$
cluster picture not defined		

where the subscripts represent the depth of the cluster $\mathfrak{R}$. It does depend on p and differs from the rational cluster picture when $p = 3$. Thus, although the cluster picture is particularly useful for Galois theoretical problems, the rational cluster picture appears to be a more suitable object for the study of regular models of the curve.

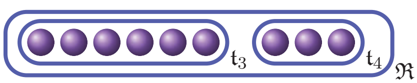
Finally, note that E_p has an almost rational cluster picture. For any two distinct roots $r, r' \in \mathfrak{R}$, the smallest v-adic disc $D_{r,r'}$ containing them also contains the whole $\mathfrak{R}$. The element $0 \in \mathbb{Q}_p^{nr}$ belongs to $D_{r,r'}$ when $p \neq 3$, while $|D_{r,r'} \cap \mathfrak{R}| = 3 = |v(r)|_p$, if $p = 3$.

The advantages of the rational cluster picture discussed in this subsection can also be observed in the following example where we study a more complex family of hyperelliptic curves having almost rational cluster picture.

1.4. Example

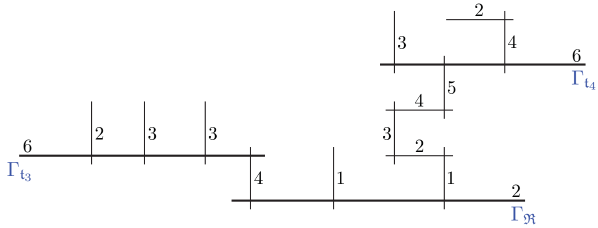
In this subsection we are going to present an example of a family of hyperelliptic curves C_p satisfying the hypothesis of Theorems 1.7 and 1.9. Via those results we will then describe the special fibre of the minimal regular model and a basis of integral differentials of C_p. All the computations involved are explained in detail in Examples 3.32, 4.25 and 6.5.

For any prime number p, let $a \in \mathbb{Z}_p$, $b \in \mathbb{Z}_p^\times$ such that the polynomial $x^2 + ax + b$ is not a square modulo p. Let $C_p/\mathbb{Q}_p$ be the hyperelliptic curve of genus 4 given by $y^2 = f(x)$, where $f(x) = (x^6 + ap^4x^3 + bp^8)((x-p)^3 - p^{11})$. The curve $C_p/\mathbb{Q}_p^{nr}$ has an almost rational cluster picture and is y-regular when $p = 2$. Its rational cluster picture is



https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

where $\rho_{\mathfrak{t}_3} = \frac{4}{3}$, $\rho_{\mathfrak{t}_4} = \frac{11}{3}$, and $\rho_{\mathfrak{R}} = 1$. From Theorem 1.7 we can construct a regular model with strict normal crossings of C_p with special fibre



over $\bar{\mathbb{F}}_p$. Computing the self-intersection of each irreducible component we easily see that this model coincides with the minimal regular model $\mathcal{C}^{\min}$. Theorem 4.23 also describes the action of the Galois group $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ on the special fibre $\mathcal{C}_s^{\min}$ of $\mathcal{C}^{\min}$. If the roots of $x^2 + ax + b \mod p$ are in $\mathbb{F}_p$ then the absolute Galois group acts trivially on each component, otherwise it swaps the 2 irreducible components of multiplicity 3 intersecting $\Gamma_{\mathfrak{t}_3}$.

From Theorem 1.9 it follows that, for any p, a basis of integral differentials of $C_p/\mathbb{Q}_p^{nr}$ is given by

$$\mu_0 = p^4 \cdot \frac{dx}{2y}, \quad \mu_1 = p^3(x-p) \cdot \frac{dx}{2y}, \quad \mu_2 = p(x-p)x \cdot \frac{dx}{2y}, \quad \mu_3 = (x-p)x^2 \cdot \frac{dx}{2y}.$$

In fact, this is also a basis of integral differentials of $C_p/\mathbb{Q}_p$ since they are all defined over $\mathbb{Q}_p$ (see Proposition B.2).

Below we will present related works of other authors concerning regular models and integral differentials of hyperelliptic curves. Note that the example presented here is not covered by [14] and [1] since the curve C_p is not semistable and not Δ_v-regular. In fact, if $p = 3$ the curve C_p does not even have tamely potential semistable reduction. The results in [5] assume $p > 2$ and C_p with tamely potential semistable reduction, hence they cannot be used when $p = 2, 3$. Finally, there is no classification for genus 4 curves.

1.5. Related works of other authors

Let K be a discretely valued field with residue field k of characteristic p and let C/K be a hyperelliptic curve of genus g.

In genus 1, when k is perfect, thanks to Tate's algorithm, one can describe the minimal regular model and the space of integral differentials of an elliptic curve C (see e.g., [24, IV.8.2], [9, Theorem 9.4.35]).

If $K = \mathbb{C}(t)$ and C has genus 2, then Namikawa and Ueno [18] and Liu [12] give a full classification of the possible configurations of the special fibre of the minimal regular model of C.

If $p \neq 2$, then Liu and Lorenzini show in [13] that regular models of C can be seen as double cover of well-chosen regular models of $\mathbb{P}^1_K$. Since the latter can be found by using the MacLane valuations [15] approach in [21], this argument gives a way to describe any regular model of a hyperelliptic curve. At the moment there is no known closed form description of a regular model based on this approach and it has not been generalised to the $p = 2$ case.

If $p > 2$, k finite, and C is semistable, then in [14] the authors explicitly construct a minimal regular model in terms of the cluster picture of C. Under the same assumptions, Kunzweiler [8] gives a basis of integral differentials rephrasing [6, Proposition 5.5] in terms of the cluster invariants introduced in [14]. These results can be recovered from Theorem 4.23 (see Corollary 4.27) and Theorem 6.3.

If $p > 2$ and C is semistable over some tamely ramified extension L/K, then Faraggi and Nowell [5] find the special fibre of the minimal regular model of C with strict normal crossings taking the quotient of the stable model of C_L and resolving the (tame) singularities. However, since they do not describe the charts of the model, their result does not immediately yield all arithmetic invariants, such as a basis of integral differentials.

The last work we want to recall represents an important ingredient of the strategy we will use in this paper (described more precisely in the next subsection). T. Dokchitser in [1] shows that the toric resolution of C gives a regular model in case of Δ_v-regularity [1, Definition 3.9]. This result, used also in [5], holds for general curves and in any residue characteristic. In his paper, Dokchitser also describes a basis of integral differentials since his model is given as open cover of affine schemes. In Corollary 3.25 and Theorem 6.1, we will rephrase his results for hyperelliptic curves by using rational cluster picture invariants from Section 3.

1.6. Strategy and outline of the paper

In [1], Dokchitser not only describes a regular model of C in case of Δ_v-regularity, but also constructs a proper flat model $\mathcal{C}_\Delta$ without any assumptions on C. Assume C is y-regular and has an almost rational cluster picture over K^{nr} with rational centres $w_1, \dots, w_m \in K^{nr}$. Our approach to construct the minimal regular model with normal crossings of C is composed by the following steps:

- Consider the x-translated hyperelliptic curves $C^{w_h}/K^{nr}{:}y^2 = f(x+w_h)$, for $h = 1, \dots, m$. For each h, [1, Theorem 3.14] constructs a proper flat model $\mathcal{C}_\Delta^{w_h}$, possibly singular.
- We glue regular open subschemes of these models along common opens, and show that the result is a proper flat regular model $\mathcal{C}$ of $C_{K^{nr}}$ with strict normal crossings.
- We give a complete description of what components of the special fibre of $\mathcal{C}$ have to be blown down to obtain the minimal model with normal crossings $\mathcal{C}^{\min}$ of $C_{K^{nr}}$.
- Finally, we describe the action of the absolute Galois group G_k of k on the special fibre of $\mathcal{C}^{\min}$.

We will explicitly describe both the models $\mathcal{C}_\Delta^{w_h}$ and $\mathcal{C}$. This allows us to study the global sections of its relative dualising sheaf $\omega_{\mathcal{C}/O_K}(\mathcal{C})$.

In Section 2, we present some results on Newton polygons used in the following sections. In Section 3, we recall the basic objects and notation of [14] and define the rational cluster picture. Moreover, we relate it with the notions given in Section 2. This comparison allows us to rephrase the objects in [1] in terms of rational clusters invariants in Section 4. In the same section we also state the theorems which describe the special fibres of a proper flat model (Theorem 4.18) and of the minimal regular model with normal crossings (Theorem 4.23) of C. The construction of these models, from which the two theorems above follow, is presented in Section 5. Finally, in Section 6, Theorems 6.3 and 6.4 describe a basis of integral differentials of C, in terms of rational clusters invariants defined in Section 3.

1.7. Notation

In the following, we present the main notation used for fields, hyperelliptic curves and Newton polytopes.

K, v	complete field with normalised discrete valuation v
O_k, π, k, p	ring of integers, uniformiser, residue field, char(k)
$\bar{K}, \bar{k}$	fixed algebraic closure of K, residue field of $\bar{K}$
K^s, K^{nr}	separable closure, maximal unramified extension of K in $\bar{K}$
$O_{K^{nr}}, k^s$	ring of integers of K^{nr}, residue field of K^{nr}
F	extension of K in $\bar{K}$, unramified in Section 4
G_K, G_k	absolute Galois groups $\mathrm{Gal}(K^s/K)$, $\mathrm{Gal}(k^s/k)$
$f(x)$	$= \sum a_i x^i$, polynomial in $K[x]$, separable from Section 3
$\mathtt{NP}(f)$	Newton polygon of f, lower convex hull of $\{(i, v(a_i)) \mid i\}$
$f\vert_L, \overline{f\vert_L}$	restriction and reduction of f to an edge L of $\mathtt{NP}(f)$ (Definition 2.5)
$g(x,y)$	$= y^2 - f(x)$, polynomial in $K[x,y]$ defining C

C	hyperelliptic curve defined over K by $g(x, y) = 0$
$f_w(x), f_h(x)$	$= f(x + w), f(x + w_h)$, for a given rational centre w_h
$g_w(x, y), g_h(x, y)$	$= y^2 - f_w(x), y^2 - f_h(x)$
C^w	$\simeq C$, hyperelliptic curve given by $g_w(x, y) = 0$
Δ^w, Δ^w_v	Newton polytopes attached to C^w as in [1, §1.1] (Notation 4.1)
$F^w_{\mathfrak{t}}, L^w_{\mathfrak{t}}, V^w_{\mathfrak{t}}, V^w_0$	v-faces and v-edges of Δ^w (Notation 4.4)
$s^\lambda_1, s^\lambda_2, r_\lambda$	$s^\lambda_1, s^\lambda_2 \in \mathbb{Q}$, $r_\lambda \in \mathbb{Z}_{\geq 0}$, attached to a v-edge of Δ^w (Notation 4.2)

For a separable polynomial $f \in k[x]$ or a hyperelliptic curve $C/K : y^2 = f(x)$ as above, the following is the main notation for clusters.

$c_f, \mathfrak{R}$	leading coefficient and set of roots of f
Σ_f, Σ_C	cluster picture, the set of clusters of f, C (Definition 3.2)
$\mathfrak{s} \in \Sigma_C$	cluster, $\mathfrak{s} = \mathcal{D} \cap \mathfrak{R}$, for a v-adic disc $\mathcal{D}$ (Definition 3.1)
$G_\mathfrak{s}, K_\mathfrak{s}, k_\mathfrak{s}$	$G_\mathfrak{s} = \mathrm{Stab}_{G_K}(\mathfrak{s})$; $K_\mathfrak{s} = (K^s)^{G_\mathfrak{s}}$; $k_\mathfrak{s}$ residue field of $K_\mathfrak{s}$
$d_\mathfrak{s}$	$= \min_{r,r' \in \mathfrak{s}} v(r - r')$ is the depth of a cluster $\mathfrak{s}$ (Definition 3.1)
$\mathfrak{s}' < \mathfrak{s} = P(\mathfrak{s}')$	$\mathfrak{s}'$ is a child of $\mathfrak{s}$ and $\mathfrak{s}$ is the parent of $\mathfrak{s}'$ (Definition 3.3)
$\mathfrak{s} \wedge \mathfrak{t}$	smallest cluster containing $\mathfrak{s}$ and $\mathfrak{t}$ (Definition 3.3)
$\rho_\mathfrak{s}$	$= \max_{w \in F} \min_{r \in \mathfrak{s}} v(r - w)$, radius of $\mathfrak{s} \in \Sigma_{C_F}$ (Definitions 3.8 and 4.6)
$b_\mathfrak{s}$	denominator of $\rho_\mathfrak{s}$ (Definition 4.6)
$w_\mathfrak{s}$	rational centre of $\mathfrak{s}$ (Definition 3.8)
$\epsilon_\mathfrak{s}$	$= v(c_f) + \sum_{r \in \mathfrak{R}} \rho_{r \wedge \mathfrak{s}}$ (Definitions 3.19 and 4.6)
$\Sigma^{\mathrm{rat}}_f, \Sigma^{\mathrm{rat}}_C$	rational cluster picture (Definition 3.9)
$\mathfrak{s} \in \Sigma^{\mathrm{rat}}_C$	rational cluster (Definition 3.9)
Σ_F	$= \Sigma^{\mathrm{rat}}_{C_F}$, for some extension F/K (Definition 4.6)
Σ^z_f, Σ^z_C	cluster picture centred at z (Definition 3.34)
$\mathfrak{s} \in \Sigma^z_C$	cluster centred at z (Definition 3.33)
$\rho^z_\mathfrak{s}, \epsilon^z_\mathfrak{s}$	$\rho^z_\mathfrak{s} = \min_{r \in \mathfrak{s}} v(r - z)$, $\epsilon^z_\mathfrak{s} = v(c_f) + \sum_{r \in \mathfrak{R}} \rho^z_{r \wedge \mathfrak{s}}$ (Definition 3.35)
Σ^W, Σ^{nr}	$\Sigma^W = \bigcup_{w \in W} \Sigma^w_C$, $\Sigma^{nr} \subset \Sigma_{K^{nr}}$ non-removable clusters (Definition 4.20)
$D_\mathfrak{s}, m_\mathfrak{s}$	$D_\mathfrak{s} = 1$ if $b_\mathfrak{s}\epsilon_\mathfrak{s}$ odd, 2 if $b_\mathfrak{s}\epsilon_\mathfrak{s}$ even; $m_\mathfrak{s} = (3 - D_\mathfrak{s})b_\mathfrak{s}$ (Definition 4.6)
$p_\mathfrak{s}$	$= 1$ if $\lvert\mathfrak{s}\rvert$ is odd, 2 if $\lvert\mathfrak{s}\rvert$ is even (Definition 4.6)
$\gamma_\mathfrak{s}$	$= 2$ if $\lvert\mathfrak{s}\rvert$ is even and $\epsilon_\mathfrak{s} - \lvert\mathfrak{s}\rvert\rho_\mathfrak{s}$ is odd, 1 otherwise (Definition 4.6)
$p^0_\mathfrak{s}$	$= 1$ if $\mathfrak{s}$ is minimal and $\mathfrak{s} \cap K_\mathfrak{s} \neq \varnothing$, 2 otherwise (Definition 4.6)
$\gamma^0_\mathfrak{s}$	$= 2$ if $p^0_\mathfrak{s} = 2$ and $\epsilon_\mathfrak{s}$ is odd, 1 otherwise (Definition 4.6)
$s_\mathfrak{s}, s^0_\mathfrak{s}$	$s_\mathfrak{s} = \frac{1}{2}(\lvert\mathfrak{s}\rvert\rho_\mathfrak{s} + p_\mathfrak{s}\rho_\mathfrak{s} - \epsilon_\mathfrak{s})$, $s^0_\mathfrak{s} = -\epsilon_\mathfrak{s}/2 + \rho_\mathfrak{s}$ (Definition 4.6)
$\overline{g_\mathfrak{s}}, \overline{g^0_\mathfrak{s}}, \overline{f^W_\mathfrak{s}}, \overline{f_\mathfrak{s}}, \tilde{f}_\mathfrak{s}$	polynomials in one variable over $k_\mathfrak{s}$ (Definitions 4.14 and 4.22)

In Section 5 we explicitly construct proper flat models of hyperelliptic curves and study the conditions for having (minimal) regular models with normal crossings. Here you can find the most used objects and notation.

Σ	$= \{\mathfrak{s}_1, \ldots, \mathfrak{s}_m\}$, set of rationally minimal clusters (Section 5.1)
$\mathfrak{s}_h$	a rationally minimal cluster, element of Σ (Section 5.1)
W	$= \{w_1, \ldots, w_m\}$, where w_h is a rational centre of $\mathfrak{s}_h$ (Section 5.1)
w_h	fixed rational centre of $\mathfrak{s}_h$, element of W (Section 5.1)
w_{hl}	$= w_h - w_l$ for fixed rational centres w_h, w_l (Section 5.1)
u_{hl}, ρ_{hl}	$u_{hl} \in O^\times_K$, $\rho_{hl} \in \mathbb{Z}$ such that $w_{hl} = u_{hl}\pi^{\rho_{hl}}$; $u_{hh} = 0$ (Section 5.1)
M	matrix associated to a proper rational cluster $\mathfrak{t} \in \Sigma^W$ (Definition 5.1, Lemma 5.2)

$\overset{M}{=}$	change of variable $(x,y,\pi)\overset{M}{=}(X,Y,Z)\bullet M^{-1}$ given by M (Section 5.2)
δ_M,σ_M,X_M	integer, cone, toric scheme attached to a matrix M (Definitio 5.1)
$m_{**},\tilde{m}_{**}$	entries of the matrices M and M^{-1} (Section 5.2)
X_Δ^h	$=\bigcup_{\mathfrak{t},M}X_M$, toric scheme constructed from $\Delta_v^{w_h}$ (Definition 5.1)
$\mathcal{C}_\Delta^w$	proper model of C^w constructed from Δ_v^w by [1, 3.14]
$\mathcal{C}_\Delta^{w_h}$	closure of $C\simeq C^{w_h}$ in X_Δ^h (Section 5.2)
R	$=O_K[X^{\pm1},Y,Z]/(\pi-X^{\tilde{m}_{13}}Y^{\tilde{m}_{23}}Z^{\tilde{m}_{33}})$ (Section 5.2)
T_M^{hl}	$\in R$, satisfying $x-w_{hl}\overset{M}{=}X^*Y^*Z^*T_M^{hl}$ (Section 5.2)
T_M^h	$=\prod_{l\neq h}T_M^{hl}\in R$ (Section 5.2)
$\mathcal{F}_M^h$	$\in R$, equals $Y^*Z^*\cdot g_h((X,Y,Z)\bullet M^{-1})$ (Section 5.2)
V_M^h	$=\operatorname{Spec}R[(T_M^h)^{-1}]\subset X_M$, (Section 5.2)
U_M^h	$=\operatorname{Spec}R[(T_M^h)^{-1}]/(\mathcal{F}_M^h)\subset V_M^h$, chart of $\mathcal{C}$ (Section 5.2)
$\mathring{X}_\Delta^h,\mathring{\mathcal{C}}_\Delta^{w_h}$	$\mathring{X}_\Delta^h=\bigcup_{\mathfrak{t},M}V_M^h\subseteq X_\Delta^h$, $\mathring{\mathcal{C}}_\Delta^{w_h}=\bigcup_{\mathfrak{t},M}U_M^h\subset X_\Delta^h$ (Section 5.2)
$\mathcal{X},\mathcal{C}$	$\mathcal{X}=\bigcup_h X_\Delta^h$, $\mathcal{C}=\bigcup_h\mathring{\mathcal{C}}_\Delta^{w_h}$ (Section 5.3)
$\hat{\mathfrak{t}}^W,\tilde{\mathfrak{t}}^W,\tilde{\mathfrak{t}}$	sets attached to a rational cluster $\mathfrak{t}$ (Definition 5.15, before Proposition 5.18 and Definition 4.13)
$\bar{X}_{F_\mathfrak{t}^w}$	1-dimensional closed subscheme of $\mathcal{C}_{\Delta,s}^w$ given by $F_\mathfrak{t}^w$ (Section 5.6)
$\mathring{X}_{F_\mathfrak{t}^w}$	$=\bar{X}_{F_\mathfrak{t}^w}\cap\mathring{\mathcal{C}}_\Delta^w$ (Section 5.6)
$\Gamma_\mathfrak{t}$	$\subseteq\mathcal{C}_s$, glueing of $\mathring{X}_{F_\mathfrak{t}^w}$ for all $w\in W$ such that $\mathfrak{t}\in\Sigma_C^w$ (Section 5.6)

2. Newton polygon

Let K be a complete field with a normalised valuation v, ring of integers O_K, uniformiser π, and residue field k of characteristic p. We fix $\bar{K}$, an algebraic closure of K, of residue field $\bar{k}$, and we denote by K^s the separable closure of K in $\bar{K}$. Denote by K^{nr} the maximal unramified extension of K in K^s, by $O_{K^{nr}}$ its ring of integers, and by k^s its residue field. Note that k^s is the separable closure of k in $\bar{k}$. Extend the valuation v to $\bar{K}$. Finally, write G_K, G_k for the Galois groups $\mathrm{Gal}(K^s/K)$, $\mathrm{Gal}(k^s/k)$, respectively.

Notation 2.1. *Let $O_{\bar{K}}=\{a\in\bar{K}\mid v(a)\geq 0\}$. Throughout this paper, given an element $a\in O_{\bar{K}}$, we will write $a\mod\pi$ for the reduction of a in $\bar{k}$. Similarly, given a polynomial $h\in O_{\bar{K}}[x_1,\dots,x_n]$, namely $h=\sum a_{i_1,\dots,i_n}\cdot x_1^{i_1}\cdots x_n^{i_n}$, we will write $h\mod\pi$ for the polynomial $\sum(a_{i_1,\dots,i_n}\mod\pi)\cdot x_1^{i_1}\cdots x_n^{i_n}\in\bar{k}[x_1,\dots,x_n]$.*

Let $f\in K[x]$ be a non-zero polynomial of degree d, say

$$f(x)=\sum_{i=0}^{d}a_ix^i.$$

The *Newton polygon* of f, denoted $\mathtt{NP}(f)$, is

$$\mathtt{NP}(f)=\text{lower convex hull }\{(i,v(a_i))\mid i=0,\dots,d,\ a_i\neq 0\}\subset\mathbb{R}^2.$$

We recall the following well-known result (see e.g., [17, II.6.3,6.4]).

Theorem 2.2. *Let $i_0<\ldots<i_s=d$ be the set of indices in $\{0,\dots,d\}$ such that the points $(i_0,v(a_{i_0})),\dots,(i_s,v(a_{i_s}))$ are the vertices of $\mathtt{NP}(f)$. For any $j=1,\dots,s$, denote by ρ_j the slope of the edge of $\mathtt{NP}(f)$ which links the points $(i_{j-1},v(a_{i_{j-1}}))$ and $(i_j,v(a_{i_j}))$. Then f has a unique factorisation over K as a product*

$$f=a_d\cdot g_0\cdot g_1\cdots g_s,$$

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

where $g_0 = x^{i_0}$ and, for all $j = 1, \ldots, s$,

- *the polynomials $g_j \in K[x]$ are monic of degree $d_j = i_j - i_{j-1}$,*
- *all the roots of g_j have valuation $-\rho_j$ in $\bar{K}$.*

In particular, $\mathtt{NP}(g_j)$ *is a segment of slope ρ_j.*

Corollary 2.3. *With the notation of Theorem 2.2, the polynomial f has exactly d_j roots of valuation $-\rho_j$ for all $j = 1, \ldots, s$.*

Corollary 2.4. *If $f = \sum a_i x^i$ is irreducible of degree d and $a_0 \neq 0$, then* $\mathtt{NP}(f)$ *is a segment linking the points $(0, v(a_0))$ and $(d, v(a_d))$.*

Definition 2.5 (Restriction and reduction). *Let $f = \sum_{i=0}^{d} a_i x^i \in K[x]$ and consider an edge L of its Newton polygon* $\mathtt{NP}(f)$. *Let $(i_1, v(a_{i_1})), (i_2, v(a_{i_2}))$, $i_1 < i_2$ be the two endpoints of L. Denote by ρ the slope of L and by n the denominator of ρ. Define the restriction of f to L as*

$$f|_L := \sum_{i=0}^{(i_2-i_1)/n} a_{ni+i_1} x^i \in K[x].$$

Moreover, we define the reduction of f with respect to L to be the polynomial

$$\overline{f|_L} := \pi^{-c} f|_L(\pi^{-n\rho} x) \bmod \pi \in k[x],$$

where $c = v(a_{i_1}) = v(a_{i_2}) + (i_1 - i_2)\rho$.

Remark 2.6. *These definitions coincide with the ones given in [1, Definitions 3.4, 3.5] when the number of variables is 1 (for suitable choices of basis of the lattices used in the definitions).*

Until the end of the section let $f \in K[x]$, consider a factorisation $f = a_d \cdot g_0 \cdot g_1 \cdots g_s$ as in Theorem 2.2. Denote by L_j the edge of slope ρ_j of $\mathtt{NP}(f)$, for any $j = 1 \ldots s$.

Remark 2.7. *By the lower convexity of* $\mathtt{NP}(f)$, *for all $j = 1, \ldots, s$, note that $\overline{f|_{L_j}} = \bar{c}_j \cdot \overline{g_j|_{\mathtt{NP}(g_j)}}$ for some $\bar{c}_j \in k^\times$. In particular they define the same k-scheme in $\mathbb{G}_{m,k}$. More precisely, for any $j = 1, \ldots, s$, let*

$$u_j = a_d \cdot \prod_{i=j+1}^{s} g_i(0).$$

Then $\bar{c}_j = u_j / \pi^{v(u_j)} \mod \pi$.

Definition 2.8. *We say that f is* $\mathtt{NP}$*-regular if the k-scheme*

$$X_{L_j} : \{\overline{f|_{L_j}} = 0\} \subset \mathbb{G}_{m,k}$$

is smooth for all $j = 1, \ldots, s$.

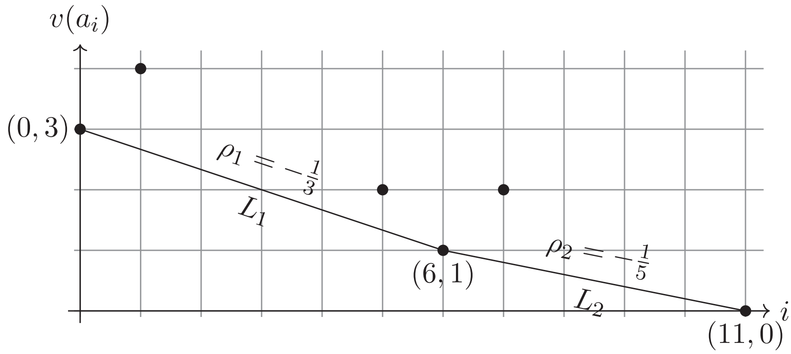
Lemma 2.9. *The polynomial $f = a_d \cdot g_0 \cdot g_1 \cdots g_s$ is* $\mathtt{NP}$*-regular if and only if g_j is* $\mathtt{NP}$*-regular for every $j = 1, \ldots, s$.*

Proof. The Lemma follows from Remark 2.7. □

We conclude this section with two examples.

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

Example 2.10. *Let* $f = x^{11} + 9x^7 - 3x^6 + 9x^5 + 81x - 27 \in \mathbb{Q}_3[x]$. *Then the Newton polygon of* f *is*



Corollary 2.3 implies that f *has* 6 *roots of valuation* $\frac{1}{3}$ *and* 5 *roots of valuation* $\frac{1}{5}$. *Furthermore, the two polynomials* g_1 *and* g_2 *in the factorisation* $f = g_1 \cdot g_2$ *of Theorem 2.2 turn out to be*

$$g_1 = x^6 + 9, \qquad g_2 = x^5 + 9x - 3.$$

Finally,

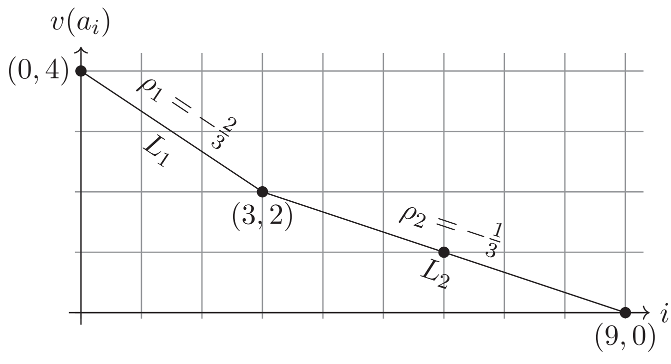
$$f|_{L_1} = -3x^2 - 27 = -3 \cdot g_1|_{\mathtt{NP}(g_1)}, \qquad f|_{L_2} = x - 3 = g_2|_{\mathtt{NP}(g_2)};$$

and

$$\overline{f|_{L_1}} = -x^2 - 1 = -(x^2+1) = -\overline{g_1|_{\mathtt{NP}(g_1)}}, \qquad \overline{f|_{L_2}} = x - 1 = \overline{g_2|_{\mathtt{NP}(g_2)}} \qquad \text{in } \mathbb{F}_3[x].$$

Thus f *is* NP*-regular.*

Example 2.11. *We now show an example of a polynomial that is not* NP*-regular. Let* $f = x^9 + 12x^6 + 36x^3 + 81 \in \mathbb{Q}_3[x]$. *Then the Newton polygon of* f *is*



Corollary 2.3 implies that f *has* 3 *roots of valuation* $\frac{2}{3}$ *and* 6 *roots of valuation* $\frac{1}{3}$. *Furthermore, the two polynomials* g_1 *and* g_2 *in the factorisation* $f = g_1 \cdot g_2$ *of Theorem 2.2 are*

$$g_1 = x^3 + 9, \qquad g_2 = x^6 + 3x^3 + 9.$$

Finally,

$$f|_{L_1} = 36x + 81 \qquad f|_{L_2} = x^2 + 12x + 36,$$

$$g_1|_{\mathtt{NP}(g_1)} = x + 9, \qquad g_2|_{\mathtt{NP}(g_2)} = x^2 + 3x + 9;$$

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

and

$$\overline{f|_{L_1}} = x + 1 = \overline{g_1|_{\mathrm{NP}(g_1)}}, \qquad \overline{f|_{L_2}} = (x+2)^2 = \overline{g_2|_{\mathrm{NP}(g_2)}} \qquad \text{in } \mathbb{F}_3[x].$$

Then f is not NP*-regular. In fact, in accordance with Lemma 2.9, g_2 is not* NP*-regular.*

3. Rational clusters

In this subsection we introduce simple combinatorial objects, that we call *rational clusters*, attached to a separable polynomial $f \in K[x]$. Via this new terminology, we will give a characterisation for the NP-regularity, from which the definition of *almost rational cluster picture*, key condition for the next sections, will follow. In fact, rational clusters are the main objects we will use for the construction of models and the description of integral differentials of hyperelliptic curves in Sections 5 and 6.

From now on, let $f \in K[x]$ be a separable polynomial and denote by $\mathfrak{R}$ the set of its roots in K^s and by c_f its leading coefficient. Then

$$f(x) = c_f \prod_{r \in \mathfrak{R}} (x - r).$$

Definition 3.1 ([14, Definition 1.1]). *A cluster (for f) is a non-empty subset $\mathfrak{s} \subseteq \mathfrak{R}$ of the form $\mathcal{D} \cap \mathfrak{R}$, where $\mathcal{D}$ is a v-adic disc $\mathcal{D} = \{x \in \bar{K} \mid v(x - z) \geq d\}$ for some $z \in \bar{K}$ and $d \in \mathbb{Q}$. If $|\mathfrak{s}| > 1$ we say that $\mathfrak{s}$ is proper and define its depth $d_{\mathfrak{s}}$ to be*

$$d_{\mathfrak{s}} = \min_{r, r' \in \mathfrak{s}} v(r - r').$$

Note that every proper cluster is cut out by a disc of the form

$$\mathcal{D} = \{x \in \bar{K} \mid v(x - r) \geq d_{\mathfrak{s}}\}$$

for any $r \in \mathfrak{s}$.

Definition 3.2 ([14, Definition 1.26]). *The cluster picture of f is the collection of its clusters, together with their depths.*

We denote by Σ_f the set of all clusters of f and by $\mathring{\Sigma}_f$ the subset of Σ_f of proper clusters.

Definition 3.3 ([14, Definition 1.3]). *If $\mathfrak{s}' \subsetneq \mathfrak{s}$ is maximal subcluster, then we say that $\mathfrak{s}'$ is a child of $\mathfrak{s}$ and $\mathfrak{s}$ is the parent of $\mathfrak{s}'$, and we write $\mathfrak{s}' < \mathfrak{s}$. For any $\mathfrak{s}', \mathfrak{s} \in \Sigma_f$, we write $\mathfrak{s}' \leq \mathfrak{s}$ if either $\mathfrak{s}' < \mathfrak{s}$ or $\mathfrak{s}' = \mathfrak{s}$. Since every cluster $\mathfrak{s} \neq \mathfrak{R}$ has one and only one parent we write $P(\mathfrak{s})$ to refer to the unique parent of $\mathfrak{s}$.*

We say that a proper cluster $\mathfrak{s}$ is minimal if it does not have any proper child.

For two clusters (or roots) $\mathfrak{s}_1, \mathfrak{s}_2$, we write $\mathfrak{s}_1 \wedge \mathfrak{s}_2$ for the smallest cluster that contains them.

Definition 3.4 ([14, Definition 1.4]). *A cluster $\mathfrak{s}$ is odd/even if its size is odd/even. If $|\mathfrak{s}| = 2$, then we say $\mathfrak{s}$ is a twin. A cluster $\mathfrak{s}$ is übereven if it has only even children.*

Definition 3.5 ([14, Definition 1.9]). *A centre $z_{\mathfrak{s}}$ of a proper cluster $\mathfrak{s}$ is any element $z_{\mathfrak{s}} \in K^s$ such that $\mathfrak{s} = \mathcal{D} \cap \mathfrak{R}$, where*

$$\mathcal{D} = \{x \in \bar{K} \mid v(x - z_{\mathfrak{s}}) \geq d_{\mathfrak{s}}\}.$$

Equivalently, $v(r - z_{\mathfrak{s}}) \geq d_{\mathfrak{s}}$ for all $r \in \mathfrak{s}$. The centre of a non-proper cluster $\mathfrak{s} = \{r\}$ is r.

Definition 3.6 ([14, Definition 1.6]). *For a proper cluster $\mathfrak{s}$ set*

$$\nu_{\mathfrak{s}} := v(c_f) + \sum_{r \in \mathfrak{R}} d_{r \wedge \mathfrak{s}}.$$

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

Definition 3.7. *We say that Σ_f is nested if one of the following equivalent conditions is satisfied:*

(i) *there exists $z \in K^s$ such that z is a centre for all proper clusters $\mathfrak{s} \in \Sigma_f$;*
(ii) *there is only one minimal cluster in Σ_f;*
(iii) *every non-minimal proper cluster has exactly one proper child.*

Definition 3.8. *A rational centre of a cluster $\mathfrak{s}$ is any element $w_{\mathfrak{s}} \in K$ such that*

$$\min_{r\in\mathfrak{s}} v(r - w_{\mathfrak{s}}) = \max_{w\in K} \min_{r\in\mathfrak{s}} v(r - w).$$

If $\mathfrak{s} = \{r\}$, with $r \in K$, then $w_{\mathfrak{s}} = r$.

If $w_{\mathfrak{s}}$ is a rational centre of a proper cluster $\mathfrak{s}$, we define the radius of $\mathfrak{s}$ to be

$$\rho_{\mathfrak{s}} = \min_{r\in\mathfrak{s}} v(r - w_{\mathfrak{s}}).$$

Definition 3.9. *A rational cluster is a cluster cut out by a v-adic disc of the form $\mathcal{D} = \{x \in \bar{K} \mid v(x - w) \geq d\}$ with $w \in K$ and $d \in \mathbb{Q}$.*

The rational cluster picture is the collection of all rational clusters of f together with their radii.

We denote by $\Sigma_f^{\mathrm{rat}} \subseteq \Sigma_f$ the set of rational clusters and by $\mathring{\Sigma}_f^{\mathrm{rat}}$ the subset of Σ_f^{rat} of proper rational clusters.

Lemma 3.10. *Let $\mathfrak{s}$ be a proper cluster. Then $d_{\mathfrak{s}} \geq \rho_{\mathfrak{s}}$.*

Proof. First we want to show that

$$\min_{r,r'\in\mathfrak{s}} v(r - r') = \max_{z\in K^s} \min_{r\in\mathfrak{s}} v(r - z).$$

Clearly $\min_{r,r'\in\mathfrak{s}} v(r - r') \leq \max_{z\in K^s} \min_{r\in\mathfrak{s}} v(r - z)$. Let $z_{\mathfrak{s}} \in K^s$ such that

$$\max_{z\in K^s} \min_{r\in\mathfrak{s}} v(r - z) = \min_{r\in\mathfrak{s}} v(r - z_{\mathfrak{s}}).$$

Then, for any $r, r' \in \mathfrak{s}$, one has

$$v(r - r') \geq \min\{v(r - z_{\mathfrak{s}}), v(r' - z_{\mathfrak{s}})\} \geq \min_{r\in\mathfrak{s}} v(r - z_{\mathfrak{s}}),$$

and so

$$\min_{r,r'\in\mathfrak{s}} v(r - r') \geq \max_{z\in K^s} \min_{r\in\mathfrak{s}} v(r - z),$$

as required. From

$$d_{\mathfrak{s}} = \min_{r,r'\in\mathfrak{s}} v(r - r') = \max_{z\in K^s} \min_{r\in\mathfrak{s}} v(r - z) \geq \max_{w\in K} \min_{r\in\mathfrak{s}} v(r - w) = \rho_{\mathfrak{s}},$$

the Lemma follows. □

Thanks to the previous lemma, the next definition gives, for any cluster $\mathfrak{s}$, the smallest rational cluster containing it.

Definition 3.11. *Given a proper cluster $\mathfrak{s} \in \Sigma_f$, we define the rationalisation $\mathfrak{s}^{\mathrm{rat}}$ of $\mathfrak{s}$ to be the smallest rational cluster containing $\mathfrak{s}$. By definition*

$$\mathfrak{s}^{\mathrm{rat}} = \mathfrak{R} \cap \{x \in \bar{K} \mid v(x - w_{\mathfrak{s}}) \geq \rho_{\mathfrak{s}}\},$$

where $w_{\mathfrak{s}}$ is a rational centre of $\mathfrak{s}$ and $\rho_{\mathfrak{s}}$ is its radius.

The next Lemma will be used in Section 5 to prove the minimality of the regular model with normal crossings we construct.

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

Lemma 3.12. *Let $\mathfrak{s} \in \Sigma_f^{\mathrm{rat}}$ be a proper cluster with rational centre $w_\mathfrak{s}$. Let $\mathfrak{s}' \in \Sigma_C^{\mathrm{rat}}$ be the child of $\mathfrak{s}$ with rational centre $w_\mathfrak{s}$ (let $\mathfrak{s}' = \varnothing$ if it does not exist). Then $(|\mathfrak{s}| - |\mathfrak{s}'|)\rho_\mathfrak{s} \in \mathbb{Z}$.*

Proof. As $\mathfrak{s} \in \Sigma_f^{\mathrm{rat}}$, one has $\mathfrak{s} = \mathfrak{s}^{\mathrm{rat}}$. Let $b_\mathfrak{s}$ be the denominator of $\rho_\mathfrak{s}$. Then $b_\mathfrak{s}$ divides the degree of the minimal polynomial of r, for any $r \in \mathfrak{s}$ satisfying $v(w_\mathfrak{s} - r) = \rho_\mathfrak{s}$. Then $(|\mathfrak{s}| - |\mathfrak{s}'|)\rho_\mathfrak{s} \in \mathbb{Z}$, where

$$\mathfrak{s}' = \mathfrak{R} \cap \{x \in \bar{K} \mid v(x - w_\mathfrak{s}) > \rho_\mathfrak{s}\},$$

as required. □

By definition, a rational cluster is G_K-invariant. Apart from that necessary condition, it is not easy to see whether a proper cluster $\mathfrak{s}$ is also a rational cluster in general. The following remark gives a sufficient condition and shows we have a simple characterisation when $K(\mathfrak{s})/K$ is tamely ramified.

Remark 3.13. *If a proper cluster $\mathfrak{s} \in \Sigma_f$ satisfies $d_\mathfrak{s} = \rho_\mathfrak{s}$, then a rational centre $w_\mathfrak{s} \in K$ of its is also a centre. Hence $\mathfrak{s}$ is a rational cluster and, in particular, is G_K-invariant. On the other hand, if a proper cluster $\mathfrak{s} \in \Sigma_f$ is G_K-invariant and $K(\mathfrak{s})/K$ is tamely ramified, then $\mathfrak{s}$ has a centre $z_\mathfrak{s} \in K$ by [14, Lemma B.1]. Thus $\rho_\mathfrak{s} = d_\mathfrak{s}$ and $\mathfrak{s} \in \Sigma_f^{\mathrm{rat}}$.*

Lemma 3.14. *Let $\mathfrak{s}$ be a proper cluster with rational centre $w_\mathfrak{s}$ and let $\mathfrak{t} \in \Sigma_f$ satisfying $\mathfrak{t} \supseteq \mathfrak{s}$. Then $w_\mathfrak{s}$ is a rational centre of $\mathfrak{t}$ and $\rho_\mathfrak{t} \leq \rho_\mathfrak{s}$. Furthermore, if $\mathfrak{s}$ is a rational cluster and $\mathfrak{t} \supsetneq \mathfrak{s}$, then $\rho_\mathfrak{t} < \rho_\mathfrak{s}$.*

Proof. It suffices to prove the Lemma for $\mathfrak{t} = P(\mathfrak{s})$. Hence we first want to show that $\min_{r \in P(\mathfrak{s})} v(r - w_\mathfrak{s}) = \rho_{P(\mathfrak{s})}$ and $\rho_{P(\mathfrak{s})} \leq \rho_\mathfrak{s}$. Note that

$$\min_{r \in P(\mathfrak{s})} v(r - w_\mathfrak{s}) \leq \max_{w \in K} \min_{r \in P(\mathfrak{s})} v(r - w) = \rho_{P(\mathfrak{s})}.$$

Moreover,

$$\rho_{P(\mathfrak{s})} = \max_{w \in K} \min_{r \in P(\mathfrak{s})} v(r - w) \leq \max_{w \in K} \min_{r \in \mathfrak{s}} v(r - w) = \rho_\mathfrak{s}.$$

If $r \in \mathfrak{s}$ then $v(w_\mathfrak{s} - r) \geq \rho_\mathfrak{s}$, by definition of $\rho_\mathfrak{s}$. On the other hand, if $r \in P(\mathfrak{s}) \smallsetminus \mathfrak{s}$ then fixing $r' \in \mathfrak{s}$ we have

$$v(r - w_\mathfrak{s}) = v(r - r' + r' - w_\mathfrak{s}) \geq \min\{v(r - r'), v(r' - w_\mathfrak{s})\} \geq \min\{d_{P(\mathfrak{s})}, \rho_\mathfrak{s}\} \geq \rho_{P(\mathfrak{s})},$$

by the previous lemma. Thus $\min_{r \in P(\mathfrak{s})} v(r - w_\mathfrak{s}) = \rho_{P(\mathfrak{s})}$, as required.

Now suppose $\mathfrak{s} \in \Sigma_f^{\mathrm{rat}}$ with $\mathfrak{t} \supsetneq \mathfrak{s}$. From Definition 3.8, it follows that

$$\{x \in \bar{K} \mid v(x - w_\mathfrak{s}) \geq \rho_\mathfrak{s}\} \cap \mathfrak{R} = \mathfrak{s} \subsetneq \mathfrak{t} \subseteq \{x \in \bar{K} \mid v(x - w_\mathfrak{s}) \geq \rho_\mathfrak{t}\} \cap \mathfrak{R},$$

as $w_\mathfrak{s}$ is a rational centre of $\mathfrak{t}$. Thus $\rho_\mathfrak{t} < \rho_\mathfrak{s}$. □

Definition 3.15. *We say that a proper rational cluster $\mathfrak{s} \in \Sigma_f^{\mathrm{rat}}$ is (rationally) minimal if it does not have any proper rational child.*

From Lemma 3.14 it follows that if $W \subseteq K$ such that every minimal rational cluster has a rational centre in W, then all clusters have a rational centre in W. This fact will be key for the construction of the model in Section 5. Another important result is Lemma 3.18, that describes the depth and the radius of $\mathfrak{s} \wedge \mathfrak{s}'$ for two rational clusters $\mathfrak{s}, \mathfrak{s}'$. To prove it, we need the following two lemmas.

Lemma 3.16. *Every cluster $\mathfrak{s}$ with $\rho_\mathfrak{s} < d_\mathfrak{s}$ has no rational subcluster $\mathfrak{s}' \subsetneq \mathfrak{s}$.*

Proof. Suppose by contradiction there exists $\mathfrak{s}' \in \Sigma_C^{\mathrm{rat}}$, $\mathfrak{s}' \subsetneq \mathfrak{s}$, and fix a rational centre $w_{\mathfrak{s}'}$ of $\mathfrak{s}'$. Then $w_{\mathfrak{s}'}$ is a rational centre of $\mathfrak{s}$ by the previous lemma. If $|\mathfrak{s}'| = 1$, then $w_{\mathfrak{s}'}$ is also a centre of $\mathfrak{s}$ and this contradicts $\rho_{\mathfrak{s}} < d_{\mathfrak{s}}$; so, assume $\mathfrak{s}'$ proper. Let $r' \in \mathfrak{s}'$ such that $v(r' - w_{\mathfrak{s}'}) = \rho_{\mathfrak{s}'}$ and $r \in \mathfrak{s}$ such that $v(r - w_{\mathfrak{s}'}) = \rho_{\mathfrak{s}}$. But then $d_{\mathfrak{s}} \leq v(r - w_{\mathfrak{s}'} + w_{\mathfrak{s}'} - r') = \rho_{\mathfrak{s}}$ again by Lemma 3.14. □

In particular, the Lemma above shows that if $\mathfrak{s} \in \Sigma_f$ and $\mathfrak{s}' \in \Sigma_f^{\mathrm{rat}}$ is a maximal rational subcluster of $\mathfrak{s}$, with $\mathfrak{s}' \subsetneq \mathfrak{s}$, then $\mathfrak{s}'$ is a child of $\mathfrak{s}$. Moreover, the parent of a rational cluster is rational.

Lemma 3.17. *Let $\mathfrak{s}, \mathfrak{s}' \in \Sigma_f^{\mathrm{rat}}$ such that $\mathfrak{s}' \not\subseteq \mathfrak{s}$. If $w_{\mathfrak{s}}$ is a rational centre of $\mathfrak{s}$ then*

$$\min_{r \in \mathfrak{s}'} v(r - w_{\mathfrak{s}}) = \rho_{\mathfrak{s} \wedge \mathfrak{s}'}.$$

Proof. By Lemma 3.14 we have

$$\min_{r \in \mathfrak{s} \wedge \mathfrak{s}'} v(r - w_{\mathfrak{s}}) = \rho_{\mathfrak{s} \wedge \mathfrak{s}'}.$$

Therefore $\min_{r \in \mathfrak{s}'} v(w_{\mathfrak{s}} - r) \geq \rho_{\mathfrak{s} \wedge \mathfrak{s}'}$. Suppose by contradiction that

$$\min_{r \in \mathfrak{s}'} v(r - w_{\mathfrak{s}}) \ =: \rho > \rho_{\mathfrak{s} \wedge \mathfrak{s}'}.$$

It follows from Lemma 3.14 that

$$\min_{r \in \mathfrak{s}} v(r - w_{\mathfrak{s}}) = \rho_{\mathfrak{s}} > \rho_{\mathfrak{s} \wedge \mathfrak{s}'}$$

as $\mathfrak{s}' \not\subseteq \mathfrak{s}$. But then there exists $\tilde{r} \in (\mathfrak{s} \wedge \mathfrak{s}') \smallsetminus (\mathfrak{s} \cup \mathfrak{s}')$ such that $v(\tilde{r} - w_{\mathfrak{s}}) = \rho_{\mathfrak{s} \wedge \mathfrak{s}'}$. Consider the rational cluster

$$\mathfrak{t} := \mathfrak{R} \cap \left\{ x \in \bar{K} \mid v(x - w_{\mathfrak{s}}) \geq \min\{\rho, \rho_{\mathfrak{s}}\} \right\} \in \Sigma_f^{\mathrm{rat}}.$$

Then $\mathfrak{s}, \mathfrak{s}' \subseteq \mathfrak{t}$, but since $\tilde{r} \notin \mathfrak{t}$ we have $\mathfrak{s} \wedge \mathfrak{s}' \not\subseteq \mathfrak{t}$ that contradicts the minimality of $\mathfrak{s} \wedge \mathfrak{s}'$. □

Lemma 3.18. *Let $\mathfrak{t} \in \Sigma_f$ with at least two children in Σ_f^{rat}. Then $d_{\mathfrak{t}} = \rho_{\mathfrak{t}} \in \mathbb{Z}$ and $\mathfrak{t} \in \Sigma_f^{\mathrm{rat}}$. More precisely, if $\mathfrak{s}, \mathfrak{s}' \in \Sigma_f^{\mathrm{rat}}$ such that $\mathfrak{s} \subsetneq \mathfrak{s} \wedge \mathfrak{s}' \supsetneq \mathfrak{s}'$, then*

$$\rho_{\mathfrak{s} \wedge \mathfrak{s}'} = v(w_{\mathfrak{s}} - w_{\mathfrak{s}'}) = d_{\mathfrak{s} \wedge \mathfrak{s}'},$$

where $w_{\mathfrak{s}}$ and $w_{\mathfrak{s}'}$ are rational centres of $\mathfrak{s}$ and $\mathfrak{s}'$ respectively.

Proof. If $d_{\mathfrak{t}} = \rho_{\mathfrak{t}}$, then $\mathfrak{t} \in \Sigma_f^{\mathrm{rat}}$ by Remark 3.13. Hence it suffices to prove the second statement as $v(w_{\mathfrak{s}} - w_{\mathfrak{s}'}) \in \mathbb{Z}$. For our assumptions $\mathfrak{s}' \not\subseteq \mathfrak{s}$. Then by Lemma 3.17 there exists $r \in \mathfrak{s}'$ so that $v(r - w_{\mathfrak{s}}) = \rho_{\mathfrak{s} \wedge \mathfrak{s}'}$. Thus,

$$v(w_{\mathfrak{s}} - w_{\mathfrak{s}'}) = \min\{v(w_{\mathfrak{s}} - r), v(r - w_{\mathfrak{s}'})\} = \rho_{\mathfrak{s} \wedge \mathfrak{s}'},$$

as $v(r - w_{\mathfrak{s}'}) \geq \rho_{\mathfrak{s}'} > \rho_{\mathfrak{s} \wedge \mathfrak{s}'}$ by Lemma 3.14. Finally, $d_{\mathfrak{s} \wedge \mathfrak{s}'} = \rho_{\mathfrak{s} \wedge \mathfrak{s}'}$ follows from Lemma 3.16. □

Definition 3.19. *For a proper cluster $\mathfrak{s}$ set*

$$\epsilon_{\mathfrak{s}} := v(c_f) + \sum_{r \in \mathfrak{R}} \rho_{r \wedge \mathfrak{s}}.$$

Example 3.20. *Let $f = x^{11} - 3x^6 + 9x^5 - 27 \in \mathbb{Q}_3[x]$. The set of roots of f is*

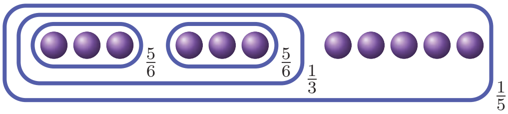
$$\mathfrak{R} = \{\sqrt[3]{3}, \zeta_3\sqrt[3]{3}, \zeta_3^2\sqrt[3]{3}, -\sqrt[3]{3}, -\zeta_3\sqrt[3]{3}, -\zeta_3^2\sqrt[3]{3}, \sqrt[5]{3}, \zeta_5\sqrt[5]{3}, \zeta_5^2\sqrt[5]{3}, \zeta_5^3\sqrt[5]{3}, \zeta_5^4\sqrt[5]{3}\},$$

where ζ_q is a primitive qth root of unity for $q = 3, 5$. Then the proper clusters of f are

$$\mathfrak{s}_1 = \{\sqrt[3]{3}, \zeta_3\sqrt[3]{3}, \zeta_3^2\sqrt[3]{3}\}, \quad \mathfrak{s}_2 = \{-\sqrt[3]{3}, -\zeta_3\sqrt[3]{3}, -\zeta_3^2\sqrt[3]{3}\}, \quad \mathfrak{s}_3 = \mathfrak{s}_1 \cup \mathfrak{s}_2, \quad \mathfrak{R}$$

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

with $d_{\mathfrak{s}_1} = d_{\mathfrak{s}_2} = \frac{5}{6}$, $d_{\mathfrak{s}_3} = \frac{1}{3}$ *and* $d_{\mathfrak{R}} = \frac{1}{5}$. *The graphic representation of the cluster picture of* f *is then*



where the subscripts of clusters (represented as circles) are their depths.

Furthermore, note that 0 *is a rational centre for all proper clusters and we have* $\rho_{\mathfrak{s}_1} = \rho_{\mathfrak{s}_2} = \rho_{\mathfrak{s}_3} = \frac{1}{3}$ *and* $\rho_{\mathfrak{R}} = \frac{1}{5}$.

Finally, for every cluster $\mathfrak{s}$ *we can also compute* $\nu_{\mathfrak{s}}$ *and* $\epsilon_{\mathfrak{s}}$, *that are*

$$\nu_{\mathfrak{s}_1} = \nu_{\mathfrak{s}_2} = \frac{9}{2}, \quad \nu_{\mathfrak{s}_3} = \epsilon_{\mathfrak{s}_1} = \epsilon_{\mathfrak{s}_2} = \epsilon_{\mathfrak{s}_3} = 3, \quad \nu_{\mathfrak{R}} = \epsilon_{\mathfrak{R}} = \frac{11}{5}.$$

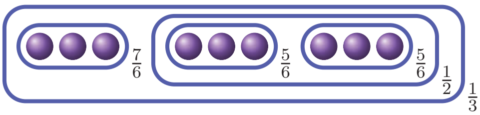
Example 3.21. *Let* $f = x^9 + 12x^6 + 36x^3 + 81 \in \mathbb{Q}_3[x]$ *and fix an isomorphism* $\overline{\mathbb{Q}}_3 \simeq \mathbb{C}$. *Then the set of roots of* f *is*

$$\mathfrak{R} = \{\sqrt[3]{3^2}, \zeta_3\sqrt[3]{3^2}, \zeta_3^2\sqrt[3]{3^2}, \zeta_9\sqrt[3]{3}, \zeta_9^2\sqrt[3]{3}, \zeta_9^4\sqrt[3]{3}, \zeta_9^5\sqrt[3]{3}, \zeta_9^7\sqrt[3]{3}, \zeta_9^8\sqrt[3]{3}\},$$

where $\zeta_q = e^{2\pi i/q}$ *is a primitive qth root of unity for* $q = 3, 9$. *Then the proper clusters of* f *are*

$$\mathfrak{s}_1 = \{\sqrt[3]{3^2}, \zeta_3\sqrt[3]{3^2}, \zeta_3^2\sqrt[3]{3^2}\}, \quad \mathfrak{s}_2 = \{\zeta_9\sqrt[3]{3}, \zeta_9^4\sqrt[3]{3}, \zeta_9^7\sqrt[3]{3}\},$$
$$\mathfrak{s}_3 = \{\zeta_9^2\sqrt[3]{3}, \zeta_9^5\sqrt[3]{3}, \zeta_9^8\sqrt[3]{3}\}, \quad \mathfrak{s}_4 = \mathfrak{s}_2 \cup \mathfrak{s}_3, \quad \mathfrak{R}$$

with $d_{\mathfrak{s}_1} = \frac{7}{6}$, $d_{\mathfrak{s}_2} = d_{\mathfrak{s}_3} = \frac{5}{6}$, $d_{\mathfrak{s}_4} = \frac{1}{2}$, *and* $d_{\mathfrak{R}} = \frac{1}{3}$. *The cluster picture of* f *is then*



It is easy to see that 0 *is a rational centre for all proper clusters and that* $\rho_{\mathfrak{s}_1} = \frac{2}{3}$, $\rho_{\mathfrak{s}_2} = \rho_{\mathfrak{s}_3} = \rho_{\mathfrak{s}_4} = \rho_{\mathfrak{R}} = \frac{1}{3}$. *Finally,*

$$\nu_{\mathfrak{s}_1} = \frac{11}{2}, \quad \nu_{\mathfrak{s}_2} = \nu_{\mathfrak{s}_3} = 5, \quad \nu_{\mathfrak{s}_4} = 4, \quad \nu_{\mathfrak{R}} = 3; \qquad \epsilon_{\mathfrak{s}_1} = 4, \quad \epsilon_{\mathfrak{s}_2} = \epsilon_{\mathfrak{s}_3} = \epsilon_{\mathfrak{s}_4} = \epsilon_{\mathfrak{R}} = 3.$$

The goal of this section is to describe the NP-regularity of $f \in K[x]$ (and its translations) in terms of conditions on its cluster picture.

Notation 3.22. *If* $p > 0$, *we denote by* $|\cdot|_p$ *the standard p-adic absolute value attached to* $\mathbb{Q}$, *that is* $|a|_p = p^{-v_p(a)}$ *for all* $a \in \mathbb{Q}$. *If* $p = 0$, *then we write* $|\cdot|_p$ *for the function on* $\mathbb{Q}$ *identically equal to* 1, *that is* $|a|_p = 1$ *for all* $a \in \mathbb{Q}$.

Lemma 3.23. *Suppose that* $x \nmid f$ *and that* NP(f) *is a segment* L *of slope* $-\rho$. *Let* n *be the denominator of* ρ. *Then* f *is* NP*-regular if and only if all proper clusters* $\mathfrak{s} \in \Sigma_f$ *with* $|\mathfrak{s}| > |\rho|_p$ *satisfy* $d_{\mathfrak{s}} = \rho$.

More precisely:

(i) *If* $\mathfrak{s} \in \Sigma_f$ *with* $|\mathfrak{s}| > |\rho|_p$ *but* $d_{\mathfrak{s}} > \rho$, *then* $\overline{f|_L}$ *has a non-zero multiple root* $\bar{u} = \frac{r^n}{\pi^{n\rho}} \mod \pi$, *for some (any)* $r \in \mathfrak{s}$.

(ii) *The multiplicity of a root* $\bar{u} \in \bar{k}^\times$ *of* $\overline{f|_L}$ *equals* $|\mathfrak{s}^0|/n$, *where*

$$\mathfrak{s}^0 = \left\{ r \in \mathfrak{R} \mid \bar{u} = \frac{r^n}{\pi^{n\rho}} \mod \pi \right\}.$$

(iii) *All multiple roots of* $\overline{f|_L}$ *come from clusters* $\mathfrak{s}$ *as described in (i).*

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

Proof. Let q be the highest power of p dividing n (set $q=1$ if $p=0$). Let $m=n/q$ so that $p\nmid m$. Let $\mathfrak{R}=\{r_i\mid i=1,\dots,D\}$ be the (multi-)set of roots of f, where $D:=\deg f$. Fix some choice of $\sqrt[n]{\pi}$ and define $\bar{u}_i\in\bar{k}^\times$ as $\bar{u}_i=r_i/\pi^\rho \mod \pi$, for all $i=1,\dots,D$. Firstly, note that there exists a proper cluster $\mathfrak{s}$ with $|\mathfrak{s}|>|\rho|_p$ and $d_\mathfrak{s}>\rho$ if and only if there exists a subset $I\subseteq\{1,\dots,D\}$ of size $|I|>q$ such that $\bar{u}_{i_1}=\bar{u}_{i_2}$ for all $i_1,i_2\in I$. Indeed, given $\mathfrak{s}$, then $I=\{i\in\{1,\dots,D\}\mid r_i\in\mathfrak{s}\}$, while given I, then $\mathfrak{s}=\{r_i\mid \bar{u}_i=\bar{u}_{i_0},\ \text{for any } i_0\in I\}$. Secondly, recall that f is not $\mathtt{NP}$-regular if and only if $\overline{f|_L}$ has a multiple root in $\bar{k}^\times$. Therefore we will prove that $\overline{f|_L}$ has a non-zero multiple root if and only if there exists a subset $I\subseteq\{1,\dots,D\}$ with size $|I|>q$ and such that $\bar{u}_{i_1}=\bar{u}_{i_2}$ for all $i_1,i_2\in I$.

Note that for the lower convexity of $\mathtt{NP}(f)=L$, we have

$$\overline{f|_L}(x^n)=\pi^{-(v(c_f)+D\rho)}f(\pi^\rho x) \mod \pi.$$

Hence $\{\bar{u}_i\mid i=1,\dots,D\}$ is the multiset of roots of $\overline{f|_L}(x^n)$. Then there exists an n-to-1 map

$$\begin{aligned}\bar{\phi}:\{\bar{u}_i\}&\longrightarrow\{\bar{w}_j\},\\ \bar{u}_i&\mapsto \bar{u}_i^m.,\end{aligned}$$

where $\{\bar{w}_j\mid j=1,\dots,D/n\}$ is the multiset of roots of $\overline{f|_L}$. Note that $\bar{w}_j\neq 0$ for all $j=1,\dots,D/n$, so all roots of $\overline{f|_L}$ are non-zero.

Now, suppose that f is not $\mathtt{NP}$-regular. We want to show that there exists a subset $I\subset\{1,\dots,D\}$ with $|I|>q$ such that $\bar{u}_{i_1}=\bar{u}_{i_2}$ for all $i_1,i_2\in I$. Since f is not $\mathtt{NP}$-regular, its reduction $\overline{f|_L}$ has a (non-zero) multiple root. Then there exist $j_1,j_2\in\{1,\dots,D/n\}$ so that $\bar{w}_{j_1}=\bar{w}_{j_2}=:\bar{w}$. Hence, by the definition of $\bar{\phi}$, for some (any) $\bar{u}\in\bar{\phi}^{-1}(\bar{w})$, there are at least $2q$ $\bar{u}_i$'s with $\bar{u}_i=\bar{u}$. Let I denote the set of their indices. Then $|I|\geq 2q>q$ and $\bar{u}_{i_1}=\bar{u}_{i_2}$ for all $i_1,i_2\in I$, as required.

On the other hand, suppose that there exists a subset $I\subset\{1,\dots,D\}$ with $|I|>q$ and such that $\bar{u}_{i_1}=\bar{u}_{i_2}$ for all $i_1,i_2\in I$. We want to show that $\overline{f|_L}$ has a multiple root, that is there exist two indices $j_1,j_2\in\{1,\dots,D/n\}$ such that $\bar{w}_{j_1}=\bar{w}_{j_2}$. Suppose not and let $j\in\{1,\dots,D/n\}$ such that $\bar{w}_j=\bar{u}_i^m=\bar{\phi}(\bar{u}_i)$ for some (all) $i\in I$. Then the polynomial $x^n-\bar{w}_j=(x^m-\bar{w}_j)^q\in\bar{k}[x]$, factor of $\overline{f|_L}(x^n)$, should have a root of order $|I|>q$. This would imply $x^m-\bar{w}_j$ is inseparable, a contradiction as $p\nmid m$.

The parts (i), (ii) and (iii) of the Lemma follow from above:

(i) Given a proper cluster $\mathfrak{s}\in\Sigma_f$ with $|\mathfrak{s}|>|\rho|_p$ and $d_\mathfrak{s}>\rho$, we showed that $\overline{f|_L}$ has a non-zero multiple root $\bar{w}_j=\bar{u}_i^n=r_i^n/\pi^{n\rho} \mod \pi$, where r_i is any root in $\mathfrak{s}$.

(ii) By the definition of $\bar{\phi}$, given $\bar{w}\in\bar{k}$, the number of $\bar{w}_j$'s such that $\bar{w}_j=\bar{w}$ equals $|\mathfrak{s}^0|/n$, where $\mathfrak{s}^0=\{r_i\mid \bar{u}_i^n=\bar{w}\}$.

(iii) Given a (non-zero) multiple root $\bar{w}$ of $\overline{f|_L}$ we showed that there exists $I\subseteq\{1,\dots,D\}$, with $|I|>q$ and $\bar{u}_{i_1}=\bar{u}_{i_2}$ for any $i_1,i_2\in I$, such that $\bar{u}_i^n=\bar{w}$ for all $i\in I$. The set $\mathfrak{s}=\{r_i\mid \bar{u}_i=\bar{u}_{i_0},\ \text{for any } i_0\in I\}$ is a proper cluster as in (i). $\square$

Theorem 3.24. *Let $w\in K$ and $f_w(x)=f(x+w)$. For all clusters $\mathfrak{s}\in\Sigma_f$ define $\lambda_\mathfrak{s}=\min_{r\in\mathfrak{s}}v(r-w)$, and let b be the denominator of $\lambda_\mathfrak{s}$. Then f_w is $\mathtt{NP}$-regular if and only if all proper clusters $\mathfrak{s}\in\Sigma_f$ with $|\mathfrak{s}|>|\lambda_\mathfrak{s}|_p$ have $d_\mathfrak{s}=\lambda_\mathfrak{s}$.*

More precisely:

(i) *Let $\mathfrak{s}\in\Sigma_f$ with $|\mathfrak{s}|>|\lambda_\mathfrak{s}|_p$ but $d_\mathfrak{s}>\lambda_\mathfrak{s}$, and let $r\in\mathfrak{s}$ with $v(r-w)=\lambda_\mathfrak{s}$. Then $\overline{f_w|_L}$ has a non-zero multiple root $\bar{u}=\frac{(r-w)^b}{\pi^{b\lambda_\mathfrak{s}}} \mod \pi$, where L is the edge of $\mathtt{NP}(f_w)$ of slope $-\lambda_\mathfrak{s}$.*

(ii) *Let L be an edge of $\mathtt{NP}(f_w)$ of slope $-\lambda$. Let l be the denominator of λ. The multiplicity of a root $\bar{u}\in\bar{k}^\times$ of $\overline{f_w|_L}$ equals $|\mathfrak{s}^0|/l$, where*

$$\mathfrak{s}^0=\Big\{r\in\mathfrak{R}\mid v(r-w)=\lambda \quad\text{and}\quad \bar{u}=\tfrac{(r-w)^l}{\pi^{l\lambda}} \mod \pi\Big\}.$$

(iii) *For every edge L of $\mathtt{NP}(f_w)$, the multiple roots of $\overline{f_w|_L}$ come from proper clusters $\mathfrak{s}$ for f as described in (i).*

Proof. Let $\mathfrak{R}_w$ be the set of roots of f_w. Note that we have a natural bijection $\mathfrak{R} \to \mathfrak{R}_w$, $r \mapsto r - w$, which induces a bijective function $\psi : \Sigma_f \to \Sigma_{f_w}$, sending

$$\mathfrak{s} = \mathfrak{R} \cap \{x \in \bar{K} \mid v(x-z) > d\} \quad \mapsto \quad \psi(\mathfrak{s}) = \mathfrak{R}_w \cap \{x \in \bar{K} \mid v(x+w-z) > d\}.$$

In particular, if $\mathfrak{s} \in \Sigma_f$, $|\mathfrak{s}| = |\psi(\mathfrak{s})|$, $d_{\mathfrak{s}} = d_{\psi(\mathfrak{s})}$ and

$$\lambda_{\mathfrak{s}} = \min_{r \in \mathfrak{s}} v(r-w) = \min_{r \in \psi(\mathfrak{s})} v(r).$$

Hence it suffices to show the theorem for $w = 0$.

Assume $w = 0$. Let $f = c_f \cdot g_0 \cdot g_1 \dots g_t$ be a factorisation of Theorem 2.2. Note that if $t = 0$, then either $f \in K$ or $f \in Kx$. In both cases, f is clearly $\mathtt{NP}$-regular and has no proper clusters. Then assume $t > 0$ and let $-\rho_i$ be the slope of $\mathtt{NP}(g_i)$ for any $i = 1, \dots, t$. Denote by $\mathfrak{R}$ the set of roots of f and by $\mathfrak{R}_i$ the set of roots of g_i for $i = 0, \dots, t$. Note that the $\mathfrak{R}_i$'s are pairwise disjoint. From Remark 2.7, for every edge L of $\mathtt{NP}(f)$ there exists i such that $\overline{f|_L} = \bar{c}_i \cdot \overline{g_i|_{\mathtt{NP}(g_i)}}$ for some $\bar{c}_i \in k^\times$. Hence, by Lemma 2.9 and Lemma 3.23, we need to prove that there exists a proper cluster $\mathfrak{s} \in \Sigma_f$ such that $|\mathfrak{s}| > |\lambda_{\mathfrak{s}}|_p$ and $d_{\mathfrak{s}} > \lambda_{\mathfrak{s}}$ if and only if for some $i = 1, \dots, t$ there exists a proper cluster $\mathfrak{s}_i \in \Sigma_{g_i}$ such that $|\mathfrak{s}_i| > |\lambda_{\mathfrak{s}_i}|_p = |\rho_i|_p$ and $d_{\mathfrak{s}_i} > \lambda_{\mathfrak{s}_i} = \rho_i$. We will show that one can choose $\mathfrak{s} = \mathfrak{s}_i$.

First, note that if $\mathfrak{s}$ is a proper cluster, then $\mathfrak{s} \not\subseteq \mathfrak{R}_0$, as $|\mathfrak{R}_0| \leq 1$. Furthermore, if $\mathfrak{s} \in \Sigma_f$ contains roots of different valuations, that is $\mathfrak{s} \not\subseteq \mathfrak{R}_i$ for all i, then

$$d_{\mathfrak{s}} = \min_{r, r' \in \mathfrak{s}} v(r - r') = \min_{r \in \mathfrak{s}} v(r) = \lambda_{\mathfrak{s}} = \min\{\rho_i \mid \mathfrak{R}_i \cap \mathfrak{s} \neq \varnothing\}.$$

Now suppose there exists a proper cluster $\mathfrak{s} \in \Sigma_f$ such that $|\mathfrak{s}| > |\lambda_{\mathfrak{s}}|_p$ and $d_{\mathfrak{s}} > \lambda_{\mathfrak{s}}$. For the observation above, the inequality $d_{\mathfrak{s}} > \lambda_{\mathfrak{s}}$ implies that $\mathfrak{s} \subseteq \mathfrak{R}_i$ for some $i = 1, \dots, t$. Let $\mathcal{D}$ be the v-adic disc such that $\mathfrak{s} = \mathcal{D} \cap \mathfrak{R}$. Since $\mathfrak{s} \subseteq \mathfrak{R}_i$, one has $\mathfrak{s} = \mathcal{D} \cap \mathfrak{R}_i$ which means that $\mathfrak{s} \in \Sigma_{g_i}$, as required.

Finally suppose that for some $i = 1, \dots, s$, there exists a proper cluster $\mathfrak{s}_i \in \Sigma_{g_i}$ such that $|\mathfrak{s}_i| > |\rho_i|_p$ and $d_{\mathfrak{s}_i} > \rho_i$. Let $r_i \in \mathfrak{s}_i$. Then

$$\mathfrak{s}_i = \{x \in \bar{K} \mid v(x - r_i) \geq d_{\mathfrak{s}_i}\} \cap \mathfrak{R}_i.$$

Consider the cluster $\mathfrak{s} := \{x \in \bar{K} \mid v(x - r_i) \geq d_{\mathfrak{s}_i}\} \cap \mathfrak{R}$ of f. Clearly $\mathfrak{s}_i \subseteq \mathfrak{s}$. Therefore

$$\lambda_{\mathfrak{s}_i} = \min_{r \in \mathfrak{s}_i} v(r) \geq \min_{r \in \mathfrak{s}} v(r) = \lambda_{\mathfrak{s}},$$

which implies

$$d_{\mathfrak{s}} = d_{\mathfrak{s}_i} > \rho_i = \lambda_{\mathfrak{s}_i} \geq \lambda_{\mathfrak{s}},$$

where $d_{\mathfrak{s}} = d_{\mathfrak{s}_i}$ by construction. Again, from the observation above, the inequality $d_{\mathfrak{s}} > \lambda_{\mathfrak{s}}$ implies that $\mathfrak{s}$ is contained in $\mathfrak{R}_j$ for some j. As $\mathfrak{s} \cap \mathfrak{R}_i \supseteq \mathfrak{s}_i \cap \mathfrak{R}_i = \mathfrak{s}_i$, we must have $\mathfrak{s} \subseteq \mathfrak{R}_i$. Thus $\mathfrak{s} = \mathfrak{s}_i$, that concludes the proof. □

Corollary 3.25. *Let $f \in K[x]$ be a separable polynomial. Let $w \in K$ and $f_w(x) = f(x+w)$. Then f_w is $\mathtt{NP}$-regular if and only if all proper clusters $\mathfrak{s} \in \Sigma_f$ have rational centre w and those with $|\mathfrak{s}| > |\rho_{\mathfrak{s}}|_p$ satisfy $d_{\mathfrak{s}} = \rho_{\mathfrak{s}}$.*

Proof. If f_w is $\mathtt{NP}$-regular, then, from the previous theorem, all proper clusters $\mathfrak{s} \in \Sigma_f$ with $|\mathfrak{s}| > |\lambda_{\mathfrak{s}}|_p$ have $d_{\mathfrak{s}} = \lambda_{\mathfrak{s}}$, where $\lambda_{\mathfrak{s}} = \min_{r \in \mathfrak{s}} v(r-w)$. First let $\mathfrak{s} \in \Sigma_f$ proper and assume $|\mathfrak{s}| > |\lambda_{\mathfrak{s}}|_p$. Then

$$d_{\mathfrak{s}} = \lambda_{\mathfrak{s}} = \min_{r \in \mathfrak{s}} v(r-w) \leq \max_{z \in K} \min_{r \in \mathfrak{s}} v(r-z) = \rho_{\mathfrak{s}} \leq d_{\mathfrak{s}},$$

so $d_{\mathfrak{s}} = \lambda_{\mathfrak{s}} = \rho_{\mathfrak{s}}$, and w is a rational centre of $\mathfrak{s}$. Now assume $|\mathfrak{s}| \leq |\lambda_{\mathfrak{s}}|_p$. In particular, $p > 0$ and $\lambda_{\mathfrak{s}} \notin \mathbb{Z}$, and so

$$\min_{r \in \mathfrak{s}} v(r-w) = \lambda_{\mathfrak{s}} \neq v(w - w_{\mathfrak{s}}),$$

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

where $w_{\mathfrak{s}}$ is a rational centre of $\mathfrak{s}$. Let $r \in \mathfrak{s}$ such that $v(r-w) = \lambda_{\mathfrak{s}}$. Then

$$\rho_{\mathfrak{s}} \leq v(r - w + w - w_{\mathfrak{s}}) = \min\{\lambda_{\mathfrak{s}}, v(w - w_{\mathfrak{s}})\} \leq \lambda_{\mathfrak{s}}.$$

Clearly

$$\rho_{\mathfrak{s}} = \max_{z\in K} \min_{r\in\mathfrak{s}} v(r - z) \geq \min_{r\in\mathfrak{s}} v(r - w) = \lambda_{\mathfrak{s}},$$

that implies $\rho_{\mathfrak{s}} = \lambda_{\mathfrak{s}} = \min_{r\in\mathfrak{s}} v(r-w)$. Hence w is a rational centre of $\mathfrak{s}$.

On the other hand, suppose that all proper clusters $\mathfrak{s} \in \Sigma_f$ have rational centre $w \in K$ and those with $|\mathfrak{s}| > |\rho_{\mathfrak{s}}|_p$ satisfy $d_{\mathfrak{s}} = \rho_{\mathfrak{s}}$. Then $\rho_{\mathfrak{s}} = \min_{r\in\mathfrak{s}} v(r-w)$ for any $\mathfrak{s} \in \Sigma_f$. Thus f_w is $\mathtt{NP}$-regular again by Theorem 3.24. ☐

The next definition, which is the main (and only, if $p \neq 2$) condition for our explicit construction of the minimal regular model of a hyperelliptic curve given by $y^2 = f(x)$, follows from the statement of Corollary 3.25.

Definition 3.26. *We say that f has an almost rational cluster picture if all proper clusters $\mathfrak{s} \in \Sigma_f$ with $|\mathfrak{s}| > |\rho_{\mathfrak{s}}|_p$ have $d_{\mathfrak{s}} = \rho_{\mathfrak{s}}$.*

Corollary 3.25 shows that f has a translation which is $\mathtt{NP}$-regular if and only if f has an almost rational cluster picture and there exists $w \in K$ that is a rational centre of all clusters.

In the following we give different characterisations of the previous definition.

Corollary 3.27. *Suppose that $K(\mathfrak{R})/K$ is a tamely ramified extension. Then f has an almost rational cluster picture if and only if every proper cluster $\mathfrak{s} \in \Sigma_f$ is G_K-invariant.*

Proof. Since $K(\mathfrak{R})/K$ is tamely ramified, every cluster $\mathfrak{s} \in \Sigma_f$ has $|\rho_{\mathfrak{s}}|_p \leq 1$. Therefore, the Corollary follows from Remark 3.13. ☐

Corollary 3.28. *Suppose that $K(\mathfrak{R})/K$ is a tamely ramified extension. Then f_w is $\mathtt{NP}$-regular for some $w \in K$ if and only if Σ_f is nested.*

Proof. First note that every cluster $\mathfrak{s} \in \Sigma_f$ has $|\rho_{\mathfrak{s}}|_p \leq 1$, as $K(\mathfrak{R})/K$ is tamely ramified. Therefore, from Corollary 3.25, we need to prove that Σ_f is nested if and only if all clusters $\mathfrak{s} \in \Sigma_f$ have $d_{\mathfrak{s}} = \rho_{\mathfrak{s}}$ and rational centre w, for some $w \in K$. But this follows from Remark 3.13. ☐

Corollary 3.29. *The polynomial f has an almost rational cluster picture if and only if for every $r \in \mathfrak{R} \setminus K$, there exists $w \in K$ so that $r_w^b := \frac{(r-w)^b}{\pi^{b\cdot v(r-w)}} \mod \pi$ is a simple root of $f_w|_L$, where b is the denominator of $v(r-w)$, $f_w(x) = f(x+w)$ and L is the edge of $\mathtt{NP}(f_w)$ of slope $-v(r-w)$.*

Proof. Fix $\tilde{r} \in \mathfrak{R} \setminus K$ and let $\mathfrak{s}$ be the smallest proper cluster containing $\tilde{r}$. Let $w_{\mathfrak{s}}$ be a rational centre of $\mathfrak{s}$. Note that $v(\tilde{r} - w_{\mathfrak{s}}) = \rho_{\mathfrak{s}} = \min_{r\in\mathfrak{s}} v(r - w_{\mathfrak{s}})$, for the choice of $\mathfrak{s}$, as $\tilde{r} \notin K$. Moreover, for any proper cluster $\mathfrak{t}$ containing $\tilde{r}$, we have $\mathfrak{s} \subseteq \mathfrak{t}$. In particular, $w_{\mathfrak{s}}$ is a rational centre of all such clusters. Let L be the edge of $\mathtt{NP}(f_{w_{\mathfrak{s}}})$ of slope $-\rho_{\mathfrak{s}}$. Theorem 3.24 shows that $\tilde{r}_{w_{\mathfrak{s}}}^{b_{\mathfrak{s}}}$ is a multiple root of $f_{w_{\mathfrak{s}}}|_L$ if and only if there exists $\mathfrak{t} \in \Sigma_f$ such that $\tilde{r} \in \mathfrak{t}$, $|\mathfrak{t}| > |\rho_{\mathfrak{t}}|_p$ and $d_{\mathfrak{t}} > \rho_{\mathfrak{t}}$. Therefore, if f has an almost rational cluster picture, then $\tilde{r}_{w_{\mathfrak{s}}}^{b_{\mathfrak{s}}}$ is a simple root.

Suppose there exists $\mathfrak{t} \in \Sigma_f$ such that $|\mathfrak{t}| > |\rho_{\mathfrak{t}}|_p$ and $d_{\mathfrak{t}} > \rho_{\mathfrak{t}}$. Then $\mathfrak{t} \cap K = \varnothing$. By Theorem 3.24, it remains to show that for any $w \in K$, we have $|\mathfrak{t}| > |\lambda_{\mathfrak{t}}|_p$ and $d_{\mathfrak{t}} > \lambda_{\mathfrak{t}}$, where $\lambda_{\mathfrak{t}} = \min_{r\in\mathfrak{t}} v(r-w)$. First note $d_{\mathfrak{t}} > \rho_{\mathfrak{t}} \geq \lambda_{\mathfrak{t}}$. Moreover, in the proof of Corollary 3.25, we saw that if $|\mathfrak{t}| \leq |\lambda_{\mathfrak{t}}|_p$ then $\rho_{\mathfrak{t}} = \lambda_{\mathfrak{t}}$ and so $|\mathfrak{t}| \leq |\rho_{\mathfrak{t}}|_p$; but $|\mathfrak{t}| > |\rho_{\mathfrak{t}}|_p$, thus $|\mathfrak{t}| > |\lambda_{\mathfrak{t}}|_p$. ☐

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

Lemma 3.30. *Suppose f has an almost rational cluster picture. Let $\mathfrak{s} \in \Sigma_f$ proper. If $d_\mathfrak{s} > \rho_\mathfrak{s}$, then $p > 0$ and $|\mathfrak{s}|$ is a p-power. In particular, if $w_\mathfrak{s}$ is a rational centre of $\mathfrak{s}$, for any $r \in \mathfrak{s}$, the elements $r - w_\mathfrak{s}$ are all the roots of a monic polynomial with coefficients in K^s, and constant term c such that $|v(c)|_p \geq 1$.*

Proof. Let $\mathfrak{s} \in \Sigma_f$ proper, with $d_\mathfrak{s} > \rho_\mathfrak{s}$. Since f has an almost rational cluster picture, we must have $|\mathfrak{s}| \leq |\rho_\mathfrak{s}|_p$. Since $\mathfrak{s}$ is proper, $p > 0$. Let $b_\mathfrak{s}$ be the denominator of $\rho_\mathfrak{s}$. Then $v_p(b_\mathfrak{s}) > 1$. Fix a rational centre $w_\mathfrak{s}$ of $\mathfrak{s}$ and a root $r \in \mathfrak{s}$ such that $v(r - w_\mathfrak{s}) = \rho_\mathfrak{s}$. Consider $\mathfrak{s}' = \{x \in \mathfrak{R} \mid v(x - r) > \rho_\mathfrak{s}\}$. Then $\mathfrak{s} \subseteq \mathfrak{s}' \leq \mathfrak{s}^{\mathrm{rat}}$ and $|\mathfrak{s}'| \leq |\rho_\mathfrak{s}|_p$ (as $d_{\mathfrak{s}'} > \rho_\mathfrak{s} = \rho_{\mathfrak{s}'}$). Let L be the Galois closure of $K(r)$. Let H be the wild inertia subgroup of $\mathrm{Gal}(L/K)$ and L^H the corresponding fixed field. Let $\sigma_1, \dots, \sigma_n \in H$ such that $\sigma_1(r - w_\mathfrak{s}), \dots, \sigma_n(r - w_\mathfrak{s})$ are the roots of the minimal polynomial of $r - w_\mathfrak{s}$ over L^H. Hence $\sigma_i(r) \in \mathfrak{R}$ and $\sigma_i(r) \neq \sigma_j(r)$ for any $i, j = 1, \dots, n$, $i \neq j$. From

$$\prod_{i=1}^{n} \sigma_i(r - w_\mathfrak{s}) \in L^H \qquad \text{and} \qquad v\Bigg(\prod_{i=1}^{n} \sigma_i(r - w_\mathfrak{s})\Bigg) = n \cdot \rho_\mathfrak{s},$$

it follows that $|\rho_\mathfrak{s}|_p \mid n$, and so $|\rho_\mathfrak{s}|_p \leq n$, since L^H/K is tamely ramified. By definition of H (see for example [17, Definition 9.3]) we have

$$v\Big(\tfrac{\sigma_i(r-w_\mathfrak{s})}{r-w_\mathfrak{s}} - 1\Big) > 0, \qquad \text{and so} \qquad v\big(\sigma_i(r) - r\big) = v\big(\sigma_i(r - w_\mathfrak{s}) - (r - w_\mathfrak{s})\big) > \rho_\mathfrak{s}$$

for any $i = 1, \dots, n$. Therefore $\sigma_i(r) \in \mathfrak{s}'$ for all i and so $n \leq |\mathfrak{s}'|$. Thus $n = |\mathfrak{s}'| = |\rho_\mathfrak{s}|_p$ and $\mathfrak{s} \subseteq \mathfrak{s}' = \{\sigma_i(r) \mid i = 1, \dots, n\}$. Finally, as $\mathfrak{s}'$ contains only conjugates of $r \in \mathfrak{s}$, the cluster $\mathfrak{s}'$ is union of orbits of $\mathfrak{s}$. In particular, $|\mathfrak{s}| \mid |\mathfrak{s}'| = |\rho_\mathfrak{s}|_p$, and so $|\mathfrak{s}|$ is a p-power. The rest of the Lemma follows. □

Proposition 3.31. *The polynomial f has an almost rational cluster picture if and only if for every proper cluster $\mathfrak{s} \in \Sigma_f$ one of the following is satisfied:*

(a) *the smallest disc containing $\mathfrak{s}$ also contains a rational point;*
(b) *$p > 0$ and after a translation by an element of K, the elements in $\mathfrak{s}$ are all the roots of a monic polynomial with coefficients in K^s of p-power degree and constant term c such that $|v(c)|_p \geq 1$.*

Proof. First of all, note that point (a) is equivalent to requiring $d_\mathfrak{s} = \rho_\mathfrak{s}$. Therefore, by Lemma 3.30 it only remains to show that if $\mathfrak{s} \in \Sigma_f$ with $d_\mathfrak{s} > \rho_\mathfrak{s}$ and (b) is satisfied, then $|\mathfrak{s}| \leq |\rho_\mathfrak{s}|_p$. Let $F \in K^s[x]$ be the polynomial in (b) and let $w \in K$ such that $r - w$, for $r \in \mathfrak{s}$, are all the roots of F. We have $\rho_\mathfrak{s} \geq \min_{r \in \mathfrak{s}} v(r - w)$. Fix $r \in \mathfrak{s}$ such that $\rho_\mathfrak{s} \geq v(r - w) =: \rho$. Since $d_\mathfrak{s} > \rho_\mathfrak{s} \geq v(r - w)$, we have $v(r' - w) = v(r - w) = \rho$ for any $r' \in \mathfrak{s}$. Then

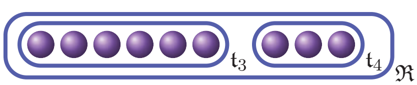
$$|\mathfrak{s}| = \deg F = |1/\deg F|_p \leq |v(c)/\deg F|_p = |\rho|_p.$$

We will prove that $\rho = \rho_\mathfrak{s}$, so that $|\mathfrak{s}| \leq |\rho|_p = |\rho_\mathfrak{s}|_p$, as required. We already know that $\rho_\mathfrak{s} \geq \rho$. Suppose by contradiction that $\rho_\mathfrak{s} > \rho$. Let $w_\mathfrak{s}$ be a rational centre of $\mathfrak{s}$ and let $r_\mathfrak{s} \in \mathfrak{s}$ such that $v(r_\mathfrak{s} - w_\mathfrak{s}) = \rho_\mathfrak{s}$. Hence

$$v(w - w_\mathfrak{s}) = v(w - r_\mathfrak{s} + r_\mathfrak{s} - w_\mathfrak{s}) = \min\{\rho, \rho_\mathfrak{s}\} = \rho.$$

But then $\rho \in \mathbb{Z}$, which contradicts $|\mathfrak{s}| \leq |\rho|_p$. □

Example 3.32. *Let p be a prime number and let $a \in \mathbb{Z}_p$, $b \in \mathbb{Z}_p^\times$ such that the polynomial $x^2 + ax + b$ is not a square modulo p. Let $f \in \mathbb{Q}_p[x]$ given by $f(x) = (x^6 + ap^4x^3 + bp^8)((x - p)^3 - p^{11})$. For any prime p the rational cluster picture of f is*

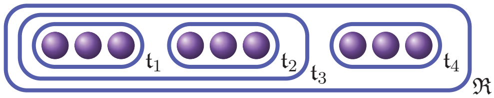


where $\rho_{\mathfrak{t}_3} = \frac{4}{3}$, $\rho_{\mathfrak{t}_4} = \frac{11}{3}$, and $\rho_\mathfrak{R} = 1$.

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

If $p \neq 3$, then the proper clusters of Σ_f coincide with the rational clusters above and $d_{\mathfrak{s}} = \rho_{\mathfrak{s}}$ for any $\mathfrak{s} = \mathfrak{t}_3, \mathfrak{t}_4, \mathfrak{R}$. In particular, f has an almost rational cluster picture when $p \neq 3$.

Suppose $p = 3$. Then the cluster picture of f is



where $d_{\mathfrak{t}_1} = d_{\mathfrak{t}_2} = \frac{11}{6}$, $d_{\mathfrak{t}_3} = \rho_{\mathfrak{t}_1} = \rho_{\mathfrak{t}_2} = \frac{4}{3}$, $d_{\mathfrak{t}_4} = \frac{25}{6}$ and $d_{\mathfrak{R}} = 1$. Thus f has an almost rational cluster picture for all p.

We conclude this section by showing that the *cluster picture centred at $w \in K$* completely determines the Newton polygon of the translation of f by w.

Definition 3.33. *Let $z \in \bar{K}$. A cluster centred at z is a cluster cut out by a v-adic disc of the form $\mathcal{D} = \{x \in \bar{K} \mid v(x - z) \geq d\}$ for some $d \in \mathbb{Q}$.*

Definition 3.34. *Let $z \in \bar{K}$. Define Σ_f^z to be the set of all clusters centred at z. Write Σ_f^z for the set $\Sigma_f^z \smallsetminus \{\{z\}\}$. Note that Σ_f^z is nested, that is every cluster $\mathfrak{s} \in \Sigma_f^z$ has at most one child in Σ_f^z.*

Definition 3.35. *Let $z \in \bar{K}$, and let $\mathfrak{s} \in \Sigma_f \setminus \{\{z\}\}$. The radius of $\mathfrak{s}$ with respect to the centre z is*

$$\rho_{\mathfrak{s}}^z = \min_{r \in \mathfrak{s}} v(r - z).$$

The cluster picture centred at z of f is the collection of all clusters in Σ_f^z together with their radii with respect to z. Finally set

$$\epsilon_{\mathfrak{s}}^z := v(c_f) + \sum_{r \in \mathfrak{R}} \rho_{r \wedge \mathfrak{s}}^z.$$

Remark 3.36. *From the definitions above, if $\mathfrak{s}$ is a cluster centred at $z \in K^{\mathrm{s}}$, then $\mathfrak{s} = \mathfrak{R} \cap \{x \in \bar{K} \mid v(x - z) \geq \rho_{\mathfrak{s}}^z\}$. But this does not mean z is a centre for $\mathfrak{s}$, that is false in general. For example, $\mathfrak{R}$ is clearly a cluster centred at any $z \in K^{\mathrm{s}}$, but there are elements of K^{s} which are not centres of $\mathfrak{R}$, for example any $z \in K^{\mathrm{s}}$ with valuation $v(z) < \min_{r \in \mathfrak{R}} v(r)$.*

Remark 3.37. *Let $\mathfrak{s} \in \Sigma_f$ be a proper cluster with centre z and rational centre w. Then $\mathfrak{s} \in \Sigma_f^z$, $d_{\mathfrak{s}} = \rho_{\mathfrak{s}}^z$, $\nu_{\mathfrak{s}} = \epsilon_{\mathfrak{s}}^z$, $\rho_{\mathfrak{s}} = \rho_{\mathfrak{s}}^w$, and $\epsilon_{\mathfrak{s}} = \epsilon_{\mathfrak{s}}^w$. Furthermore, $\mathfrak{s} \in \Sigma_f^{\mathrm{rat}}$ if and only if $\mathfrak{s} \in \Sigma_f^w$.*

The following result gives a complete description of the Newton polygon of the translation of f by $w \in K$, knowing the cluster picture centred at w of f.

Lemma 3.38. *Let $w \in K$ and let $f_w(x) = f(x + w)$. Then there is a 1-to-1 correspondence between the clusters in Σ_f^w and the edges of $\mathtt{NP}(f_w)$. More explicitly, let $\mathfrak{s}_1 \subset \cdots \subset \mathfrak{s}_n = \mathfrak{R}$ be the clusters in Σ_f^w and let $\mathfrak{s}_0 = \{w\}$ if $\{w\} \in \Sigma_f^w$ or $\mathfrak{s}_0 = \varnothing$ otherwise. Then $\mathtt{NP}(f_w)$ has vertices Q_i, $i = 0, \ldots, n$, where*

- $Q_n = (|\mathfrak{R}|, \epsilon_{\mathfrak{R}}^w - |\mathfrak{R}|\rho_{\mathfrak{R}}^w) = (\deg f, v(c_f))$,
- $Q_i = (|\mathfrak{s}_i|, \epsilon_{\mathfrak{s}_i}^w - |\mathfrak{s}_i|\rho_{\mathfrak{s}_i}^w) = (|\mathfrak{s}_i|, \epsilon_{\mathfrak{s}_{i+1}}^w - |\mathfrak{s}_i|\rho_{\mathfrak{s}_{i+1}}^w)$, *for* $i = 1, \ldots, n - 1$,
- $Q_0 = (|\mathfrak{s}_0|, \epsilon_{\mathfrak{s}_1}^w - |\mathfrak{s}_0|\rho_{\mathfrak{s}_1}^w)$.

and edges L_i, $i = 1, \ldots, n$, of slope $-\rho_{\mathfrak{s}_i}^w$ linking Q_{i-1} and Q_i.

Furthermore, for any $i = 1, \ldots, n$ we have

$$\overline{f_w|_{L_i}}(x^{b_i}) = \tfrac{u}{\pi^{v(u)}} \prod_{r \in \mathfrak{s}_i \setminus \mathfrak{s}_{i-1}} \left(x + \tfrac{w-r}{\pi^{\rho_i}}\right) \mod \pi, \qquad u = c_f \prod_{r \in \mathfrak{R} \setminus \mathfrak{s}} (w - r),$$

where $\rho_i = \rho_{\mathfrak{s}_i}^w$, and b_i is the denominator of ρ_i.

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

Proof. Without loss of generality we can assume $w = 0$ so that $f_w = f$. First note that the coordinates of Q_n are trivial. Now consider a factorisation $f = c_f \cdot g_0 \cdot g_1 \cdots g_s$ of Theorem 2.2. Recall the polynomials g_j are monic and $g_0 \mid x$. Let $\mathfrak{R}_j$ be the set of roots of g_j. It follows from the definition of cluster centred at 0 that

$$n = s, \quad \text{and} \quad \mathfrak{s}_i = \bigcup_{j=0}^{i} \mathfrak{R}_j \quad \text{for all } i = 0, \ldots, n.$$

Therefore $\mathfrak{s}_0 = \mathfrak{R}_0$ and $\mathfrak{R}_i = \mathfrak{s}_i \setminus \mathfrak{s}_{i-1}$ for any $i = 1, \ldots, n$.

Let $i = 1, \ldots, n-1$. Then the x-coordinate of Q_i follows as

$$|\mathfrak{s}_i| = \sum_{j=0}^{i} |\mathfrak{R}_j| = \sum_{j=0}^{i} \deg g_j = \deg \prod_{j=0}^{i} g_j.$$

The y-coordinate of Q_i equals the sum of $v(c_f)$ and the valuation of the constant term of $\prod_{j=i+1}^{n} g_j$, so

$$Q_i = \left(|\mathfrak{s}_i|, v(c_f) + \sum_{j=i+1}^{n} |\mathfrak{R}_j| v(r_j) \right),$$

where r_j is any root in $\mathfrak{R}_j$. But since $\mathfrak{s}_i = \bigcup_{j=0}^{i} \mathfrak{R}_j$, we have $v(r_j) = \rho^0_{\mathfrak{s}_j}$. Therefore

$$v(c_f) + \sum_{j=i+1}^{n} |\mathfrak{R}_j| v(r_j) = v(c_f) + \sum_{j=i+1}^{n} (|\mathfrak{s}_j| - |\mathfrak{s}_{j-1}|)\rho^0_{\mathfrak{s}_j} = \epsilon^0_{\mathfrak{s}_i} - |\mathfrak{s}_i|\rho^0_{\mathfrak{s}_i}.$$

Moreover,

$$\epsilon^0_{\mathfrak{s}_i} - |\mathfrak{s}_i|\rho^0_{\mathfrak{s}_i} = \epsilon^0_{\mathfrak{s}_{i+1}} - |\mathfrak{s}_i|\rho^0_{\mathfrak{s}_{i+1}}$$

from the easy computation $\epsilon^0_{\mathfrak{s}_i} - \epsilon^0_{\mathfrak{s}_{i+1}} = |\mathfrak{s}_i|\big(\rho^0_{\mathfrak{s}_i} - \rho^0_{\mathfrak{s}_{i+1}}\big)$. Finally the x-coordinate of Q_0 is trivial, while its y-coordinate equals

$$v(c_f) + \sum_{j=1}^{n} |\mathfrak{R}_j| v(r_j) = v(c_f) + \sum_{j=1}^{n} (|\mathfrak{s}_j| - |\mathfrak{s}_{j-1}|)\rho^0_{\mathfrak{s}_j} = \epsilon^0_{\mathfrak{s}_1} - |\mathfrak{s}_0|\rho^0_{\mathfrak{s}_1},$$

that concludes the first part of the proof as $|\mathfrak{s}_0| = |\mathfrak{R}_0| = \deg g_0$.

The computation of $f|_{L_i}$ follows from Remark 2.7. Indeed, let $i = 1, \ldots, n$, and define $\bar{c}_i = u/\pi^{v(u)} \bmod \pi$, where $u = c_f \prod_{j=i+1}^{n} g_j(0)$. Then $\overline{f|_{L_i}}(x^{b_i}) = \bar{c}_i \cdot \overline{g_i|_{\mathrm{NP}(g_i)}}(x^{b_i})$, where b_i is the denominator of $\rho^0_{\mathfrak{s}_i}$. But

$$\overline{g_i|_{\mathrm{NP}(g_i)}}(x^{b_i}) = g_i\big(\pi^{\rho^0_{\mathfrak{s}_i}} x\big)/\pi^{\rho^0_{\mathfrak{s}_i} \deg g_i} \mod \pi.$$

Thus the Lemma follows as $\mathfrak{R}_i = \mathfrak{s}_i \setminus \mathfrak{s}_{i-1}$. □

Notation 3.39. *Let $\mathfrak{s} \in \Sigma^w_f$. Following the notation of Lemma 3.38, let $i \in \{1, \ldots, n\}$ be such that $\mathfrak{s} = \mathfrak{s}_i$. We will write $L^w_{\mathfrak{s}}$ for the edge L_i.*

4. Description of a regular model

From now on, assume $\mathrm{char}(K) \neq 2$ and let C/K be a hyperelliptic curve, that is a geometrically connected, smooth, projective curve, equipped with a separable morphism $C \to \mathbb{P}^1_K$ of degree 2. Let $y^2 = f(x)$ be a Weierstrass equation of C. Suppose $\deg f > 1$. Let g be the genus of C. Accordingly with [14] we define the *cluster picture* of C as the cluster picture of f. Analogously, all definitions and notations attached to f given in Section 3 (e.g. Σ_f, Σ^{rat}_f, Σ^z_f) are given for C in the same way (e.g. Σ_C, Σ^{rat}_C, Σ^z_C). In particular, we will say that C has an almost rational cluster picture if f does (Definition 3.26).

For the following sections we will use the main definitions, notations and results of [1, §3]. In particular, we recall (without stating) the definitions of Newton polytopes Δ and Δ_v attached to a polynomial

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

$g \in K[x, y]$, v-vertices/edges/faces of Δ, the denominator δ_λ of a v-face/edge λ, the slopes s_1^λ, s_2^λ of a v-edge λ.

Notation 4.1. *We denote by Δ_v^w and Δ^w respectively the polytopes Δ_v and Δ attached to the polynomial $g_w(x, y) = y^2 - f(x + w)$. The piecewise affine function $v : \Delta^w \to \mathbb{R}$ determining the bijection $\Delta^w \to \Delta_v^w$, $P \mapsto (P, v(P))$, will be denoted by v (with a little abuse of notation). For a v-face F of Δ^w, denote by $v_F : \Delta^w \to \mathbb{R}$ the linear function equal to v on F. Since the projection $\Delta_v^w \to \Delta^w$ is a bijection, given a vertex/edge/face λ of Δ_v^w we will denote by the same symbol λ the corresponding v-vertex/edge/face of Δ^w. Since they are mainly used for indexing, this will not cause confusion.*

Notation 4.2. *Given a v-edge λ of Δ^w, we will denote by r_λ the smallest non-negative integer such that we can fix $\frac{n_i}{d_i} \in \mathbb{Q}$, for $i = 0, \ldots, r_\lambda + 1$ so that*

$$s_1^\lambda = \frac{n_0}{d_0} > \frac{n_1}{d_1} > \ldots > \frac{n_{r_\lambda}}{d_{r_\lambda}} > \frac{n_{r_\lambda+1}}{d_{r_\lambda+1}} = s_2^\lambda, \quad \text{with} \quad \begin{vmatrix} n_i n_{i+1} \\ d_i d_{i+1} \end{vmatrix} = 1.$$

Thanks to Lemma 3.38 we can explicitly relate the Newton polytope Δ_v^w of $g_w(x, y)$ and the cluster picture centred at w of C.

Lemma 4.3. *Let $w \in K$. Then there is a 1-to-1 correspondence between the clusters in Σ_C^w and the faces of the Newton polytope Δ_v^w. More explicitly, let $\mathfrak{s}_1 \subset \cdots \subset \mathfrak{s}_n = \mathfrak{R}$ be the clusters in Σ_C^w and let $\mathfrak{s}_0 = \{w\}$ if $\{w\} \in \Sigma_C^w$ or $\mathfrak{s}_0 = \varnothing$ otherwise. Then Δ_v^w has vertices T, Q_i, $i = 0, \ldots, n$, where*

- $T = (0, 2, 0)$,
- $Q_n = (|\mathfrak{R}|, 0, v(c_f))$,
- $Q_i = (|\mathfrak{s}_i|, 0, \epsilon^w_{\mathfrak{s}_{i+1}} - |\mathfrak{s}_i|\rho^w_{\mathfrak{s}_{i+1}})$ *for $i = 0, \ldots, n-1$,*

and edges L_i ($i = 1, \ldots, n$), linking Q_{i-1} and Q_i, and V_j ($j = 0, \ldots, n$), linking Q_j and T. Furthermore, (possible choices for) the slopes of the v-edges of Δ^w are:

- $$s_1^{V_n} = \delta_{V_n} \frac{-v(c_f) + (|\mathfrak{R}| - 2g)\rho^w_{\mathfrak{R}}}{2} \quad \text{and} \quad s_2^{V_n} = \lfloor s_1^{V_n} - 1 \rfloor;$$
- $$\begin{aligned} s_1^{V_i} &= \delta_{V_i} \left(-\frac{\epsilon^w_{\mathfrak{s}_i}}{2} + \left(\left\lfloor \frac{|\mathfrak{s}_i|}{2} \right\rfloor + 1 \right) \rho^w_{\mathfrak{s}_i} \right), \\ s_2^{V_i} &= \delta_{V_i} \left(-\frac{\epsilon^w_{\mathfrak{s}_{i+1}}}{2} + \left(\left\lfloor \frac{|\mathfrak{s}_i|}{2} \right\rfloor + 1 \right) \rho^w_{\mathfrak{s}_{i+1}} \right) \end{aligned} \qquad \text{for all } i = 1, \ldots, n-1;$$
- $$s_1^{V_0} = \delta_{V_0} \left(\frac{\epsilon^w_{\mathfrak{s}_1}}{2} - \rho^w_{\mathfrak{s}_1} \right) \quad \text{and} \quad s_2^{V_0} = \lfloor s_1^{V_0} - 1 \rfloor;$$
- $$s_1^{L_i} = \delta_{L_i} \left(-\frac{\epsilon^w_{\mathfrak{s}_i}}{2} + \left(\left\lfloor \frac{|\mathfrak{s}_i|}{2} \right\rfloor + 1 \right) \rho^w_{\mathfrak{s}_i} \right) \quad \text{and} \quad s_2^{L_i} = \lfloor s_1^{L_i} - 1 \rfloor,$$

  for all $i = 1, \ldots, n$. In particular, as δ_{L_i} is the denominator of $\rho^w_{\mathfrak{s}_i}$,

$$r_{L_i} = \begin{cases} 1 & \text{if } \delta_{L_i}\epsilon^w_{\mathfrak{s}_i} \text{ is odd,} \\ 0 & \text{if } \delta_{L_i}\epsilon^w_{\mathfrak{s}_i} \text{ is even.} \end{cases}$$

Finally, for suitable choices of basis of the lattices in [1, 3.4, 3.5], we have

$$\overline{g_w|_{L_i}}(x^{b_i}) = -\frac{u}{\pi^{v(u)}} \prod_{r \in \mathfrak{s}_i \setminus \mathfrak{s}_{i-1}} \left(x + \frac{w-r}{\pi^{\rho_i}}\right) \mod \pi, \qquad u = c_f \prod_{r \in \mathfrak{R} \setminus \mathfrak{s}_i} (w - r),$$

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

for any $i = 1, \ldots, n$, where $\rho_i = \rho^w_{\mathfrak{s}_i}$, and b_i is the denominator of ρ_i;

$$\overline{g_w|_{V_j}}(y) = y^{|\bar{V}_j(\mathbb{Z})_{\mathbb{Z}}|-1} - \tfrac{u}{\pi^{v(u)}} \mod \pi, \qquad u = c_f \prod_{r \in \mathfrak{R} \setminus \mathfrak{s}_j} (w - r),$$

for any $j = 0, \ldots, n$, where $|\bar{V}_j(\mathbb{Z})_{\mathbb{Z}}|$ is the number of integer points P on the v-edge V_j with $v(P) \in \mathbb{Z}$, endpoints included.

Proof. The structure of Δ^w_v follows from Lemma 3.38. For the computation of the slopes, we only need to individuate, for all the v-edges, the two points P_0 and P_1 of [1, Definition 3.12]. It is easy to see that the followings are admissible choices.

- For V_i and L_i ($i = 1, \ldots, n$), choose $P_0 = (|\mathfrak{s}_i|, 0)$ and $P_1 = \left(\left\lfloor \frac{|\mathfrak{s}_i|-1}{2} \right\rfloor, 1\right)$.
- For V_0, choose $P_0 = (0, 2)$ and $P_1 = (1, 1)$;

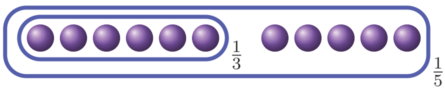
The second part of the Lemma then follows from the first one. The computations of the reductions also follows from Lemma 3.38 by choosing the lattices $Q_{i-1} + (b_i, 0)\mathbb{Z}$ for $g_w|_{L_i}$ and $Q_i + (-|\mathfrak{s}_i|/a, 2/a)\mathbb{Z}$ for $g_w|_{V_j}$, where $a = |\bar{V}_j(\mathbb{Z})_{\mathbb{Z}}| - 1$. □

Notation 4.4. *Let C be as above and let $w \in K$. For every cluster $\mathfrak{s} \in \Sigma^w_C$ denote by $F^w_{\mathfrak{s}}$ the v-face of the Newton polytope Δ^w of $g_w(x, y) = y^2 - f(x + w)$ that corresponds to $\mathfrak{s}$.*

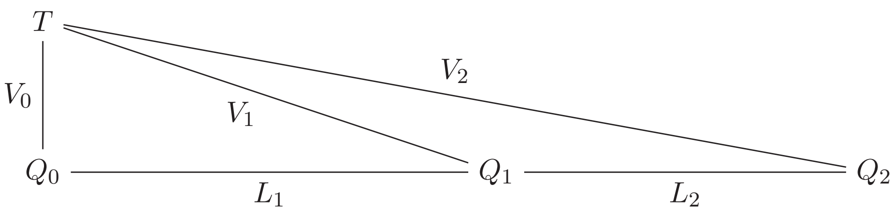
Following the notation of Lemma 4.3, let $i \in \{1, \ldots, n\}$ be such that $\mathfrak{s} = \mathfrak{s}_i$. We will write $L^w_{\mathfrak{s}}$, $V^w_{\mathfrak{s}}$, V^w_0 for the v-edges L_i, V_i, V_0, respectively.

Example 4.5. *Let C be the hyperelliptic curve over $\mathbb{Q}_3$ given by the equation $y^2 = f(x)$ where $f(x) = x^{11} - 3x^6 + 9x^5 - 27$ is the polynomial of Example 3.20.*

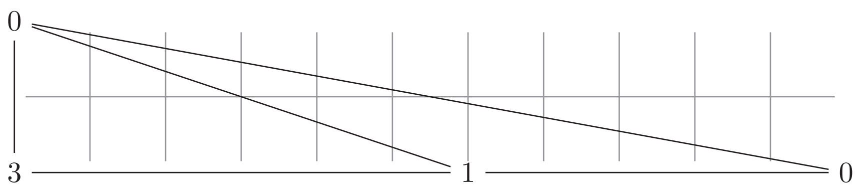
Its cluster picture centred at 0 is



where the subscripts represent the radii with respect to 0. As we can see, Σ^0_C consists of two clusters: $\mathfrak{s}_1$ of size 6, radius $\frac{1}{3}$ and $\epsilon^0_{\mathfrak{s}_1} = 3$, and $\mathfrak{s}_2 = \mathfrak{R}$ of size 11, radius $\frac{1}{5}$ and $\epsilon^0_{\mathfrak{s}_2} = \frac{11}{5}$. Therefore the picture of Δ^0 broken into v-faces will be



where $T = (0, 2)$, $Q_0 = (0, 0)$, $Q_1 = (6, 0)$, and $Q_2 = (11, 0)$. Denoting the values of v on vertices, the picture becomes



To state the theorems which describe the special fibre of the proper flat model $\mathcal{C}$ of C we will construct in Section 5, we need some definitions.

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

Definition 4.6. *Let F/K be an unramified extension and let $\Sigma_F = \Sigma_{C_F}^{\mathrm{rat}}$ (i.e. set of clusters cut out by discs with centre in F). For any proper $\mathfrak{s} \in \Sigma_F$ let $G_{\mathfrak{s}} = \mathrm{Stab}_{G_K}(\mathfrak{s})$ and $K_{\mathfrak{s}} = (K^{\mathrm{s}})^{G_{\mathfrak{s}}}$. We define the following quantities:*

$\mathfrak{s} \in \Sigma_F$, *proper*		
radius	$\rho_{\mathfrak{s}}$	$\max_{w\in F} \min_{r\in\mathfrak{s}} v(r-w)$
	$b_{\mathfrak{s}}$	*denominator of* $\rho_{\mathfrak{s}}$
	$\epsilon_{\mathfrak{s}}$	$v(c_f) + \sum_{r\in\mathfrak{R}} \rho_{r\wedge\mathfrak{s}}$
	$D_{\mathfrak{s}}$	1 *if* $b_{\mathfrak{s}}\epsilon_{\mathfrak{s}}$ *odd,* 2 *if* $b_{\mathfrak{s}}\epsilon_{\mathfrak{s}}$ *even*
multiplicity	$m_{\mathfrak{s}}$	$(3-D_{\mathfrak{s}})b_{\mathfrak{s}}$
parity	$p_{\mathfrak{s}}$	1 *if* $\lvert\mathfrak{s}\rvert$ *is odd,* 2 *if* $\lvert\mathfrak{s}\rvert$ *is even*
slope	$s_{\mathfrak{s}}$	$\frac{1}{2}(\lvert\mathfrak{s}\rvert\rho_{\mathfrak{s}} + p_{\mathfrak{s}}\rho_{\mathfrak{s}} - \epsilon_{\mathfrak{s}})$
	$\gamma_{\mathfrak{s}}$	2 *if* $\mathfrak{s}$ *is even and* $\epsilon_{\mathfrak{s}} - \lvert\mathfrak{s}\rvert\rho_{\mathfrak{s}}$ *is odd,* 1 *otherwise*
	$p_{\mathfrak{s}}^0$	1 *if* $\mathfrak{s}$ *is minimal and* $\mathfrak{s}\cap K_{\mathfrak{s}} \neq \varnothing$, 2 *otherwise*
	$s_{\mathfrak{s}}^0$	$-\epsilon_{\mathfrak{s}}/2 + \rho_{\mathfrak{s}}$
	$\gamma_{\mathfrak{s}}^0$	2 *if* $p_{\mathfrak{s}}^0 = 2$ *and* $\epsilon_{\mathfrak{s}}$ *is odd, 1 otherwise*

Lemma 4.7. *Keep the notation of the previous definition and let $\mathfrak{s} \in \Sigma_K$. Then $\mathfrak{s} \in \Sigma_F$ but the quantities in Definition 4.6 do not depend on F.*

Proof. A cluster $\mathfrak{s} \in \Sigma_F$ belongs to Σ_K if and only if $\sigma(\mathfrak{s}) = \mathfrak{s}$ for any $\sigma \in G_K$. Then the result follows from Lemma A.1. □

Remark 4.8. *Lemma 4.3 shed some light on the quantities we defined in Definition 4.6. Let $\mathfrak{s} \in \Sigma_F$. Fix a rational centre $w_{\mathfrak{s}} \in F$ of $\mathfrak{s}$ such that $w_{\mathfrak{s}} \in K_{\mathfrak{s}}$ if $p_{\mathfrak{s}}^0 = 1$. Denoting $V = V_{\mathfrak{s}}^{w_{\mathfrak{s}}}$, $L = L_{\mathfrak{s}}^{w_{\mathfrak{s}}}$, and $V_0 = V_0^{w_{\mathfrak{s}}}$, we have:*

- *$b_{\mathfrak{s}} = \delta_L$ and $r_L = 2 - D_{\mathfrak{s}}$.*
- *$\gamma_{\mathfrak{s}} = \delta_V$, $p_{\mathfrak{s}}/\gamma_{\mathfrak{s}} = \bar{V}(\mathbb{Z})_{\mathbb{Z}} - 1$ and $s_1^V = \gamma_{\mathfrak{s}} s_{\mathfrak{s}}$. If V is internal, that is $\mathfrak{s} \neq \mathfrak{R}$, then $s_2^V = \gamma_{\mathfrak{s}}(s_{\mathfrak{s}} - p_{\mathfrak{s}}\frac{\rho_{\mathfrak{s}} - \rho_{P(\mathfrak{s})}}{2})$.*
- *If $\mathfrak{s}$ is minimal and so V_0 is an edge of $F_{\mathfrak{s}}^{w_{\mathfrak{s}}}$, then $\gamma_{\mathfrak{s}}^0 = \delta_{V_0}$, $p_{\mathfrak{s}}^0/\gamma_{\mathfrak{s}}^0 = \bar{V}_0(\mathbb{Z})_{\mathbb{Z}} - 1$ and $s_1^{V_0} = -\gamma_{\mathfrak{s}}^0 s_{\mathfrak{s}}^0$.*

Lemma 4.9. *Let $\mathfrak{s} \in \Sigma_C^{\mathrm{rat}}$ with rational centre $w \in K$. Then $D_{\mathfrak{s}} = 1$ if and only if $v_{F_{\mathfrak{s}}^w}((a,1)) \notin \mathbb{Z}$, for every $a \in \mathbb{Z}$.*

Proof. If $D_{\mathfrak{s}} = 1$ then $r_{L_{\mathfrak{s}}^w} = 1$ by Lemma 4.3, and so $v_{F_{\mathfrak{s}}^w}((a,1)) \notin \mathbb{Z}$, for every $a \in \mathbb{Z}$. Now let $c, d \in \mathbb{Z}$ such that $\rho_{\mathfrak{s}} \cdot c + d = 1/b_{\mathfrak{s}}$. If $D_{\mathfrak{s}} = 2$, then $b_{\mathfrak{s}}\epsilon_{\mathfrak{s}} \in 2\mathbb{Z}$, so

$$v_{F_{\mathfrak{s}}^w}(cb_{\mathfrak{s}}\epsilon_{\mathfrak{s}}/2, 1) = \frac{v_{F_{\mathfrak{s}}^w}((cb_{\mathfrak{s}}\epsilon_{\mathfrak{s}}, 0))}{2} = \frac{\epsilon_{\mathfrak{s}} - (cb_{\mathfrak{s}}\epsilon_{\mathfrak{s}})\rho_{\mathfrak{s}}}{2} = \frac{db_{\mathfrak{s}}\epsilon_{\mathfrak{s}}}{2} \in \mathbb{Z},$$

as required. □

Definition 4.10. *We say that C is y-regular if $p \nmid D_{\mathfrak{s}}$ for every proper $\mathfrak{s} \in \Sigma_C^{\mathrm{rat}}$, that is if either $p \neq 2$ or $D_{\mathfrak{s}} = 1$ for any proper $\mathfrak{s} \in \Sigma_C^{\mathrm{rat}}$.*

Remark 4.11. *Let F/K be an unramified extension. Then from Lemma 4.7, if C_F is y-regular then C is y-regular.*

Lemma 4.12. *The hyperelliptic curve C is Δ_v-regular if and only if C is y-regular and f is* NP*-regular.*

Proof. Let $g(x, y) = y^2 - f(x)$. If C is y-regular and f is NP-regular, then C is Δ_v-regular by Lemma 4.3 and Lemma 4.9.

Conversely, if C is Δ_v-regular, then f is NP-regular, and all clusters have rational centre 0 by Corollary 3.25. It remains to show that if $p = 2$ then $D_{\mathfrak{s}} = 1$ for every proper $\mathfrak{s} \in \Sigma_C^{\mathrm{rat}}$. Suppose there exists $\mathfrak{s} \in \Sigma_C^{\mathrm{rat}}$ such that $D_{\mathfrak{s}} = 2$. Consider the variety $\bar{X}_{F^0_{\mathfrak{s}}}$ ([1, Definition 3.7]). By Lemma 4.9, the smoothness of $\bar{X}_{F^0_{\mathfrak{s}}}$ implies there exists $\mathfrak{s}' \in \Sigma_C^{\mathrm{rat}}$, such that $|\mathfrak{s}| - |\mathfrak{s}'| = 1$. Hence $\rho_{\mathfrak{s}} \in \mathbb{Z}$ from Lemma 3.12. Therefore $\bar{F}^0_{\mathfrak{s}}(\mathbb{Z}) = \bar{F}^0_{\mathfrak{s}}(\mathbb{Z})_{\mathbb{Z}}$, by Lemma 4.9. But this gives a contradiction as it forces either $\overline{g|_{V^0_{\mathfrak{s}'}}}$ or $\overline{g|_{V^0_{\mathfrak{s}}}}$ to be a square. $\square$

Definition 4.13. *Let $\mathfrak{s} \in \Sigma_F$ be a proper cluster and let $c \in \{0, \dots, b_{\mathfrak{s}} - 1\}$ such that $c\rho_{\mathfrak{s}} - \frac{1}{b_{\mathfrak{s}}} \in \mathbb{Z}$. Define*

$$\tilde{\mathfrak{s}} = \{\mathfrak{s}' \in \Sigma_F \cup \{\varnothing\} \mid \mathfrak{s}' < \mathfrak{s} \text{ and } \tfrac{|\mathfrak{s}'|}{b_{\mathfrak{s}}} - c\epsilon_{\mathfrak{s}} \notin 2\mathbb{Z}\},$$

where $\varnothing < \mathfrak{s}$ if $\mathfrak{s}$ is minimal and $p^0_{\mathfrak{s}} = 2$.

The genus $g(\mathfrak{s})$ of a rational cluster $\mathfrak{s} \in \Sigma_F$ is defined as follows:

- *If $D_{\mathfrak{s}} = 1$, then $g(\mathfrak{s}) = 0$.*
- *If $D_{\mathfrak{s}} = 2$, then $2g(\mathfrak{s}) + 1$ or $2g(\mathfrak{s}) + 2$ equals*

$$\frac{|\mathfrak{s}| - \sum_{\mathfrak{s}' \in \Sigma_F, \mathfrak{s}' < \mathfrak{s}} |\mathfrak{s}'|}{b_{\mathfrak{s}}} + |\tilde{\mathfrak{s}}|.$$

Definition 4.14. *Let $\Sigma_C^{\min}$ be the set of rationally minimal clusters of C and let $\Sigma \subseteq \Sigma_C^{\min}$ non-empty. For each cluster $\mathfrak{s} \in \Sigma$, fix a rational centre $w_{\mathfrak{s}}$; if possible, choose $w_{\mathfrak{s}} \in \mathfrak{s}$. Let W be the set of these rational centres and define $\Sigma^W = \bigcup_{w \in W} \Sigma_C^w$. For any proper cluster $\mathfrak{s} \in \Sigma^W$ fix a rational centre $w_{\mathfrak{s}} \in W$. Denote $r_{\mathfrak{s}} = \frac{w_{\mathfrak{s}} - r}{\pi^{\rho_{\mathfrak{s}}}}$ for $r \in \mathfrak{R}$. Define reductions $\overline{f^W_{\mathfrak{s}}}(x) \in k[x]$, $\overline{g_{\mathfrak{s}}} \in k[y]$, and for $\mathfrak{s} \in \Sigma$ also $\overline{g^0_{\mathfrak{s}}} \in k[y]$ by*

$$\overline{f^W_{\mathfrak{s}}}(x^{b_{\mathfrak{s}}}) = \frac{u}{\pi^{v(u)}} \prod_{r \in \mathfrak{s} \setminus \bigcup_{\mathfrak{s}' < \mathfrak{s}} \mathfrak{s}'} (x + r_{\mathfrak{s}}) \mod \pi, \qquad u = c_f \prod_{r \in \mathfrak{R} \setminus \mathfrak{s}} r_{\mathfrak{s}},$$

$$\overline{g_{\mathfrak{s}}}(y) = y^{p_{\mathfrak{s}}/\gamma_{\mathfrak{s}}} - \frac{u}{\pi^{v(u)}} \mod \pi, \qquad u = c_f \prod_{r \in \mathfrak{R} \setminus \mathfrak{s}} r_{\mathfrak{s}},$$

$$\overline{g^0_{\mathfrak{s}}}(y) = y^{p^0_{\mathfrak{s}}/\gamma^0_{\mathfrak{s}}} - \frac{u}{\pi^{v(u)}} \mod \pi, \qquad u = c_f \prod_{r \in \mathfrak{R} \setminus \{w_{\mathfrak{s}}\}} r_{\mathfrak{s}}.$$

where the union runs through all $\mathfrak{s}' \in \Sigma^W$, $\mathfrak{s}' < \mathfrak{s}$. Finally define the k-schemes

(1) $X^W_{\mathfrak{s}} : \{\overline{f^W_{\mathfrak{s}}} = 0\} \subset \mathbb{G}_{m,k}$;
(2) $X_{\mathfrak{s}} : \{\overline{g_{\mathfrak{s}}} = 0\} \subset \mathbb{G}_{m,k}$;
(3) $X^0_{\mathfrak{s}} : \{\overline{g^0_{\mathfrak{s}}} = 0\} \subset \mathbb{G}_{m,k}$ *if $\mathfrak{s} \in \Sigma$.*

Notation 4.15. *Given a scheme $\mathcal{X}/O_K$ we will denote by $\mathcal{X}_\eta$ its generic fibre $\mathcal{X} \times_{\mathrm{Spec}\, O_K} \mathrm{Spec}\, K$, and by $\mathcal{X}_s$ its special fibre $\mathcal{X} \times_{\mathrm{Spec}\, O_K} \mathrm{Spec}\, k$.*

Notation 4.16. *If $C = C_1 \cup \cdots \cup C_r$ is a chain of $\mathbb{P}^1_k$s of length r and multiplicities $m_i \in \mathbb{Z}$ (meeting transversely), then $\infty \in C_i$ is identified with $0 \in C_{i+1}$, and $0, \infty \in C$ are respectively $0 \in C_1$ and $\infty \in C_r$. Finally, if $r = 0$, then $C = \mathrm{Spec}\, k$ and $0 = \infty$.*

Notation 4.17. *Let $\alpha \in \mathbb{Z}_+$, $a, b \in \mathbb{Q}$, with $a > b$, and fix $\frac{n_i}{d_i} \in \mathbb{Q}$ so that*

$$\alpha a = \frac{n_0}{d_0} > \frac{n_1}{d_1} > \dots > \frac{n_r}{d_r} > \frac{n_{r+1}}{d_{r+1}} = \alpha b, \quad \text{with} \quad \begin{vmatrix} n_i & n_{i+1} \\ d_i & d_{i+1} \end{vmatrix} = 1,$$

and r minimal. We write $\mathbb{P}^1(\alpha, a, b)$ for a chain of $\mathbb{P}^1_k$s of length r and multiplicities $\alpha d_1, \dots, \alpha d_r$. Furthermore, we denote by $\mathbb{P}^1(\alpha, a)$ the chain $\mathbb{P}^1(\alpha, a, \lfloor \alpha a - 1 \rfloor / \alpha)$.

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

Theorem 4.18 and Theorem 4.23 will follow from Section 5.

Theorem 4.18. *Let C/K be a hyperelliptic curve given by a Weierstrass equation $y^2 = f(x)$. Suppose $\deg f > 1$ and let Σ, W and Σ^W as in Definition 4.14. Then there exists a proper flat model $\mathcal{C}/O_K$ (constructed in Section 5) of C such that its special fibre $\mathcal{C}_s/k$ consists of 1-dimensional schemes given below in (1), (2), (3), (4), (5), glued along 0-dimensional transversal intersections:*

(1) *Every proper cluster $\mathfrak{s} \in \Sigma^W$ gives a 1-dimensional closed subscheme $\Gamma_{\mathfrak{s}}$ of multiplicity $m_{\mathfrak{s}}$. $\Gamma_{\mathfrak{s}}$ is not integral if and only if $D_{\mathfrak{s}} = 2$, $\tilde{\mathfrak{s}} \cap (\Sigma^W \cup \{\varnothing\}) = \varnothing$ and $\overline{f_{\mathfrak{s}}^W}$ is a square. When this happens, if $p = 2$ then $\Gamma_{\mathfrak{s}}$ is not reduced and $(\Gamma_{\mathfrak{s}})_{\mathrm{red}}$ is irreducible of multiplicity 2 in $\Gamma_{\mathfrak{s}}$, if $p \neq 2$ then $\Gamma_{\mathfrak{s}}$ is reducible, namely $\Gamma_{\mathfrak{s}} = \Gamma_{\mathfrak{s}}^+ \cup \Gamma_{\mathfrak{s}}^-$, with $\Gamma_{\mathfrak{s}}^{\pm} = \mathbb{P}^1_k$.*

(2) *Every proper cluster $\mathfrak{s} \in \Sigma^W$ with $D_{\mathfrak{s}} = 1$ gives the closed subscheme $X_{\mathfrak{s}}^W \times \mathbb{P}^1_k$, of multiplicity $b_{\mathfrak{s}}$, where $X_{\mathfrak{s}}^W \times \{0\} \subset \Gamma_{\mathfrak{s}}$.*

(3) *Every proper cluster $\mathfrak{s} \in \Sigma^W$ such that $\mathfrak{s} \neq \mathfrak{R}$, gives the closed subscheme $X_{\mathfrak{s}} \times \mathbb{P}^1(\gamma_{\mathfrak{s}}, s_{\mathfrak{s}}, s_{\mathfrak{s}} - p_{\mathfrak{s}} \cdot \frac{\rho_{\mathfrak{s}} - \rho_{P(\mathfrak{s})}}{2})$ where $X_{\mathfrak{s}} \times \{0\} \subset \Gamma_{\mathfrak{s}}$ and $X_{\mathfrak{s}} \times \{\infty\} \subset \Gamma_{P(\mathfrak{s})}$.*

(4) *Every cluster $\mathfrak{s} \in \Sigma$ gives the closed subscheme $X_{\mathfrak{s}}^0 \times \mathbb{P}^1(\gamma_{\mathfrak{s}}^0, -s_{\mathfrak{s}}^0)$ where $X_{\mathfrak{s}}^0 \times \{0\} \subset \Gamma_{\mathfrak{s}}$ (the chains are open-ended).*

(5) *Finally, the cluster $\mathfrak{R}$ gives the closed subscheme $X_{\mathfrak{R}} \times \mathbb{P}^1(\gamma_{\mathfrak{R}}, s_{\mathfrak{R}})$ where $X_{\mathfrak{R}} \times \{0\} \subset \Gamma_{\mathfrak{R}}$ (the chains are open-ended).*

If $\Gamma_{\mathfrak{s}}$ is reducible, the two points in $X_{\mathfrak{s}} \times \{0\}$ (and $X_{\mathfrak{s}}^0 \times \{0\}$ if $\mathfrak{s} \in \Sigma$) belong to different irreducible components of $\Gamma_{\mathfrak{s}}$. Similarly, if $\mathfrak{s} \neq \mathfrak{R}$ and $\Gamma_{P(\mathfrak{s})}$ is reducible, the two points of $X_{\mathfrak{s}} \times \{\infty\}$ belong to different irreducible components of $\Gamma_{P(\mathfrak{s})}$.

Furthermore, if C has an almost rational cluster picture and is y-regular, then, by choosing $\Sigma = \Sigma_C^{\min}$, the model $\mathcal{C}$ is regular with strict normal crossings. In that case, if $\mathfrak{s}$ is übereven and $\epsilon_{\mathfrak{s}}$ is even, then $\Gamma_{\mathfrak{s}} \simeq X_{\mathfrak{s}} \times \mathbb{P}^1_k$, otherwise $\Gamma_{\mathfrak{s}}$ is irreducible of genus $g(\mathfrak{s})$.

Remark 4.19. *Consider the proper flat model $\mathcal{C}/O_K$ of Theorem 4.18. Via the canonical immersion $\mathcal{C}_s \hookrightarrow \mathcal{C}$, the singular points of $\mathcal{C}$ are images of*

- *singular points of the subscheme given in (1) when $D_{\mathfrak{s}} = 2$ and either $p = 2$, or $\mathfrak{s} = \mathfrak{t}^{\mathrm{rat}}$ for some $\mathfrak{t} \in \Sigma_C$ with $|\mathfrak{t}| > |\rho_{\mathfrak{t}}|_p$ and $d_{\mathfrak{t}} > \rho_{\mathfrak{t}}$, or $\mathfrak{s} = \mathfrak{s}_1 \wedge \mathfrak{s}_2$ for some $\mathfrak{s}_1 \in \Sigma$ and $\mathfrak{s}_2 \in \Sigma_C^{\min} \smallsetminus \Sigma$;*
- *non-reduced points of the subscheme given in (2) when $D_{\mathfrak{s}} = 1$ and either $\mathfrak{s} = \mathfrak{t}^{\mathrm{rat}}$ for some $\mathfrak{t} \in \Sigma_C$ with $|\mathfrak{t}| > |\rho_{\mathfrak{t}}|_p$ and $d_{\mathfrak{t}} > \rho_{\mathfrak{t}}$, or $\mathfrak{s} = \mathfrak{s}_1 \wedge \mathfrak{s}_2$ for some $\mathfrak{s}_1 \in \Sigma$ and $\mathfrak{s}_2 \in \Sigma_C^{\min} \smallsetminus \Sigma$;*
- *non-reduced points of subschemes given in (3), (4), (5) (that may exist only if $p = 2$).*

Note that $\mathcal{C}$ is not necessarily normal, hence it may have infinitely many singular points.

Definition 4.20. *Let $\mathfrak{s} \in \Sigma_{K^{nr}}$. We say that*

- *$\mathfrak{s}$ is removable if either $|\mathfrak{s}| = 1$, or $\mathfrak{s}$ has a child $\mathfrak{s}' \in \Sigma_{K^{nr}}$ of size $2g+1$ ($\mathfrak{s} = \mathfrak{R}$ in this case).*
- *$\mathfrak{s}$ is contractible if*

  (1) *$|\mathfrak{s}| = 2$ and $\rho_{\mathfrak{s}} \notin \mathbb{Z}$, $\epsilon_{\mathfrak{s}}$ odd, $\rho_{P(\mathfrak{s})} \leq \rho_{\mathfrak{s}} - \frac{1}{2}$; or*

  (2) *$\mathfrak{s} = \mathfrak{R}$ of size $2g+2$, with a child $\mathfrak{s}' \in \Sigma_{K^{nr}}$ of size $2g$, and $\rho_{\mathfrak{s}} \notin \mathbb{Z}$, $v(c_f)$ odd, $\rho_{\mathfrak{s}'} \geq \rho_{\mathfrak{s}} + \frac{1}{2}$; or*

  (3) *$\mathfrak{s} = \mathfrak{R}$ of size $2g+2$, union of its 2 odd proper children $\mathfrak{s}_1, \mathfrak{s}_2 \in \Sigma_{K^{nr}}$, with $v(c_f)$ odd, $\rho_{\mathfrak{s}_i} \geq \rho_{\mathfrak{s}} + 1$ for $i = 1, 2$.*

Notation 4.21. *Write $\Sigma^{nr} \subseteq \Sigma_{K^{nr}}$ for the subset of non-removable clusters.*

Definition 4.22. *Choose rational centres $w_{\mathfrak{s}} \in K^{nr}$ for every $\mathfrak{s} \in \Sigma^{nr}$, in such a way that $w_{\mathfrak{s}} \in \mathfrak{s}$ when $p_{\mathfrak{s}}^0 = 1$, and $\sigma(w_{\mathfrak{s}}) = w_{\sigma(\mathfrak{s})}$ for all $\sigma \in \mathrm{Gal}(K^{nr}/K)$. Denote $r_{\mathfrak{s}} = \frac{w_{\mathfrak{s}} - r}{\pi^{\rho_{\mathfrak{s}}}}$ for $r \in \mathfrak{R}$ and define $\overline{g_{\mathfrak{s}}}, \overline{g_{\mathfrak{s}}^0} \in k^s[y]$*

as in Definition 4.14, and $\overline{f_{\mathfrak{s}}}(x) \in k^s[x]$, *by*

$$x^{2-p^0_{\mathfrak{s}}}\overline{f_{\mathfrak{s}}}(x^{b_{\mathfrak{s}}}) = \frac{u}{\pi^{v(u)}} \prod_{r\in\mathfrak{s}\setminus\bigcup_{\mathfrak{s}'<\mathfrak{s}}\mathfrak{s}'} (x+r_{\mathfrak{s}}) \mod \pi, \quad u = c_f \prod\nolimits_{r\in\mathfrak{R}\setminus\mathfrak{s}} r_{\mathfrak{s}},$$

where the union runs through all $\mathfrak{s}' \in \Sigma^{nr}$, $\mathfrak{s}' < \mathfrak{s}$. *Let* $G_{\mathfrak{s}} = \mathrm{Stab}_{G_K}(\mathfrak{s})$, $K_{\mathfrak{s}} = (K^s)^{G_{\mathfrak{s}}}$, *and let* $k_{\mathfrak{s}}$ *be the residue field of* $K_{\mathfrak{s}}$. *Then* $\overline{f_{\mathfrak{s}}} \in k_{\mathfrak{s}}[x]$, $\overline{g_{\mathfrak{s}}} \in k_{\mathfrak{s}}[y]$, *and for* $\mathfrak{s}$ *minimal* $\overline{g^0_{\mathfrak{s}}} \in k_{\mathfrak{s}}[y]$.

Let $\mathfrak{s}_0 \in \Sigma^{nr}$ *be minimal and contained in* $\mathfrak{s}$. *Denote* $\mathfrak{s} = \tilde{\mathfrak{s}} \setminus \{\{r\} < \mathfrak{s} \mid r \neq w_{\mathfrak{s}_0}\}$. *Note that* $\mathfrak{s}$ *does not depend on the choice of* $\mathfrak{s}_0$. *Define* $\tilde{f}_{\mathfrak{s}} \in k_{\mathfrak{s}}[x]$ *by*

$$\tilde{f}_{\mathfrak{s}}(x) = \prod_{\mathfrak{s}'\in\mathfrak{s}} \left(x - \overline{u_{\mathfrak{s}',\mathfrak{s}}}\right) \cdot \overline{f_{\mathfrak{s}}}(x),$$

where $\overline{u_{\mathfrak{s}',\mathfrak{s}}} = \frac{w_{\mathfrak{s}'}-w_{\mathfrak{s}}}{\pi^{\rho_{\mathfrak{s}}}} \mod \pi$ *if* $\mathfrak{s}' \neq \varnothing$ *and* $\overline{u_{\mathfrak{s}',\mathfrak{s}}} = 0$ *otherwise.*

In the next theorem we describe the special fibre of the minimal regular model of C with normal crossings. We use Definitions/Notations 3.1, 3.3, 3.4, 3.2, 3.8, 3.9, 3.26, 4.6, 4.10, 4.13, 4.17, 4.20, 4.21, 4.22 in the statement. Note that a full description of the model is developed in Section 5.

Theorem 4.23 (Minimal regular NC model). *Let* $C/K : y^2 = f(x)$ *be a hyperelliptic curve of genus* ≥ 1. *Suppose* $C_{K^{nr}}$ *has an almost rational cluster picture and is y-regular. Then the minimal regular model with normal crossings* $\mathcal{C}^{\min}/O_{K^{nr}}$ *of* C *has special fibre* $\mathcal{C}^{\min}_s/k^s$ *described as follows:*

(1) *Every* $\mathfrak{s} \in \Sigma^{nr}$ *gives a* 1*-dimensional subscheme* $\Gamma_{\mathfrak{s}}$ *of multiplicity* $m_{\mathfrak{s}}$. *If* $\mathfrak{s}$ *is übereven and* $\epsilon_{\mathfrak{s}}$ *is even, then* $\Gamma_{\mathfrak{s}}$ *is the disjoint union of* $\Gamma^{r_{\mathfrak{s},-}}_{\mathfrak{s}} \simeq \mathbb{P}^1$ *and* $\Gamma^{r_{\mathfrak{s},+}}_{\mathfrak{s}} \simeq \mathbb{P}^1$, *otherwise* $\Gamma_{\mathfrak{s}}$ *is irreducible of genus* $g(\mathfrak{s})$ *(write* $\Gamma^{r_{\mathfrak{s},-}}_{\mathfrak{s}} = \Gamma^{r_{\mathfrak{s},+}}_{\mathfrak{s}} = \Gamma_{\mathfrak{s}}$ *in this case). The indices* $r_{\mathfrak{s},-}$ *and* $r_{\mathfrak{s},+}$ *are the roots of* $\overline{g_{\mathfrak{s}}}$ *(where* $r_{\mathfrak{s},-} = r_{\mathfrak{s},+}$ *if* $\deg \overline{g_{\mathfrak{s}}} = 1$*).*

(2) *Every* $\mathfrak{s} \in \Sigma^{nr}$ *with* $D_{\mathfrak{s}} = 1$ *gives open-ended* $\mathbb{P}^1$*s of multiplicity* $b_{\mathfrak{s}}$ *from* $\Gamma_{\mathfrak{s}}$ *indexed by roots of* $\overline{f_{\mathfrak{s}}}$.

(3) *Every non-maximal element* $\mathfrak{s} \in \Sigma^{nr}$ *gives chains* $\mathbb{P}^1(\gamma_{\mathfrak{s}}, s_{\mathfrak{s}}, s_{\mathfrak{s}} - p_{\mathfrak{s}} \cdot \frac{\rho_{\mathfrak{s}}-\rho_{P(\mathfrak{s})}}{2})$ *from* $\Gamma_{\mathfrak{s}}$ *to* $\Gamma_{P(\mathfrak{s})}$ *indexed by roots of* $\overline{g_{\mathfrak{s}}}$.

(4) *Every minimal element* $\mathfrak{s} \in \Sigma^{nr}$ *gives open-ended chains* $\mathbb{P}^1(\gamma^0_{\mathfrak{s}}, -s^0_{\mathfrak{s}})$ *from* $\Gamma_{\mathfrak{s}}$ *indexed by roots of* $\overline{g^0_{\mathfrak{s}}}$.

(5) *The maximal element* $\mathfrak{s} \in \Sigma^{nr}$ *gives open-ended chains* $\mathbb{P}^1(\gamma_{\mathfrak{s}}, s_{\mathfrak{s}})$ *from* $\Gamma_{\mathfrak{s}}$ *indexed by roots of* $\overline{g_{\mathfrak{s}}}$.

(6) *Finally, blow down all* $\Gamma_{\mathfrak{s}}$ *where* $\mathfrak{s}$ *is a contractible cluster.*

In (3) *and* (5), *a chain indexed by* r *goes from* $\Gamma^r_{\mathfrak{s}}$. *In* (3) *the chain indexed by* $r_{\mathfrak{s},-}$ *goes to* $\Gamma^{r_{P(\mathfrak{s}),-}}_{P(\mathfrak{s})}$ *and the chain indexed by* $r_{\mathfrak{s},+}$ *goes to* $\Gamma^{r_{P(\mathfrak{s}),+}}_{P(\mathfrak{s})}$.

Before blowing down in (6), *the components given in* (1)–(5) *describe the special fibre of a regular model of* $C_{K^{nr}}$ *with strict normal crossings.*

The Galois group G_k *acts naturally, that is for every* $\sigma \in G_k$, $\sigma(\Gamma^r_{\mathfrak{s}}) = \Gamma^{\sigma(r)}_{\sigma(\mathfrak{s})}$, *and similarly, on the chains.*

If $\Gamma_{\mathfrak{s}}$ *is irreducible, then its function field is isomorphic to* $k^s(x)[y]$ *with the relation* $y^{D_{\mathfrak{s}}} = \tilde{f}_{\mathfrak{s}}(x)$.

Remark 4.24. *Note that if* $\Gamma_{\mathfrak{s}}$ *or* $\Gamma_{P(\mathfrak{s})}$ *is reducible then* $p_{\mathfrak{s}}/\gamma_{\mathfrak{s}} = 2$.

Example 4.25. *Let* p *be a prime number and let* $a \in \mathbb{Z}_p$, $b \in \mathbb{Z}_p^\times$ *such that the polynomial* $x^2 + ax + b$ *is not a square modulo* p. *Let* C *be the hyperelliptic curve over* $\mathbb{Q}_p$ *of genus* 4 *given by the equation* $y^2 = f(x)$, *where* $f(x) = (x^6 + ap^4x^3 + bp^8)((x-p)^3 - p^{11})$. *In Example 3.32, we described the rational cluster picture of* C *and proved that* C *has an almost rational cluster picture. Recall that* Σ^{rat}_C *consists of* 3 *clusters* $\mathfrak{t}_3, \mathfrak{t}_4, \mathfrak{R}$ *of size* 6, 3, 9 *respectively such that* $\mathfrak{t}_3 < \mathfrak{R}$ *and* $\mathfrak{t}_4 < \mathfrak{R}$. *In particular, note that* $\Sigma_{\mathbb{Q}_p^{nr}} = \Sigma^{\mathrm{rat}}_C$, *and no cluster of* $\Sigma_{\mathbb{Q}_p^{nr}}$ *is removable, so* $\Sigma^{nr} = \Sigma^{\mathrm{rat}}_C$. *The minimal elements of* Σ^{nr} *are then* $\mathfrak{t}_3$ *and* $\mathfrak{t}_4$.

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

We want to describe the special fibre of the minimal regular model with normal crossings $\mathcal{C}^{\min}$ of C. Compute the quantities in Definitions 4.6 and 4.13, and the polynomials $\overline{f_{\mathfrak{s}}}, \overline{g_{\mathfrak{s}}}, \overline{g^0_{\mathfrak{s}}}$ of Definition 4.22, for any cluster in Σ^{nr}:

	$\rho_{\mathfrak{s}}$	$b_{\mathfrak{s}}$	$\epsilon_{\mathfrak{s}}$	$D_{\mathfrak{s}}$	$m_{\mathfrak{s}}$	$p_{\mathfrak{s}}$	$s_{\mathfrak{s}}$	$\gamma_{\mathfrak{s}}$	$p^0_{\mathfrak{s}}$	$s^0_{\mathfrak{s}}$	$\gamma^0_{\mathfrak{s}}$	$g(\mathfrak{s})$	$\overline{f_{\mathfrak{s}}}(x)$	$\overline{g_{\mathfrak{s}}}(y)$	$\overline{g^0_{\mathfrak{s}}}(y)$
$\mathfrak{t}_3$	$\frac{4}{3}$	3	11	1	6	2	$-\frac{1}{6}$	2	2	$-\frac{25}{6}$	2	0	$x^2+\bar{a}x+\bar{b}$	$y+1$	$y-1$
$\mathfrak{t}_4$	$\frac{11}{3}$	3	17	1	6	1	$-\frac{7}{6}$	1	2	$-\frac{29}{6}$	2	0	$x-1$	$y-1$	$y+1$
$\mathfrak{R}$	1	1	9	1	2	1	$\frac{1}{2}$	1	2			0	1	$y-1$	

where $\bar{a}, \bar{b}$ are the reductions of a, b modulo p. Then C is also y-regular for any p. Following the steps of Theorem 4.23 the special fibre of $\mathcal{C}^{\min}$ over $\bar{\mathbb{F}}_p$ can be described as follows:

(1) *The clusters $\mathfrak{t}_3, \mathfrak{t}_4, \mathfrak{R}$ give 3 irreducible components $\Gamma_{\mathfrak{t}_3}, \Gamma_{\mathfrak{t}_4}, \Gamma_{\mathfrak{R}}$ of genus 0 of multiplicities 6, 6, 2 respectively;*

(2) *The cluster $\mathfrak{t}_3$ gives 2 open-ended $\mathbb{P}^1$s of multiplicity 3 from $\Gamma_{\mathfrak{t}_3}$, while $\mathfrak{t}_4$ gives 1 open-ended $\mathbb{P}^1$ of multiplicity 3 from $\Gamma_{\mathfrak{t}_4}$.*

(3) *From $\gamma_{\mathfrak{t}_3}s_{\mathfrak{t}_3} = -\frac{1}{3} > -\frac{1}{2} > -1 = \gamma_{\mathfrak{t}_3}\big(s_{\mathfrak{t}_3} - p_{\mathfrak{t}_3}\cdot\frac{\rho_{\mathfrak{t}_3}-\rho_{\mathfrak{R}}}{2}\big)$, the cluster $\mathfrak{t}_3$ gives 1 $\mathbb{P}^1$ of multiplicity 4 from $\Gamma_{\mathfrak{t}_3}$ to $\Gamma_{\mathfrak{R}}$. From*

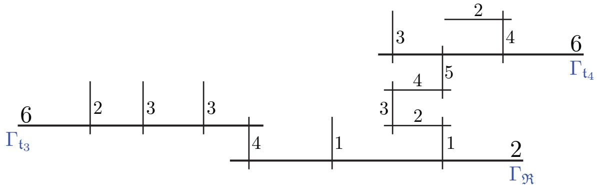
$$\gamma_{\mathfrak{t}_4}s_{\mathfrak{t}_4} = -\tfrac{7}{6} > -\tfrac{6}{5} > -\tfrac{5}{4} > -\tfrac{4}{3} > -\tfrac{3}{2} > -2 > -\tfrac{5}{2} = \gamma_{\mathfrak{t}_3}\big(s_{\mathfrak{t}_4} - p_{\mathfrak{t}_4}\cdot\tfrac{\rho_{\mathfrak{t}_4}-\rho_{\mathfrak{R}}}{2}\big)$$

the cluster $\mathfrak{t}_4$ gives 1 chain of $\mathbb{P}^1$s of multiplicities 5, 4, 3, 2, 1 from $\Gamma_{\mathfrak{t}_4}$ to $\Gamma_{\mathfrak{R}}$.

(4) *From $-\gamma^0_{\mathfrak{t}_3}s^0_{\mathfrak{t}_3} = \frac{25}{3} > 8 > 7$ the cluster $\mathfrak{t}_3$ gives 1 open-ended $\mathbb{P}^1$ of multiplicity 2 from $\Gamma_{\mathfrak{t}_3}$. From $-\gamma^0_{\mathfrak{t}_4}s^0_{\mathfrak{t}_4} = \frac{29}{3} > \frac{19}{2} > 9 > 8$, the cluster $\mathfrak{t}_4$ gives 1 open-ended chain of $\mathbb{P}^1$s of multiplicities 4, 2 from $\Gamma_{\mathfrak{t}_4}$.*

(5) *From $\gamma_{\mathfrak{R}}s_{\mathfrak{R}} = \frac{1}{2} > 0 > -1$, the cluster $\mathfrak{R}$ gives 1 open-ended $\mathbb{P}^1$ of multiplicity 1 from $\Gamma_{\mathfrak{R}}$.*

(6) *There is no contractible cluster, so the components we considered in the steps above describe the special fibre of $\mathcal{C}^{\min}$ over $\bar{\mathbb{F}}_p$:*



Finally, from the Galois action on the roots of the polynomials $\overline{f_{\mathfrak{s}}}, \overline{g_{\mathfrak{s}}}, \overline{g^0_{\mathfrak{s}}}$, for $\mathfrak{s} \in \Sigma^{nr}$, we get that G_k acts trivially if $x^2+\bar{a}x+\bar{b}$ is reducible in $\mathbb{F}_p$, while it swaps the two components of multiplicity 3 intersecting $\Gamma_{\mathfrak{t}_3}$ (coming from (2)) otherwise.

As application of Theorem 4.23 we suppose k is finite of characteristic $p > 2$ and C is semistable of genus $g \geq 2$. In this setting [14, Theorem 8.5] describes the minimal regular model of C in terms of its cluster picture Σ_C. We compare that result with the one obtained from Theorem 4.23 (Corollary 4.27).

First note that $C_{K^{nr}}$ is y-regular as $p \neq 2$. From [14, Definition 1.7], if C is semistable then

(1) the extension $K(\mathfrak{R})/K$ has ramification degree at most 2;
(2) every proper cluster is $\mathrm{Gal}(K^s/K^{nr})$-invariant;
(3) every principal cluster has $d_{\mathfrak{s}} \in \mathbb{Z}$ and $\nu_{\mathfrak{s}} \in 2\mathbb{Z}$.

It follows from Corollary 3.27 that $C_{K^{nr}}$ has an almost rational cluster picture.

In fact, (1) and (2) imply $\rho_{\mathfrak{s}} = d_{\mathfrak{s}}$ and $\epsilon_{\mathfrak{s}} = \nu_{\mathfrak{s}}$ for any proper cluster $\mathfrak{s}$ (Remark 3.13). In particular, $\Sigma^{\mathrm{rat}}_{C_{K^{nr}}} = \Sigma_C$. We will then say that $\mathfrak{s} \in \Sigma_C$ is non-removable if $\mathfrak{s}$ is proper and non-removable as rational cluster in $\Sigma_{K^{nr}}$.

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

Lemma 4.26. *Suppose k finite and $p > 2$. Assume C is semistable and let $\mathfrak{s} \in \Sigma_C$ be a non-removable cluster. Then $d_\mathfrak{s} \in \frac{1}{2}\mathbb{Z}$ and $\nu_\mathfrak{s} \in \mathbb{Z}$. Moreover, $\mathfrak{s}$ is contractible if and only if $d_\mathfrak{s} \notin \mathbb{Z}$ or $\nu_\mathfrak{s} \notin 2\mathbb{Z}$.*

Proof. Let $\mathfrak{s} \in \Sigma_C$ be a non-removable cluster. Since $K(\mathfrak{R})/K$ has ramification degree at most 2, then $d_\mathfrak{s} \in \frac{1}{2}\mathbb{Z}$.

By Theorem 4.23 the multiplicity of the 1-dimensional scheme $\Gamma_\mathfrak{s}$ is $m_\mathfrak{s}$. Furthermore, $\Gamma_\mathfrak{s}$ is an irreducible component of the special fibre of the minimal regular model of C if and only if $\mathfrak{s}$ is not contractible. Therefore if $\mathfrak{s}$ is not contractible, then $m_\mathfrak{s} = 1$, that is $D_\mathfrak{s} = 2$ and $b_\mathfrak{s} = 1$. It follows that $\nu_\mathfrak{s} \in 2\mathbb{Z}$ and $d_\mathfrak{s} \in \mathbb{Z}$. Suppose $\mathfrak{s}$ contractible. Then either $d_\mathfrak{s} \notin \mathbb{Z}$ (and $\nu_\mathfrak{s} \in \mathbb{Z}$) or $\mathfrak{s} = \mathfrak{R}$ of size $2g + 2$, with 2 odd rational children and $v(c_f)$ odd. We want to show that in the latter case, $\nu_\mathfrak{s}$ is odd. By Lemma 3.18, $d_\mathfrak{R} \in \mathbb{Z}$. Then $\nu_\mathfrak{R} = v(c_f) + (2g + 2)d_\mathfrak{R}$ is odd. □

Let $\mathfrak{s} \in \Sigma_C$ be a non-removable cluster. By Lemma 4.26, if $\mathfrak{s}$ is not contractible, then $2g(\mathfrak{s}) + 1$ or $2g(\mathfrak{s}) + 2$ equals the number of odd children of $\mathfrak{s}$. In fact, this also holds when $\mathfrak{s}$ is contractible since in that case $g(\mathfrak{s}) = 0$ and $\mathfrak{s}$ has at most 2 odd children.

Corollary 4.27 (Minimal regular model (semistable reduction)). *Suppose that k is finite and $p > 2$. Let C/K be a semistable hyperelliptic curve of genus $g \geq 2$. The minimal regular model $\mathcal{C}^{\min}/O_{K^{nr}}$ of C has special fibre $\mathcal{C}_s^{\min}/k^s$ described as follows:*

(1) *Every non-removable cluster $\mathfrak{s} \in \Sigma_C$ gives a 1-dimensional subscheme $\Gamma_\mathfrak{s}$. If $\mathfrak{s}$ is übereven, then $\Gamma_\mathfrak{s}$ is the disjoint union of $\Gamma_\mathfrak{s}^{r_{\mathfrak{s},-}} \simeq \mathbb{P}^1$ and $\Gamma_\mathfrak{s}^{r_{\mathfrak{s},+}} \simeq \mathbb{P}^1$, otherwise $\Gamma_\mathfrak{s}$ is irreducible of genus $g(\mathfrak{s})$ (write $\Gamma_\mathfrak{s}^{r_{\mathfrak{s},-}} = \Gamma_\mathfrak{s}^{r_{\mathfrak{s},+}} = \Gamma_\mathfrak{s}$ in this case). The indices $r_{\mathfrak{s},-}$ and $r_{\mathfrak{s},+}$ are the roots of $\overline{g_\mathfrak{s}}$.*

(2) *Every odd proper cluster $\mathfrak{s} \in \Sigma_C$, with size $|\mathfrak{s}| \leq 2g$, gives a chain of $\mathbb{P}^1$s of length $\left\lfloor \frac{d_\mathfrak{s} - d_{P(\mathfrak{s})} - 1}{2} \right\rfloor$ from $\Gamma_\mathfrak{s}$ to $\Gamma_{P(\mathfrak{s})}$ indexed by the root of $\overline{g_\mathfrak{s}}$.*

(3) *Every even proper cluster $\mathfrak{s} \in \Sigma_C$, with size $|\mathfrak{s}| \leq 2g$, gives a chain of $\mathbb{P}^1$s of length $\left\lfloor d_\mathfrak{s} - d_{P(\mathfrak{s})} - \frac{1}{2} \right\rfloor$ from $\Gamma_\mathfrak{s}^{r_{\mathfrak{s},-}}$ to $\Gamma_{P(\mathfrak{s})}^{r_{P(\mathfrak{s}),-}}$ indexed by $r_{\mathfrak{s},-}$ and a chain of $\mathbb{P}^1$s of same length from $\Gamma_\mathfrak{s}^{r_{\mathfrak{s},+}}$ to $\Gamma_{P(\mathfrak{s})}^{r_{P(\mathfrak{s}),+}}$ indexed by $r_{\mathfrak{s},+}$.*

(4) *Finally, blow down all $\Gamma_\mathfrak{s}$ where $\mathfrak{s}$ is a contractible cluster.*

All components have multiplicity 1, and the absolute Galois group G_k acts naturally, as in Theorem 4.23.

Proof. Let $\mathfrak{s} \in \Sigma_C$ be a non-removable cluster. From Lemma 4.26, if $\mathfrak{s}$ is not contractible, then $D_\mathfrak{s} = 2$, $\gamma_\mathfrak{s} s_\mathfrak{s} \in \mathbb{Z}$ and $\gamma_\mathfrak{s}^0 s_\mathfrak{s}^0 \in \mathbb{Z}$. Suppose $\mathfrak{s}$ contractible. If $|\mathfrak{s}| = 2$ with $d_\mathfrak{s} \notin \mathbb{Z}$ (case (1) of Definition 4.20), then $\gamma_\mathfrak{s}^0 s_\mathfrak{s}^0 \in \mathbb{Z}$ and $\gamma_\mathfrak{s} = 1$, $s_\mathfrak{s} \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ and so $s_\mathfrak{s} - d_\mathfrak{s} + d_{P(\mathfrak{s})} \in \mathbb{Z}$, as $P(\mathfrak{s})$ can not be contractible. If $\mathfrak{s} = \mathfrak{R}$ (cases (2), (3) of Definition 4.20), then $v(c_f)$ is odd, and so $\gamma_\mathfrak{s} = 2$ and $\gamma_\mathfrak{s} s_\mathfrak{s} \in \mathbb{Z}$. Therefore (2), (4) and (5) of Theorem 4.23 do not give any components.

Finally, as $\gamma_\mathfrak{s} = 1$ and $p_\mathfrak{s} \frac{d_\mathfrak{s} - d_{P(\mathfrak{s})}}{2} \in \frac{1}{2}\mathbb{Z}$ for any proper $\mathfrak{s}$ with size $|\mathfrak{s}| \leq 2g$ (i.e. non-maximal), the length of $\mathbb{P}^1(\gamma_\mathfrak{s}, s_\mathfrak{s}, s_\mathfrak{s} - p_\mathfrak{s} \cdot \frac{d_\mathfrak{s} - d_{P(\mathfrak{s})}}{2})$ is

$$\left\lfloor \gamma_\mathfrak{s} s_\mathfrak{s} - \gamma_\mathfrak{s} \left(s_\mathfrak{s} - p_\mathfrak{s} \cdot \frac{d_\mathfrak{s} - d_{P(\mathfrak{s})}}{2} \right) - \frac{1}{2} \right\rfloor = \left\lfloor p_\mathfrak{s} \cdot \frac{d_\mathfrak{s} - d_{P(\mathfrak{s})}}{2} - \frac{1}{2} \right\rfloor.$$

The Corollary then follows from Theorem 4.23. □

5. Construction of the model

We are going to construct a proper flat model $\mathcal{C}/O_K$ of C by glueing models defined in [1, §4]. For this reason we will assume the reader has familiarity with the definitions and the results presented in that paper. Let us start this section by describing the strategy we will follow.

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

Let $\Sigma_C^{\min}$ be the set of rationally minimal clusters of C and let $\Sigma \subseteq \Sigma_C^{\min}$ non-empty. For any cluster $\mathfrak{s} \in \Sigma$ fix a rational centre $w_\mathfrak{s}$ in such a way that $\mathring{\Sigma}_C^{w_\mathfrak{s}}$ consists of the proper clusters in $\Sigma_C^{w_\mathfrak{s}}$. This requirement can be satisfied by choosing $w_\mathfrak{s} \in \mathfrak{s}$ whenever possible.[3] Let W be the set of all such rational centres and define $\Sigma^W := \bigcup_{w\in W} \Sigma_C^w$. For every proper cluster $\mathfrak{t} \in \Sigma^W$ fix a rational centre $w_\mathfrak{t} \in W$ (Lemma 3.14). For every $w \in W$, consider the curve $C^w : y^2 = f(x+w)$, isomorphic to C, and construct the (proper flat) model $\mathcal{C}_\Delta^w/O_K$ by [1, §4, Theorem 3.14]. We will define an open subscheme $\mathring{\mathcal{C}}_\Delta^w$ of $\mathcal{C}_\Delta^w$ and we will show that glueing the schemes $\mathring{\mathcal{C}}_\Delta^w$, to varying of $w \in W$, along common opens, gives a proper flat model $\mathcal{C}/O_K$ of C. Furthermore, if $\Sigma = \Sigma_C^{\min}$, and C is y-regular and has an almost rational cluster picture, then $\mathring{\mathcal{C}}_\Delta^w$ is an open regular subscheme of $\mathcal{C}_\Delta^w$ and therefore $\mathcal{C}$ is also regular.

5.1. Charts

Let $\Sigma = \{\mathfrak{s}_1 \dots, \mathfrak{s}_m\} \subseteq \Sigma_C^{\min}$ be a non-empty set of rationally minimal clusters and let $W = \{w_1, \dots, w_m\}$ be a set of corresponding rational centres, such that $\mathring{\Sigma}_C^{w_h}$ consists of the proper clusters of $\Sigma_C^{w_h}$, for any $h = 1, \dots, m$. Define $\Sigma^W := \bigcup_{h=1}^m \Sigma_C^{w_h}$. For any $h, l = 1, \dots, m$, $h \neq l$, define $w_{hl} := w_h - w_l$, and write $w_{hl} = u_{hl}\pi^{\rho_{hl}}$, where $u_{hl} \in O_K^\times$ and $\rho_{hl} \in \mathbb{Z}$. Note that $\rho_{hl} = \rho_{\mathfrak{s}_h \wedge \mathfrak{s}_l} = \rho_{lh}$, by Lemma 3.18. Set $u_{hh} := 0$. Finally, for any $h, l = 1, \dots, m$, denote by $\overline{u_{hl}} \in k$ the reduction of u_{hl} modulo π.

Definition 5.1. *Let $h = 1, \dots, m$ and let $\mathfrak{t} \in \Sigma_C^{w_h}$ be a proper cluster. Recall the matrices and cones defined in [1, §4]. We say that a matrix M is associated to $\mathfrak{t}$ if either*

(i) *$M = M_{L,i}$, with $L = L_\mathfrak{t}^{w_h}$ and $i = 0, \dots, r_L$ or*
(ii) *$M = M_{V,j}$, with $V = V_\mathfrak{t}^{w_h}$ and $i = 0, \dots, r_V$ or*
(iii) *$M = M_{V_0,j}$, with $V_0 = V_0^{w_h}$ and $j = 0, \dots, r_{V_0}$, if $\mathfrak{t} = \mathfrak{s}_h$.*

For a matrix M associated to $\mathfrak{t}$ we denote by δ_M and σ_M respectively

(i) *the denominator $\delta_{L_\mathfrak{t}^{w_h}}$ and the cone $\sigma_{L_\mathfrak{t}^{w_h},i,i+1}$ if $M = M_{L_\mathfrak{t}^{w_h},i}$,*
(ii) *the denominator $\delta_{V_\mathfrak{t}^{w_h}}$ and the cone $\sigma_{V_\mathfrak{t}^{w_h},j,j+1}$ if $M = M_{V_\mathfrak{t}^{w_h},j}$,*
(iii) *the denominator $\delta_{V_0^{w_h}}$ and the cone $\sigma_{V_0^{w_h},j,j+1}$ if $M = M_{V_0^{w_h},j}$.*

Finally, define $X_M = \operatorname{Spec} O_K[\sigma_M^\vee \cap \mathbb{Z}^3]$ and write

$$X_\Delta^h = \bigcup X_M,$$

for the toric scheme defined in [1, §4.2] from the Newton polytope $\Delta_v^{w_h}$ associated to the curve C^{w_h}. Therefore, by Lemma 4.3, the union runs through every proper cluster $\mathfrak{t} \in \Sigma_C^{w_h}$ and all matrices M associated to $\mathfrak{t}$.

The following Lemma describes all possible matrices associated to $\mathfrak{t}$.

Lemma 5.2. *Let $\mathfrak{t} \in \Sigma_C^{w_h}$ be a proper cluster. Consider the v-face $F_\mathfrak{t}^{w_h}$. Let $P_0, P_1 \in \mathbb{Z}^2$ and $n_i, d_i, k_i \in \mathbb{Z}$ be as in [1, §4] and define*

$$\delta := \delta_M, \quad \gamma_i := \frac{n_0 d_i - n_i d_0}{\delta d_0} \quad \text{and} \quad T_i := \begin{pmatrix} \frac{1}{\delta} & -k_i & k_{i+1} \\ 0 & \delta & 0 \\ 0 & 0 & \delta \end{pmatrix},$$

for each matrix M associated to $\mathfrak{t}$.

[3] This is the assumption used in Theorem 4.18.

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

- *Let c be the unique element of $\{0,\dots,b_{\mathfrak{t}}-1\}$ such that $\frac{1}{b_{\mathfrak{t}}}-\rho_{\mathfrak{t}}\cdot c=d\in\mathbb{Z}$. For $L=L_{\mathfrak{t}}^{w_h}$ and for all $i=0,\dots,r_L$, choose $k_i=cn_i+d\delta d_i(\lfloor\mathfrak{t}/2\rfloor+1)$; then $M_{L,i}$ and $M_{L,i}^{-1}$ are respectively*

$$\begin{pmatrix} \delta & -c\delta d_i\left(\frac{\epsilon_{\mathfrak{t}}}{2}+\gamma_i\right) & c\delta d_{i+1}\left(\frac{\epsilon_{\mathfrak{t}}}{2}+\gamma_{i+1}\right)\\ 0 & d_i & -d_{i+1}\\ -\delta\rho_{\mathfrak{t}} & -d\delta d_i\left(\frac{\epsilon_{\mathfrak{t}}}{2}+\gamma_i\right) & d\delta d_{i+1}\left(\frac{\epsilon_{\mathfrak{t}}}{2}+\gamma_{i+1}\right)\end{pmatrix},\qquad T_i\cdot\begin{pmatrix} 1 & \left\lfloor\frac{|\mathfrak{t}|}{2}\right\rfloor+1 & 0\\ d_{i+1}\rho_{\mathfrak{t}} & \frac{d_{i+1}\epsilon_{\mathfrak{t}}}{2}+\gamma_{i+1} & d_{i+1}\\ d_i\rho_{\mathfrak{t}} & \frac{d_i\epsilon_{\mathfrak{t}}}{2}+\gamma_i & d_i\end{pmatrix},$$

where $P_0=(|\mathfrak{t}|,0)$, $P_1=(\lfloor|\mathfrak{t}|-1/2\rfloor,1)$ and $\delta=\delta_L=b_{\mathfrak{t}}$.

- *If $\mathfrak{t}$ is odd, then for $V=V_{\mathfrak{t}}^{w_h}$ and for all $j=0,\dots,r_V$, the matrices $M_{V,j}$ and $M_{V,j}^{-1}$ are respectively*

$$\begin{pmatrix} -|\mathfrak{t}| & -\frac{|\mathfrak{t}|+1}{2}d_j & \frac{|\mathfrak{t}|+1}{2}d_{j+1}\\ 2 & d_j & -d_{j+1}\\ -\epsilon_{\mathfrak{t}}+|\mathfrak{t}|\rho_{\mathfrak{t}} & n_j & -n_{j+1}\end{pmatrix},\qquad T_j\cdot\begin{pmatrix} 1 & \frac{|\mathfrak{t}|+1}{2} & 0\\ d_{j+1}\rho_{\mathfrak{t}}-2\cdot\gamma_{j+1} & \frac{d_{j+1}\epsilon_{\mathfrak{t}}}{2}-|\mathfrak{t}|\cdot\gamma_{j+1} & d_{j+1}\\ d_j\rho_{\mathfrak{t}}-2\cdot\gamma_j & \frac{d_j\epsilon_{\mathfrak{t}}}{2}-|\mathfrak{t}|\cdot\gamma_j & d_j\end{pmatrix},$$

where $P_0=(|\mathfrak{t}|,0)$, $P_1=(\lfloor|\mathfrak{t}|-1/2\rfloor,1)$, $\delta=\delta_V=1$ and $k_j=k_{j+1}=0$.

- *If $\mathfrak{t}$ is even, then for $V=V_{\mathfrak{t}}^{w_h}$ and for all $j=0,\dots,r_V$, the matrices $M_{V,j}$ and $M_{V,j}^{-1}$ are respectively*

$$\begin{pmatrix} -\delta\frac{|\mathfrak{t}|}{2} & -\left(\frac{|\mathfrak{t}|}{2}+1\right)d_j-k_j\frac{|\mathfrak{t}|}{2} & \left(\frac{|\mathfrak{t}|}{2}+1\right)d_{j+1}+k_{j+1}\frac{|\mathfrak{t}|}{2}\\ \delta & d_j+k_j & -d_{j+1}-k_{j+1}\\ -\delta\frac{\epsilon_{\mathfrak{t}}-|\mathfrak{t}|\rho_{\mathfrak{t}}}{2} & \frac{n_j}{\delta}-k_j\frac{\epsilon_{\mathfrak{t}}-|\mathfrak{t}|\rho_{\mathfrak{t}}}{2} & -\frac{n_{j+1}}{\delta}+k_{j+1}\frac{\epsilon_{\mathfrak{t}}-|\mathfrak{t}|\rho_{\mathfrak{t}}}{2}\end{pmatrix},$$

$$T_j\cdot\begin{pmatrix} 1 & \frac{|\mathfrak{t}|}{2}+1 & 0\\ d_{j+1}\rho_{\mathfrak{t}}-\gamma_{j+1} & \frac{d_{j+1}\epsilon_{\mathfrak{t}}}{2}-\frac{|\mathfrak{t}|}{2}\gamma_{j+1} & d_{j+1}\\ d_j\rho_{\mathfrak{t}}-\gamma_j & \frac{d_j\epsilon_{\mathfrak{t}}}{2}-\frac{|\mathfrak{t}|}{2}\gamma_j & d_j\end{pmatrix},$$

where $P_0=(|\mathfrak{t}|,0)$, $P_1=(\lfloor|\mathfrak{t}|-1/2\rfloor,1)$ and $\delta=\delta_V$.

- *If $f(w_h)=0$, then for $V_0=V_0^{w_h}$ and for all $j=0,\dots,r_{V_0}$, the matrices $M_{V_0,j}$ and $M_{V_0,j}^{-1}$ are respectively*

$$\begin{pmatrix} 1 & d_j & -d_{j+1}\\ -2 & -d_j & d_{j+1}\\ \epsilon_{\mathfrak{s}_h}-\rho_{\mathfrak{s}_h} & n_j & -n_{j+1}\end{pmatrix},\qquad T_j\cdot\begin{pmatrix} -1 & -1 & 0\\ d_{j+1}\rho_{\mathfrak{s}_h}+2\cdot\gamma_{j+1} & \frac{d_{j+1}\epsilon_{\mathfrak{s}_h}}{2}+\gamma_{j+1} & d_{j+1}\\ d_j\rho_{\mathfrak{s}_h}+2\cdot\gamma_j & \frac{d_j\epsilon_{\mathfrak{s}_h}}{2}+\gamma_j & d_j\end{pmatrix},$$

where $P_0=(0,2)$, $P_1=(1,1)$, $\delta=\delta_{V_0}=1$ and $k_j=k_{j+1}=0$.

- *If $f(w_h)\neq 0$, then for $V_0=V_0^{w_h}$ and for all $j=0,\dots,r_{V_0}$, the matrices $M_{V_0,j}$ and $M_{V_0,j}^{-1}$ are respectively*

$$\begin{pmatrix} 0 & d_j & -d_{j+1}\\ -\delta & -d_j-k_j & d_{j+1}+k_{j+1}\\ \delta\frac{\epsilon_{\mathfrak{s}_h}}{2} & \frac{n_j}{\delta}+k_j\frac{\epsilon_{\mathfrak{s}_h}}{2} & -\frac{n_{j+1}}{\delta}-k_{j+1}\frac{\epsilon_{\mathfrak{s}_h}}{2}\end{pmatrix},\qquad T_j\cdot\begin{pmatrix} -1 & -1 & 0\\ d_{j+1}\rho_{\mathfrak{s}_h}+\gamma_{j+1} & \frac{d_{j+1}\epsilon_{\mathfrak{s}_h}}{2} & d_{j+1}\\ d_j\rho_{\mathfrak{s}_h}+\gamma_j & \frac{d_j\epsilon_{\mathfrak{s}_h}}{2} & d_j\end{pmatrix},$$

where $P_0=(0,2)$, $P_1=(1,1)$ and $\delta=\delta_{V_0}$.

Proof. We follow the notation of [1, §4]. Choose $P_0,P_1\in\mathbb{Z}^2$ as in the proof of Lemma 4.3. First consider the edge $L_{\mathfrak{t}}^{w_h}$ of $F_{\mathfrak{t}}^{w_h}$. From Lemma 4.3 we have

$$\nu=(1,0,-\rho_t)\quad\text{ and }\quad(w_x,w_y)=(-\lfloor|\mathfrak{t}|/2\rfloor-1,1)\,.$$

Then $M_{L_{\mathfrak{t}}^{w_h},i}$ and $M_{L_{\mathfrak{t}}^{w_h},i}^{-1}$ follow from [1, §4.3] as $k_i\equiv n_i(\delta\rho_{\mathfrak{t}})^{-1}\mod\delta$ and

$$\frac{n_0}{\delta d_0}=\frac{1}{\delta}s_1^{L_{\mathfrak{t}}^{w_h}}=\nu_{F_{\mathfrak{t}}^{w_h}}(P_1)-\nu_{F_{\mathfrak{t}}^{w_h}}(P_0)=-\frac{\epsilon_{\mathfrak{t}}}{2}+(\lfloor|\mathfrak{t}|/2\rfloor+1)\,\rho_{\mathfrak{t}}$$

Now assume $\mathfrak{t}$ even and consider the edge $V_{\mathfrak{t}}^{w_h}$ of $F_{\mathfrak{t}}^{w_h}$. Since $\mathfrak{t}$ is even,

$$V_{\mathfrak{t}}^{w_h}(\mathbb{Z})=\left\{(|\mathfrak{t}|,0),\left(\frac{|\mathfrak{t}|}{2},1\right),(0,2)\right\},\qquad \nu=\left(-\frac{|\mathfrak{t}|}{2},1,-\frac{\epsilon_t}{2}+\frac{|\mathfrak{t}|}{2}\rho_{\mathfrak{t}}\right)$$

and $(w_x,w_y)=\left(-\frac{|\mathfrak{t}|}{2}-1,1\right)$ as above. Then $M_{V_{\mathfrak{t}}^{w_h},j}$ and $M^{-1}_{V_{\mathfrak{t}}^{w_h},j}$ follow again from [1, (4.3)] as

$$\frac{n_0}{\delta d_0}=\frac{1}{\delta}s_1^{V_{\mathfrak{t}}^{w_h}}=v_{F_{\mathfrak{t}}^{w_h}}(P_1)-v_{F_{\mathfrak{t}}^{w_h}}(P_0)=-\frac{\epsilon_{\mathfrak{t}}}{2}+\left(\frac{|\mathfrak{t}|}{2}+1\right)\rho_{\mathfrak{t}}.$$

Similar arguments and computations yield the remaining matrices. □

Remark 5.3. *From the Lemma above one can explicitly construct the charts of the model $\mathcal{C}_{\Delta}^{w_h}$. The description of its special fibre $\mathcal{C}_{\Delta,s}^{w_h}$ which follows from [1, Theorem 3.14], matches the one given in Theorem 4.18 in the case $W=\{w_h\}$.*

5.2. *Open subschemes*

Let $h=1,\dots,m$ and let $\mathfrak{t}\in\Sigma_C^{w_h}$ be a proper cluster. Let M be a matrix associated to $\mathfrak{t}$. Write

$$M=\begin{pmatrix}m_{11}&m_{12}&m_{13}\\m_{21}&m_{22}&m_{23}\\m_{31}&m_{32}&m_{33}\end{pmatrix}\qquad\text{and}\qquad M^{-1}=\begin{pmatrix}\tilde m_{11}&\tilde m_{12}&\tilde m_{13}\\\tilde m_{21}&\tilde m_{22}&\tilde m_{23}\\\tilde m_{31}&\tilde m_{32}&\tilde m_{33}\end{pmatrix}$$

Recall that $X_M=\operatorname{Spec}R$, where

$$R=\frac{O_K[X^{\pm1},Y,Z]}{(\pi-X^{\tilde m_{13}}Y^{\tilde m_{23}}Z^{\tilde m_{33}})}\hookrightarrow\frac{O_K[X^{\pm1},Y^{\pm1},Z^{\pm1}]}{(\pi-X^{\tilde m_{13}}Y^{\tilde m_{23}}Z^{\tilde m_{33}})}\overset{M}{\simeq}K\left[x^{\pm1},y^{\pm1}\right],$$

via the change of variable

$$\begin{pmatrix}X\\Y\\Z\end{pmatrix}=\begin{pmatrix}x^{m_{11}}y^{m_{21}}\pi^{m_{31}}\\x^{m_{12}}y^{m_{22}}\pi^{m_{32}}\\x^{m_{13}}y^{m_{23}}\pi^{m_{33}}\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}\bullet M,\qquad\begin{pmatrix}x\\y\\\pi\end{pmatrix}=\begin{pmatrix}X^{\tilde m_{11}}Y^{\tilde m_{21}}Z^{\tilde m_{31}}\\X^{\tilde m_{12}}Y^{\tilde m_{22}}Z^{\tilde m_{32}}\\X^{\tilde m_{13}}Y^{\tilde m_{23}}Z^{\tilde m_{33}}\end{pmatrix}=\begin{pmatrix}X\\Y\\Z\end{pmatrix}\bullet M^{-1}.$$

Let $l\neq h$. Set

$$T_M^{hl}(X,Y,Z):=\begin{cases}1+u_{hl}X^{\rho_{hl}\tilde m_{13}-\tilde m_{11}}Y^{\rho_{hl}\tilde m_{23}-\tilde m_{21}}Z^{\rho_{hl}\tilde m_{33}-\tilde m_{31}}&\text{if }\mathfrak{t}\supseteq\mathfrak{s}_h\wedge\mathfrak{s}_l,\\u_{hl}^{-1}X^{\tilde m_{11}-\rho_{hl}\tilde m_{13}}Y^{\tilde m_{21}-\rho_{hl}\tilde m_{23}}Z^{\tilde m_{31}-\rho_{hl}\tilde m_{33}}+1&\text{if }\mathfrak{t}\not\supseteq\mathfrak{s}_h\wedge\mathfrak{s}_l,\end{cases}$$

element of $R[Y^{-1},Z^{-1}]$. Note that

$$\text{if}\quad\mathfrak{t}\supseteq\mathfrak{s}_h\wedge\mathfrak{s}_l\quad\text{then}\quad T_M^{hl}(X,Y,Z)\overset{M}{\longmapsto}\frac{x+w_{hl}}{x},$$
$$\text{if}\quad\mathfrak{t}\not\supseteq\mathfrak{s}_h\wedge\mathfrak{s}_l\quad\text{then}\quad T_M^{hl}(X,Y,Z)\overset{M}{\longmapsto}\frac{x+w_{hl}}{w_{hl}}.$$

The following two lemmas prove that $T_M^{hl}(X,Y,Z)\in R$. Therefore, up to units, $T_M^{hl}(X,Y,Z)$ can be seen as the polynomial in $O_K[X^{\pm1},Y,Z]$ satisfying

$$x-w_{hl}\overset{M}{=}X^{n_X}Y^{n_Y}Z^{n_Z}T_M^{hl}(X,Y,Z),$$

with $n_X,n_Y,n_Z\in\mathbb{Z}$, such that $\operatorname{ord}_Y(T_M^{hl})=\operatorname{ord}_Z(T_M^{hl})=0$.

Lemma 5.4. *Let $h,l=1,\dots,m$, with $h\neq l$, let $t\in\Sigma_C^{w_h}$ be such that $\mathfrak{t}\supseteq\mathfrak{s}_h\wedge\mathfrak{s}_l$ and let M be a matrix associated to $\mathfrak{t}$. Then*

$$\rho_{hl}\tilde m_{23}-\tilde m_{21}\geq\rho_{\mathfrak{t}}\tilde m_{23}-\tilde m_{21}\geq0\quad\text{and}\quad\rho_{hl}\tilde m_{33}-\tilde m_{31}\geq\rho_{\mathfrak{t}}\tilde m_{33}-\tilde m_{31}\geq0.$$

Furthermore if $M = M_{L_{\mathfrak{t}}^{w_h},i}$ *then*

- $\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21} = 0$ *if and only if* $i = r_{L_{\mathfrak{t}}^{w_h}}$ *or* $\mathfrak{t} = \mathfrak{s}_h \wedge \mathfrak{s}_l$,
- $\rho_{hl}\tilde{m}_{33} - \tilde{m}_{31} = 0$ *if and only if* $\mathfrak{t} = \mathfrak{s}_h \wedge \mathfrak{s}_l$;

if $M = M_{V_{\mathfrak{t}}^{w_h},j}$ *then*

- $\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21} > 0$,
- $\rho_{hl}\tilde{m}_{33} - \tilde{m}_{31} = 0$ *if and only if* $\mathfrak{t} = \mathfrak{s}_h \wedge \mathfrak{s}_l$ *and* $j = 0$.

Proof. This result follows from Lemma 5.2, which gives a complete description of M and M^{-1}. We show it when $\mathfrak{t}$ is even and $M = M_{V_{\mathfrak{t}}^{w_h},j}$, and leave the other cases for the reader. First of all recall that $\rho_{hl} = \rho_{\mathfrak{s}_h \wedge \mathfrak{s}_l}$ by Lemma 3.18. Then

$$\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21} = \delta\left(d_{j+1}\left(\rho_{hl} - \rho_{\mathfrak{t}}\right) + \gamma_{j+1}\right) > \delta d_{j+1}\left(\rho_{\mathfrak{s}_h \wedge \mathfrak{s}_l} - \rho_{\mathfrak{t}}\right) \geq 0,$$

where $\gamma_j = \frac{n_0 d_j - n_j d_0}{\delta d_0}$ and $\delta = \delta_M$. Similarly,

$$\rho_{hl}\tilde{m}_{33} - \tilde{m}_{31} = \delta\left(d_j\left(\rho_{hl} - \rho_{\mathfrak{t}}\right) + \gamma_j\right) \geq \delta d_j\left(\rho_{\mathfrak{s}_h \wedge \mathfrak{s}_l} - \rho_{\mathfrak{t}}\right) \geq 0.$$

In particular $\rho_{hl}\tilde{m}_{33} - \tilde{m}_{31} = 0$ if and only if $\mathfrak{t} = \mathfrak{s}_h \wedge \mathfrak{s}_l$ and $j = 0$. □

Lemma 5.5. *Let* $\mathfrak{t} \in \Sigma_C^{w_h}$ *be a proper cluster such that* $\mathfrak{t} \not\supseteq \mathfrak{s}_h \wedge \mathfrak{s}_l$, *and let* M *be a matrix associated to* $\mathfrak{t}$. *Then*

$$\tilde{m}_{21} - \rho_{hl}\tilde{m}_{23} \geq 0 \quad \text{and} \quad \tilde{m}_{31} - \rho_{hl}\tilde{m}_{33} > 0.$$

Furthermore, $\tilde{m}_{21} - \rho_{hl}\tilde{m}_{23} = 0$ *if and only if*

- $M = M_{L_{\mathfrak{t}}^{w_h},i}$ *and* $i = r_{L_{\mathfrak{t}}^{w_h}}$, *or*
- $\mathfrak{t} < \mathfrak{s}_h \wedge \mathfrak{s}_l$, $M = M_{V_{\mathfrak{t}}^{w_h},j}$, *and* $j = r_{V_{\mathfrak{t}}^{w_h}}$.

Proof. This result follows again from Lemma 5.2. As in the previous lemma, we show it when $\mathfrak{t}$ is even and $M = M_{V_{\mathfrak{t}}^{w_h},j}$, and leave the other cases for the reader.

Let $r = r_{V_{\mathfrak{t}}^{w_h}}$. Note that $t \neq \mathfrak{R}$. Set $\delta = \delta_M$ and $\gamma_j = \frac{n_0 d_j - n_j d_0}{\delta d_0}$. Then

$$\tilde{m}_{31} - \rho_{hl}\tilde{m}_{33} = \delta\left(d_j\left(\rho_t - \rho_{hl}\right) - \gamma_j\right) > \delta d_j\left(\rho_{P(\mathfrak{t})} - \rho_{\mathfrak{s}_h \wedge \mathfrak{s}_l}\right) \geq 0.$$

since $d_j > 0$ and $\gamma_j/d_j < \gamma_{r+1}/d_{r+1} = \rho_{\mathfrak{t}} - \rho_{P(\mathfrak{t})}$. Similarly,

$$\tilde{m}_{21} - \rho_{hl}\tilde{m}_{23} = \delta\left(d_{j+1}\left(\rho_t - \rho_{hl}\right) - \gamma_{j+1}\right) \geq \delta d_{j+1}\left(\rho_{P(\mathfrak{t})} - \rho_{\mathfrak{s}_h \wedge \mathfrak{s}_l}\right) \geq 0,$$

In particular $\tilde{m}_{21} - \rho_{hl}\tilde{m}_{23} = 0$ if and only if $\mathfrak{t} < \mathfrak{s}_h \wedge \mathfrak{s}_l$ and $j = r$. □

Let

$$T_M^h(X, Y, Z) := \prod_{l \neq h} T_M^{hl}(X, Y, Z),$$

and define

$$V_M^h := \operatorname{Spec} R[T_M^h(X, Y, Z)^{-1}] \subset X_M, \qquad \text{and} \qquad X_\Delta^h := \bigcup_{\mathfrak{t}, M} V_M^h \subseteq X_\Delta^h,$$

where $\mathfrak{t}$ runs through all proper clusters in $\Sigma_C^{w_h}$ and M runs through all matrices associated to $\mathfrak{t}$. We can then define the subscheme

$$\mathring{\mathcal{C}}_\Delta^{w_h} := \mathcal{C}_\Delta^{w_h} \cap X_\Delta^h \subset X_\Delta^h,$$

where $\mathcal{C}_\Delta^{w_h}/O_K$ is the model of the hyperelliptic curve $C^{w_h}: y^2 = f(x + w_h)$ described in [1, Theorem 3.14] (see [1, §4] for the construction). Explicitly, let $g_h(x, y) := y^2 - f(x + w_h)$ and define $\mathcal{F}_M^h \in O_K[X^{\pm 1}, Y, Z]$

such that $\mathrm{ord}_Y(\mathcal{F}_M^h) = \mathrm{ord}_Z(\mathcal{F}_M^h) = 0$, with all non-zero coefficients in $O_K^\times$, satisfying

$$y^2 - f(x + w_h) \overset{M}{=} Y^{n_{Y,h}} Z^{n_{Z,h}} \mathcal{F}_M^h(X, Y, Z),$$

for unique $n_{Y,h}, n_{Z,h} \in \mathbb{Z}$. Consider the subscheme

$$U_M^h := \mathrm{Spec}\, \frac{R\left[T_M^h(X, Y, Z)^{-1}\right]}{\left(\mathcal{F}_M^h(X, Y, Z)\right)} \subset V_M^h.$$

Then

$$\mathring{\mathcal{C}}_\Delta^{w_h} = \bigcup_{\mathfrak{t}, M} U_M^h \subset X_\Delta^h,$$

where $\mathfrak{t}$ runs through all proper clusters in $\Sigma_C^{w_h}$ and M runs through all matrices associated to $\mathfrak{t}$, as before.

5.3. Glueing

Let $h, l = 1, \ldots, m$, with $h \neq l$. Consider the isomorphism

$$\phi\colon K\left[x^{\pm 1}, y^{\pm 1}, \prod_{o \neq l} (x + w_{lo})^{-1}\right] \xrightarrow{\simeq} K\left[x^{\pm 1}, y^{\pm 1}, \prod_{o \neq h} (x + w_{ho})^{-1}\right] \tag{1}$$

sending $x \mapsto x + w_{hl}$, $y \mapsto y$. If $\mathfrak{t} \supseteq \mathfrak{s}_h \wedge \mathfrak{s}_l$ and M is a matrix associated to $\mathfrak{t}$, then ϕ gives a map

$$R[Y^{-1}, Z^{-1}, T_M^l(X, Y, Z)^{-1}] \xrightarrow{M^{-1} \circ \phi \circ M} R[Y^{-1}, Z^{-1}, T_M^h(X, Y, Z)^{-1}],$$

which sends

$$F(X, Y, Z) \mapsto F(X \cdot T_M^{hl}(X, Y, Z)^{m_{11}}, Y \cdot T_M^{hl}(X, Y, Z)^{m_{12}}, Z \cdot T_M^{hl}(X, Y, Z)^{m_{13}}).$$

Hence it induces the isomorphisms

$$R[T_M^l(X, Y, Z)^{-1}] \xrightarrow{\simeq} R[T_M^h(X, Y, Z)^{-1}], \qquad V_M^h \xrightarrow{\simeq} V_M^l. \tag{2}$$

Via these maps we see that $g_h(x, y) = Y^{n_{Y,h}} Z^{n_{Z,h}} \mathcal{F}_M^h(X, Y, Z)$ also equals

$$Y^{n_{Y,l}} \cdot Z^{n_{Z,l}} \cdot (T_M^{hl})^{n_{Y,l} m_{12} + n_{Z,l} m_{13}} \mathcal{F}_M^l \left(X \cdot (T_M^{hl})^{m_{11}}, Y \cdot (T_M^{hl})^{m_{12}}, Z \cdot (T_M^{hl})^{m_{13}}\right),$$

where $T_M^{hl} = T_M^{hl}(X, Y, Z)$. Since neither Y nor Z divide $T_M^{hl}(X, Y, Z)$, we have $n_{Y,h} = n_{Y,l}$, $n_{Z,h} = n_{Z,l}$ and

$$\mathcal{F}_M^h(X, Y, Z) = (T_M^{hl})^{n_{Y,l} m_{12} + n_{Z,l} m_{13}} \mathcal{F}_M^l \left(X\, (T_M^{hl})^{m_{11}}, Y\, (T_M^{hl})^{m_{12}}, Z\, (T_M^{hl})^{m_{13}}\right).$$

Hence (2) induces the isomorphisms

$$\frac{R\left[T_M^l(X, Y, Z)^{-1}\right]}{\left(\mathcal{F}_M^l(X, Y, Z)\right)} \xrightarrow{\simeq} \frac{R\left[T_M^h(X, Y, Z)^{-1}\right]}{\left(\mathcal{F}_M^h(X, Y, Z)\right)}, \qquad U_M^h \xrightarrow{\simeq} U_M^l. \tag{3}$$

Define the subschemes

$$V^{hl} := \bigcup_{\mathfrak{t}_l, M_l} V_{M_l}^h \subseteq X_\Delta^h, \qquad U^{hl} := V^{hl} \cap \mathcal{C}_\Delta^{w_h} \subseteq \mathring{\mathcal{C}}_\Delta^{w_h},$$

where $\mathfrak{t}_l$ runs through all proper clusters in $\Sigma_C^{w_h} \cap \Sigma_C^{w_l}$ (i.e. $\mathfrak{t}_l \in \Sigma^W$, $\mathfrak{s}_h \wedge \mathfrak{s}_l \subseteq \mathfrak{t}_l$) and M_l runs through all matrices associated to $\mathfrak{t}_l$. From (1), (2) and (3) we have isomorphisms of schemes

$$V^{hl} \xrightarrow{\simeq} V^{lh}, \qquad U^{hl} \xrightarrow{\simeq} U^{lh}. \tag{4}$$

Now, $U^{hl} \subset V^{hl}$ are open subschemes respectively of $\mathring{\mathcal{C}}_\Delta^{w_h} \subset X_\Delta^h$ for any $l \neq h$. Glueing the schemes $\mathring{\mathcal{C}}_\Delta^{w_h} \subset X_\Delta^h$, to varying of $h = 1, \ldots, m$, respectively along the opens $U^{hl} \subset V^{hl}$ via (4) gives the schemes $\mathcal{C} \subset \mathcal{X}$. We will show that $\mathcal{C}/O_K$ is a proper flat[4] model of C.

[4] Note that the flatness of $\mathcal{C}$ is trivial since it is a local property.

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

5.4. Generic fibre

We start studying the generic fibre $\mathcal{C}_\eta$ of $\mathcal{C}$. Since it is the glueing of all $\mathring{\mathcal{C}}^{w_h}_{\Delta,\eta}$ through the glueing maps

$$U^{hl}_\eta \longrightarrow U^{lh}_\eta$$

induced by (4), we start focusing on $\mathring{\mathcal{C}}^{w_h}_{\Delta,\eta}$ for $h = 1, \ldots, m$. In particular, as $\mathring{\mathcal{C}}^{w_h}_{\Delta}$ is an open subscheme of $\mathcal{C}^{w_h}_{\Delta}$, we study $\mathcal{C}^{w_h}_{\Delta,\eta} \smallsetminus \mathring{\mathcal{C}}^{w_h}_{\Delta,\eta} = C^{w_h} \smallsetminus \mathring{\mathcal{C}}^{w_h}_{\Delta,\eta}$.

Lemma 5.6. *For any $h = 1, \ldots, m$,*

$$C^{w_h} \smallsetminus \mathring{\mathcal{C}}^{w_h}_{\Delta,\eta} = \operatorname{Spec} \frac{K[x,y]}{\left(g_h(x,y), \prod_{o\neq h}(x + w_{ho})\right)}.$$

Proof. For every choice of a proper cluster $\mathfrak{t} \in \Sigma^{w_h}_C$, and M associated to $\mathfrak{t}$, let

$$P_M := \left(\mathcal{C}^{w_h}_{\Delta,\eta} \smallsetminus \mathring{\mathcal{C}}^{w_h}_{\Delta,\eta}\right) \cap X_M = \operatorname{Spec} \frac{R \otimes_{O_K} K}{\left(\mathcal{F}^h_M(X,Y,Z), T^h_M(X,Y,Z)\right)}.$$

To study P_M we are going to use Lemma 5.2 and the definition of $T^h_M(X,Y,Z)$.

Suppose first $\mathfrak{t} \neq \mathfrak{R}$ and $M = M_{V^{w_h}_{\mathfrak{t}},j}$. Then $\tilde{m}_{23}, \tilde{m}_{33} > 0$, so

$$P_M = \operatorname{Spec} \frac{R[Y^{-1}, Z^{-1}]}{\left(\mathcal{F}^h_M(X,Y,Z), T^h_M(X,Y,Z)\right)} \overset{M}{\simeq} \operatorname{Spec} \frac{K[x^{\pm 1}, y^{\pm 1}]}{\left(g_h(x,y), \prod_o (x + w_{ho})\right)}, \tag{5}$$

where the product runs over all $o \neq h$. Now let $\mathfrak{t} = \mathfrak{R}$ and $M = M_{V^{w_h}_{\mathfrak{t}},j}$. If $j \neq r_{V^{w_h}_{\mathfrak{R}}}$, then P_M is as in the previous case (since $\tilde{m}_{23}, \tilde{m}_{33} > 0$). If $j = r_{V^{w_h}_{\mathfrak{R}}}$, then $\tilde{m}_{33} > 0$, $\tilde{m}_{23} = 0$, but $\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21} > 0$ by Lemma 5.4. So from the definition of $T^{hl}_M(X,Y,Z)$ we have once more the equality (5). Similarly, if $\mathfrak{t} = \mathfrak{s}_h$ and $M = M_{V^{w_h}_0,j}$, then $\tilde{m}_{33} > 0$, and $\tilde{m}_{21} - \rho_{hl}\tilde{m}_{23} > 0$ by Lemma 5.5. Hence we have (5) again.

It remains to study P_M when $M = M_{L^{w_h}_{\mathfrak{t}},i}$. If $i \neq r_{L^{w_h}_{\mathfrak{t}}}$, then $\tilde{m}_{23}, \tilde{m}_{33} > 0$ and so P_M is as in (5). Let $i = r_{L^{w_h}_{\mathfrak{t}}}$. Then $\tilde{m}_{33} > 0$ but both $\tilde{m}_{23}$ and $\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21}$ equal 0. Hence $\tilde{m}_{23} = \tilde{m}_{21} = 0$, which also implies $m_{21} = m_{23} = 0$. Therefore M defines an isomorphism $R[Z^{-1}] \simeq K[x^{\pm 1}, y]$, which induces

$$P_M = \operatorname{Spec} \frac{R[Z^{-1}]}{\left(\mathcal{F}^h_M(X,Y,Z), T^h_M(X,Y,Z)\right)} \overset{M}{\simeq} \operatorname{Spec} \frac{K[x^{\pm 1}, y]}{\left(g_h(x,y), \prod_{o\neq h}(x + w_{ho})\right)}.$$

This concludes the proof. □

Regarding $\mathcal{C}^{w_h}_{\Delta}$ as a model of C via the natural isomorphism $C \xrightarrow{\sim} C^{w_h}$, we get

$$C \smallsetminus \mathring{\mathcal{C}}^{w_h}_{\Delta,\eta} = \operatorname{Spec} \frac{K[x,y]}{\left(y^2 - f(x), \prod_{o\neq h}(x - w_o)\right)}.$$

Thus the generic fibre of $\mathcal{C}$ is isomorphic to C.

5.5. Special fibre

We now study the structure of the special fibre $\mathcal{C}_s$ of $\mathcal{C}$. As for the generic fibre, we consider

$$\mathcal{C}^{w_h}_{\Delta,s} \smallsetminus \mathring{\mathcal{C}}^{w_h}_{\Delta,s},$$

for any $h = 1, \ldots, m$. For every choice of a proper cluster $\mathfrak{t} \in \Sigma^{w_h}_C$, and M associated to $\mathfrak{t}$, let

$$S_M := \left(\mathcal{C}^{w_h}_{\Delta,s} \smallsetminus \mathring{\mathcal{C}}^{w_h}_{\Delta,s}\right) \cap X_M = \operatorname{Spec} \frac{O_K[X^{\pm 1}, Y, Z]}{\left(\mathcal{F}^h_M(X,Y,Z), T^h_M(X,Y,Z), Y^{\tilde{m}_{23}}Z^{\tilde{m}_{33}}, \pi\right)}.$$

Lemma 5.7. *Let $M = M_{L,i}$ for $L = L^{w_h}_{\mathfrak{t}}$. Let $l \neq h$. If $\mathfrak{t} = \mathfrak{s}_l \wedge \mathfrak{s}_h$, then $T^{hl}_M(X,Y,Z) = X^{-1}(X + u_{hl})$, otherwise*

(i) $T_M^{hl}(X,Y,0)=1$ *for* $i=0,\dots,r_L$;
(ii) $T_M^{hl}(X,0,Z)=1$ *for* $i=0,\dots,r_L-1$.

Proof. Fix $l\neq h$. If $\mathfrak{t}\not\supseteq \mathfrak{s}_l\wedge\mathfrak{s}_h$, then by Lemma 5.5, we have $\tilde{m}_{21}-\rho_{hl}\tilde{m}_{23}\geq 0$ and $\tilde{m}_{31}-\rho_{hl}\tilde{m}_{33}>0$. Moreover, if $\tilde{m}_{21}-\rho_{hl}\tilde{m}_{23}=0$, then $i=r_L$. Therefore the equalities in (i) and (ii) follow directly from the definition of T_M^{hl}.

On the other hand, if $\mathfrak{t}\supsetneq \mathfrak{s}_l\wedge\mathfrak{s}_h$, then by Lemma 5.4, we have $\rho_{hl}\tilde{m}_{23}-\tilde{m}_{21}\geq 0$ and $\rho_{hl}\tilde{m}_{33}-\tilde{m}_{31}>0$. Moreover, if $\rho_{hl}\tilde{m}_{23}-\tilde{m}_{21}=0$, then $i=r_L$. Therefore we have (i) and (ii) again.

Finally, assume $\mathfrak{t}=\mathfrak{s}_l\wedge\mathfrak{s}_h$. Since $\rho_{\mathfrak{t}}=\rho_{hl}\in\mathbb{Z}$, then $\rho_{hl}\tilde{m}_{13}-\tilde{m}_{11}=-1$. Hence

$$T_M^{hl}(X,Y,Z)=1+u_{hl}X^{-1}=X^{-1}\left(X+u_{hl}\right),$$

by Lemma 5.4. □

Lemma 5.8. *Suppose* $M=M_{L_{\mathfrak{t}}^{w_h},i}$. *Then*

$$S_M=\operatorname{Spec}\frac{O_K[X^{\pm 1},Y,Z]}{\left(\mathcal{F}_M^h(X,Y,Z),\prod_l\left(X+u_{hl}\right),Y^{\tilde{m}_{23}}Z^{\tilde{m}_{33}},\pi\right)}\subset\mathcal{C}_\Delta^{w_h},$$

where the product runs over all $l\neq h$ *such that* $\mathfrak{t}=\mathfrak{s}_l\wedge\mathfrak{s}_h$.

Proof. Lemma 5.2 shows that $\tilde{m}_{33}$ is always different from 0, while $\tilde{m}_{23}=0$ if and only if $i=r_{L_{\mathfrak{t}}^{w_h}}$. Thus the result follows from Lemma 5.7. □

Lemma 5.9. *Let* $f_h(x)=f(x+w_h)$ *and* $l\neq h$. *Let* $L_{hl}=L^{w_h}_{\mathfrak{s}_h\wedge\mathfrak{s}_l}$ *and let* $\mathfrak{t}_l\in\Sigma_C^{w_l}$, $\mathfrak{t}_l<\mathfrak{s}_h\wedge\mathfrak{s}_l$. *Then* $\overline{u_{lh}}$ *is a multiple root of* $\overline{f_h|_{L_{hl}}}$ *of order* $|\mathfrak{t}_l|$.

Conversely, if $\Sigma=\{\mathfrak{s}_1,\dots,\mathfrak{s}_m\}=\Sigma_C^{\min}$, *C has an almost rational cluster picture and* $\bar{\alpha}\in\bar{k}$ *is a multiple root of* $\overline{f_h|_L}$ *for some edge L of* $\mathtt{NP}(f_h)$, *then* $\bar{\alpha}=\overline{u_{lh}}$ *and* $L=L^{w_h}_{\mathfrak{s}_h\wedge\mathfrak{s}_l}$ *for some* $l\neq h$.

Proof. For any proper cluster $\mathfrak{s}\in\Sigma_f$, let $\lambda_{\mathfrak{s}}=\min_{r\in\mathfrak{s}}v(r-w_h)$. Let $\mathfrak{s}\in\Sigma_C^{w_l}$, with $\mathfrak{s}_l\subseteq\mathfrak{s}\subsetneq\mathfrak{s}_h\wedge\mathfrak{s}_l$. Then w_h is not rational centre of $\mathfrak{s}$, and for every root $r\in\mathfrak{s}$, one has

$$v(r-w_h)=v(r-w_l+w_l-w_h)=\min\{v(r-w_l),\rho_{hl}\}=\rho_{hl},$$

as $v(r-w_l)\geq\rho_{\mathfrak{s}}>\rho_{hl}$. Therefore $\lambda_{\mathfrak{s}}=\rho_{hl}\in\mathbb{Z}$. In particular, $|\lambda_{\mathfrak{s}}|_p\leq 1$. Furthermore,

$$d_{\mathfrak{s}}\geq\rho_{\mathfrak{s}}>\lambda_{\mathfrak{s}}=\rho_{hl}\qquad\text{and}\qquad\frac{r-w_h}{\pi^{\rho_{hl}}}\equiv\frac{w_{lh}}{\pi^{\rho_{hl}}}\mod\pi,$$

and so Theorem 3.24(i) implies that $\overline{u_{lh}}=\frac{w_{lh}}{\pi^{\rho_{hl}}}\mod\pi$ is a multiple root of $\overline{f_h|_{L_{hl}}}$, where $L_{hl}=L^{w_h}_{\mathfrak{s}_h\wedge\mathfrak{s}_l}$.

Let $\mathfrak{t}_l\in\Sigma_C^{w_l}$, $\mathfrak{t}_l<\mathfrak{s}_h\wedge\mathfrak{s}_l$. Since $\mathfrak{s}_l\subseteq\mathfrak{t}_l<\mathfrak{s}_h\wedge\mathfrak{s}_l$ we have

$$\mathfrak{t}_l=\left\{r\in\mathfrak{R}\mid\overline{u_{lh}}=\tfrac{r-w_h}{\pi^{\rho_{hl}}}\mod\pi\right\},$$

as $v(r-w_l)>\rho_{hl}$ if and only if $\overline{u_{lh}}=\frac{r-w_h}{\pi^{\rho_{hl}}}\mod\pi$. Thus the multiplicity of $\overline{u_{lh}}$ is $|\mathfrak{t}_l|$ by Theorem 3.24(ii).

Now let $\bar{\alpha}$ be a multiple root of $\overline{f_h|_L}$ for some edge L of $\mathtt{NP}(f_h)$ and let $\mathfrak{s}\in\Sigma_f$ associated to $\bar{\alpha}$ by Theorem 3.24(iii). We want to prove that if C has an almost rational cluster picture and $\Sigma=\Sigma_C^{\min}$, then there exists $l\neq h$ so that $\bar{\alpha}=\overline{u_{lh}}$. Note first w_h is not a rational centre of $\mathfrak{s}$. Indeed, if w_h is a rational centre of $\mathfrak{s}$, then

$$|\mathfrak{s}|>|\lambda_{\mathfrak{s}}|_p=|\rho_{\mathfrak{s}}|_p,\qquad d_{\mathfrak{s}}>\lambda_{\mathfrak{s}}=\rho_{\mathfrak{s}},$$

which contradicts the fact that C has an almost rational cluster picture. As $\{\mathfrak{s}_1,\dots,\mathfrak{s}_m\}=\Sigma_C^{\min}$, we must have that w_l is a rational centre of $\mathfrak{s}$, for some $l\neq h$. Then $\mathfrak{s}_l\subseteq\mathfrak{s}\subsetneq\mathfrak{s}_h\wedge\mathfrak{s}_l$. Since $\bar{\alpha}=\frac{r-w_h}{\pi^{\lambda_{\mathfrak{s}}}}\mod\pi$ for any $r\in\mathfrak{s}$, from above we have $\bar{\alpha}=\overline{u_{lh}}$. Finally, L is the edge of $\mathtt{NP}(f_h)$ of slope $-\lambda_{\mathfrak{s}}=-\rho_{hl}$. Thus $L=L^{w_h}_{\mathfrak{s}_h\wedge\mathfrak{s}_l}$. □

It remains to compute S_M when $M=M_{V,j}$, where $V=V_{\mathfrak{t}}^{w_h}$ or $V=V_0^{w_h}$.

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

Lemma 5.10. *Let $M = M_{V,j}$ for $V = V_{\mathfrak{t}}^{w_h}$, or $V = V_0^{w_h}$ if $\mathfrak{t} = \mathfrak{s}_h$. For any $l \neq h$ we have*

(i) $T_M^{hl}(X, Y, 0) = 1$ *except when* $\mathfrak{t} = \mathfrak{s}_l \wedge \mathfrak{s}_h$ *and* $j = 0$*;*
(ii) $T_M^{hl}(X, 0, Z) = 1$ *except when* $\mathfrak{t} < \mathfrak{s}_l \wedge \mathfrak{s}_h$ *and* $j = r_V$*.*

Proof. The Lemma immediately follows from Lemmas 5.4 and 5.5. □

Lemma 5.11. *Let $M = M_{V,j}$ with $V = V_{\mathfrak{t}}^{w_h}$, or $V = V_0^{w_h}$ if $\mathfrak{t} = \mathfrak{s}_h$. Then $S_M = \varnothing$.*

Proof. For any $l \neq h$, we want to prove that

$$S_M^{hl} := \{T_M^{hl}(X, Y, Z) = Y^{\tilde{m}_{23}} Z^{\tilde{m}_{33}} = 0\} = \varnothing. \tag{6}$$

Lemma 5.2 shows that $\tilde{m}_{33}$ is always different from 0 and that $\tilde{m}_{23} = 0$ if and only if $j = r_V$, and $V = V_{\mathfrak{R}}^{w_h}$ or $V = V_0^{w_h}$. Assume that if $\mathfrak{t} = \mathfrak{s}_l \wedge \mathfrak{s}_h$ then $j \neq 0$ and that if $\mathfrak{t} < \mathfrak{s}_l \wedge \mathfrak{s}_h$ then $j \neq r_V$. Lemma 5.10 implies (6).

If $\mathfrak{t} = \mathfrak{s}_l \wedge \mathfrak{s}_h$ and $j = 0$, then $\rho_{hl}\tilde{m}_{33} - \tilde{m}_{31} = 0$ but $\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21} > 0$. So

$$S_M^{hl} = \{T_M^{hl}(X, Y, Z) = Z^{\tilde{m}_{33}} = 0\} \subset \operatorname{Spec} R[Y^{-1}].$$

Similarly, if $\mathfrak{t} < \mathfrak{s}_l \wedge \mathfrak{s}_h$ and $j = r_V$, then $\tilde{m}_{21} - \rho_{hl}\tilde{m}_{23} = 0$, however $\tilde{m}_{31} - \rho_{hl}\tilde{m}_{33} > 0$. Then

$$S_M^{hl} = \{T_M^{hl}(X, Y, Z) = Y^{\tilde{m}_{23}} = 0\} \subset \operatorname{Spec} R[Z^{-1}].$$

In both cases, $S_M^{hl} \subseteq X_F$ as sets, where $F = F_{\mathfrak{s}_l \wedge \mathfrak{s}_h}^{w_h}$ ([1, Definition 3.7]). Let $L = L_{\mathfrak{s}_l \wedge \mathfrak{s}_h}^{w_h}$, and let $f_h(x) = f(x + w_h)$ and $g_h(x, y) = y^2 - f_h(x)$. By Lemmas 5.8 and 5.9, one has

$$S_M^{hl} \subseteq X_F \cap S_{M_{L,0}} = \varnothing,$$

as $\mathcal{F}_{M_{L,0}}^h(X, Y, 0) \mod \pi$ equals $Y^b - X^a \overline{f_h|_L}(X)$, for some $a \in \mathbb{Z}$, $b = 1, 2$ (see Lemma 5.17 for more details, whose proof is independent of this result). Thus if $V = V_{\mathfrak{t}}^{w_h}$ and $M = M_{V,j}$, then $S_M = \varnothing$. □

5.6. Components

Now that we have compared the special fibre of $\mathcal{C}$ with those of the models $\mathcal{C}_\Delta^{w_h}$, let us describe closed subschemes that form it. We will first study closed subschemes forming $\mathring{\mathcal{C}}_{\Delta,s}^{w_h}$ and then how they glue in $\mathcal{C}_s$.

Let $f_h(x) = f(x + w_h)$ and $g_h(x, y) = y^2 - f_h(x)$. According to [1, Theorem 3.14] the special fibre of $\mathcal{C}_\Delta^{w_h}$ is formed by:

- Chains of $\mathbb{P}_k^1$s coming from v-edges of Δ^{w_h}.
- 1-dimensional subschemes coming from v-faces of Δ^{w_h}.

More precisely, each v-edge E gives a scheme $X_E \times \mathbb{P}_E$, where $\mathbb{P}_E$ is a chain of $\mathbb{P}_k^1$s and $X_E \subset \mathbb{G}_{m,k}$ is given by $\overline{g_h|_E} = 0$. The multiplicities and the length of $\mathbb{P}_E$ can be completely described by the slopes of E. On the other hand, each v-face F gives a proper scheme $\bar{X}_F$ containing an open subscheme $X_F \subseteq \mathbb{G}_{m,k}^2$ given by $\overline{g_h|_F} = 0$. How the previous schemes intersect to form $\mathcal{C}_{\Delta,s}^{w_h}$ is described by [1, Theorem 3.14]. The reader is pointed to [1] for more details.

Definition 5.12. *Let $\mathfrak{t} \in \Sigma^W$ be a proper cluster. For any rational centre w of $\mathfrak{t}$, let $r_{\mathfrak{t},w} = \frac{w-r}{\pi^{\rho_{\mathfrak{t}}}}$, $u_{\mathfrak{t},w} = c_f \prod_{r \in \mathfrak{R} \setminus \mathfrak{t}} r_{\mathfrak{t},w}$ and $u_{\mathfrak{s}_h,w_h}^0 = c_f \prod_{r \in \mathfrak{R} \setminus \{w_h\}} r_{\mathfrak{s}_h,w_h}$. Define $\overline{f_{\mathfrak{t},w}^W}, \overline{g_{\mathfrak{t},w}} \in k[X]$, and $\overline{g_{\mathfrak{s}_h,w_h}^0} \in k[X]$ for any $h = 1, \ldots, m$, as follows:*

(i) *Let* $u=u_{\mathfrak{t},w}$. *Define* $\overline{f^W_{\mathfrak{t},w}}$ *by*

$$\overline{f^W_{\mathfrak{t},w}}(X^{b_{\mathfrak{t}}}) = \frac{u}{\pi^{v(u)}} \prod_{r\in\mathfrak{t}\setminus\bigcup_{\mathfrak{s}<\mathfrak{t}}\mathfrak{s}} (X+r_{\mathfrak{t},w}) \mod \pi,$$

where the union runs through all children $\mathfrak{s}$ *of* $\mathfrak{t}$ *in* Σ^W. *If* $\Sigma=\Sigma_C^{\min}$ *denote* $\overline{f^W_{\mathfrak{t},w}}$ *by* $\overline{f_{\mathfrak{t},w}}$.

(ii) *Let* $u=u_{\mathfrak{t},w}$. *Define* $\overline{g_{\mathfrak{t},w}}(X) := X^{p_{\mathfrak{t}}/\gamma_{\mathfrak{t}}} - \frac{u}{\pi^{v(u)}} \mod \pi$.

(iii) *Let* $u=u^0_{\mathfrak{s}_h,w_h}$. *Define* $\overline{g^0_{\mathfrak{s}_h,w_h}}(X) := X^{p^0_{\mathfrak{s}_h}/\gamma^0_{\mathfrak{s}_h}} - \frac{u}{\pi^{v(u)}} \mod \pi$.

Note that the polynomials defined in Definition 5.12 agree with the ones in Definition 4.14 when $w=w_{\mathfrak{t}}$.

Lemma 5.13. *Let* $\mathfrak{s},\mathfrak{t}\in\Sigma_C^{\mathrm{rat}}$, *with* $\mathfrak{s}\subsetneq\mathfrak{t}$. *Let* w',w *be rational centres of* $\mathfrak{s}$ *and* $\mathfrak{t}$ *respectively, and define* $\overline{u_{w'w}} = \frac{w'-w}{\pi^{\rho_{\mathfrak{t}}}} \mod \pi$. *Then* $\overline{u_{w'w}}$ *does not depend on the choice of a rational centre* w' *of* $\mathfrak{s}$.

Proof. Suppose that w_1,w_2 are two rational centres of $\mathfrak{s}$. Then $v(w_1-w_2)\geq\rho_{\mathfrak{s}}>\rho_{\mathfrak{t}}$, and so the Lemma follows. □

Remark 5.14. *Let* $\mathfrak{t}\in\Sigma_C^{w_h}$. *Let* $l=1,\dots,m$, $l\neq h$. *Then* $\mathfrak{t}=\mathfrak{s}_h\wedge\mathfrak{s}_l$ *if and only if it has a child* $\mathfrak{s}\in\Sigma_C^{w_l}\setminus\Sigma_C^{w_h}$. *In particular, if this happens, Lemma* 5.13 *shows that* $\overline{u_{lh}} = \frac{w-w_h}{\pi^{\rho_{\mathfrak{t}}}} \mod \pi$ *for any rational centre* w *of* $\mathfrak{s}$.

Definition 5.15. *Let* $\mathfrak{t}\in\Sigma_C^{w_h}$ *be a proper cluster. Define* $\hat{\mathfrak{t}}^W := \{\mathfrak{s}\in\Sigma^W\cup\{\varnothing\}\mid\mathfrak{s}<\mathfrak{t}\}$, *where* $\varnothing<\mathfrak{t}$ *only if* $\mathfrak{t}$ *has no child in* Σ^W. *If* $\varnothing<\mathfrak{t}$ *then we will say that* w_h *is the rational centre of* $\varnothing$.

Define $\mathbb{G}_{\mathfrak{t},w_h} := \mathbb{G}_{m,k}\setminus\bigcup_l\{\overline{u_{lh}}\}$, *where the union runs through all* $l\neq h$ *such that* $\mathfrak{s}_l\wedge\mathfrak{s}_h=\mathfrak{t}$. *Note that Remark* 5.14 *shows that* $\mathbb{G}_{\mathfrak{t},w_h} = \mathbb{A}^1_k\setminus\bigcup_{\mathfrak{s}\in\hat{\mathfrak{t}}^W}\{\overline{u_{w_{\mathfrak{s}}w_h}}\}$, *where* $\overline{u_{w_{\mathfrak{s}}w_h}} = \frac{w_{\mathfrak{s}}-w_h}{\pi^{\rho_{\mathfrak{t}}}} \mod \pi$, *and* $w_{\mathfrak{s}}$ *is any rational centre of* $\mathfrak{s}$.

Let $\mathfrak{t}\in\Sigma_C^{w_h}$ be a proper cluster. Let $V=V_{\mathfrak{t}}^{w_h}$ and $M=M_{V,j}$. In Section 5.5 we showed the special fibre of U_M^h equals $X_M\cap\mathcal{C}^{w_h}_{\Delta,s}$. Therefore the components of $\mathring{\mathcal{C}}^{w_h}_{\Delta,s}$ coming from V are the same of those of $\mathcal{C}^{w_h}_{\Delta,s}$ given by the same v-edge. Therefore V gives a closed subscheme $X_V\times\mathbb{P}_V$ of $\mathring{\mathcal{C}}^{w_h}_{\Delta,s}$, where $\mathbb{P}_V$ is a chain of $\mathbb{P}^1_k$s and $X_V:\{\overline{g_h|_V}=0\}$ over $\mathbb{G}_{m,k}$. Lemma 4.3 implies that $\overline{g_h|_V}=\overline{g_{\mathfrak{t},w_h}}$.

Let $V_0=V_0^{w_h}$ and $M=M_{V_0,j}$. Similarly to above, $X_M\cap\mathring{\mathcal{C}}^{w_h}_{\Delta,s}=X_M\cap\mathcal{C}^{w_h}_{\Delta,s}$ and so V_0 gives rise to a closed subscheme $X_{V_0}\times\mathbb{P}_{V_0}$ of $\mathring{\mathcal{C}}^{w_h}_{\Delta,s}$, where $\mathbb{P}_{V_0}$ is a chain of $\mathbb{P}^1_k$s and $X_{V_0}:\{\overline{g_h|_{V_0}}=0\}$ over $\mathbb{G}_{m,k}$. Note that $\overline{g_h|_{V_0}}=\overline{g^0_{\mathfrak{s}_h,w_h}}$.

Let $\mathfrak{t}\in\Sigma_C^{w_h}$ be a proper cluster. Let $L=L_{\mathfrak{t}}^{w_h}$ and $M=M_{L,i}$. By Lemma 5.8, the v-edge L gives a subscheme $X_L^W\times\mathbb{P}_L$ of $\mathring{\mathcal{C}}^{w_h}_{\Delta,s}$, where $\mathbb{P}_L$ is a chain of $\mathbb{P}^1_k$s of length r_L and $X_L^W:\{\overline{g_h|_L}=0\}$ in $\mathbb{G}_{\mathfrak{t},w_h}$. Note that $r_L=0$ or 1 by Lemma 4.3 and $r_L=1$ if and only if $D_{\mathfrak{t}}=1$. Let $\mathfrak{t}_h\in\Sigma_C^{w_h}$ be the unique child of $\mathfrak{t}$ with rational centre w_h or set $\mathfrak{t}_h=\varnothing$ if $\mathfrak{t}$ has no such child. We will show that

$$\overline{g_h|_L}(X) = -\prod_{\mathfrak{s}\in\hat{\mathfrak{t}}^W,\ \mathfrak{s}\neq\mathfrak{t}_h} (X+\overline{u_{w_{\mathfrak{s}}w_h}})^{|\mathfrak{s}|}\cdot\overline{f^W_{\mathfrak{t},w_h}}(X). \tag{7}$$

where $\overline{u_{w_{\mathfrak{s}}w_h}} = \frac{w_{\mathfrak{s}}-w_h}{\pi^{\rho_{\mathfrak{t}}}} \mod \pi$, and $w_{\mathfrak{s}}$ is any rational centre of $\mathfrak{s}$.

Suppose $\mathfrak{t}\neq\mathfrak{s}_h\wedge\mathfrak{s}_l$ for any $l\neq h$. Equivalently, all children of $\mathfrak{t}$ in Σ^W (at most one) belong to $\Sigma_C^{w_h}$. Then Lemma 4.3 shows that $\overline{g_h|_L}=-\overline{f^W_{\mathfrak{t},w_h}}$. Suppose now that $\mathfrak{t}=\mathfrak{s}_h\wedge\mathfrak{s}_l$ for some $l\neq h$. In this case $b_{\mathfrak{t}}=1$. We have

$$\frac{\overline{g_h|_L}(X)}{\prod_{\mathfrak{s}\in\hat{\mathfrak{t}}^W,\mathfrak{s}\neq\mathfrak{t}_h}(X+\overline{u_{w_{\mathfrak{s}}w_h}})^{|\mathfrak{s}|}} = \left(\frac{-\frac{u}{\pi^{v(u)}}\prod_{r\in\mathfrak{t}\setminus\mathfrak{t}_h}(X+r_{\mathfrak{t},w_h})}{\prod_{\mathfrak{s}\in\hat{\mathfrak{t}}^W,\mathfrak{s}\neq\mathfrak{t}_h}\prod_{r\in\mathfrak{s}}(X+r_{\mathfrak{t},w_h})} \mod \pi\right) = -\overline{f^W_{\mathfrak{t},w_h}}(X),$$

where $r_{\mathfrak{t},w_h}$ and $u=u_{\mathfrak{t},w_h}$ are as in Definition 5.12. Indeed, $\overline{u_{w_\mathfrak{s} w_h}} = r_{\mathfrak{t},w_h} \mod \pi$ for every $r \in \mathfrak{s}$ as $v(w_\mathfrak{s} - r) \geq \rho_\mathfrak{s} > \rho_\mathfrak{t}$, and since $b_\mathfrak{t}=1$, Lemma 4.3 implies that

$$\overline{g_h|_L}(x) = -\tfrac{u}{\pi^{v(u)}} \prod\nolimits_{r\in\mathfrak{t}\setminus\mathfrak{t}_h} (x + r_{\mathfrak{t},w_h}) \mod \pi.$$

In particular, Remark 5.13 and Lemma 5.9 shows that $(X+\overline{u_{hl}}) \nmid \overline{f^W_{\mathfrak{t},w_h}}(X)$, for any $l \neq h$ such that $\mathfrak{s}_l \wedge \mathfrak{s}_h = \mathfrak{t}$. Moreover, $X \nmid \overline{f^W_{\mathfrak{t},w_h}}(X)$ by definition. Therefore the scheme X^W_L is equal to the closed subscheme $X^W_{\mathfrak{t},w_h} \subset \mathbb{A}^1_k$ given by $\overline{f^W_{\mathfrak{t},w_h}} = 0$.

Let $\mathfrak{t} \in \Sigma^W$ be a proper cluster. For any $h = 1, \ldots, m$ such that $\mathfrak{s}_h \subseteq \mathfrak{t}$, let $\bar{X}_{F^{w_h}_\mathfrak{t}}$ be the 1-dimensional closed subscheme of $\mathcal{C}^{w_h}_{\Delta,s}$ given by $F^{w_h}_\mathfrak{t}$. Define

$$\mathring{X}_{F^{w_h}_\mathfrak{t}} := \bar{X}_{F^{w_h}_\mathfrak{t}} \cap \mathring{\mathcal{C}}^{w_h}_\Delta.$$

Denote by $\Gamma_\mathfrak{t}$ the 1-dimensional closed subscheme of $\mathcal{C}_s$, result of the glueing of the subschemes $\mathring{X}_{F^{w_h}_\mathfrak{t}}$ of $\mathring{\mathcal{C}}^{w_h}_{\Delta,s}$ to varying of h such that $\mathfrak{t} \in \Sigma^{w_h}_C$.

Lemma 5.16. *Let $\mathfrak{t} \in \Sigma^{w_h}_C$ be a proper cluster. The multiplicity of $\Gamma_\mathfrak{t}$ in $\mathcal{C}_s$ is $m_\mathfrak{t}$.*

Proof. Let $L = L^{w_h}_\mathfrak{t}$, $M = M_{L,0}$, and let $F = F^{w_h}_\mathfrak{t}$. The multiplicity of $\bar{X}_{F^{w_h}_\mathfrak{t}}$, and so of $X_{F^{w_h}_\mathfrak{t}}$ and $\Gamma_\mathfrak{t}$, is δ_F. Hence we only need to show that $m_\mathfrak{t} = \delta_F$. Let $d_0 \in \mathbb{Z}$ as in Lemma 5.2. Then $\delta_F = \delta_L d_0$. The result follows as $\delta_L = b_\mathfrak{t}$ and d_0, denominator of s^L_1, equals $3 - D_\mathfrak{t}$ by Lemma 4.3. □

Lemma 5.17. *Let $L = L^{w_h}_\mathfrak{t}$, $F = F^{w_h}_\mathfrak{t}$ and $M = M_{L,0}$. Let $c \in \{0, \ldots, b_\mathfrak{t} - 1\}$ such that $1/b_\mathfrak{t} - \rho_\mathfrak{t} \cdot c \in \mathbb{Z}$. Then $\mathcal{F}^h_M(X, Y, 0) \mod \pi$ equals the polynomial*

$$\overline{g_h|_F}(X,Y) = Y^{D_\mathfrak{t}} - \prod_{\mathfrak{s}\in\hat{\mathfrak{t}}^W} (X - \overline{u_{w_\mathfrak{s} w_h}})^{\frac{|\mathfrak{s}|}{b_\mathfrak{t}} - c\epsilon_\mathfrak{t}} \overline{f^W_{\mathfrak{t},w_h}}(X),$$

where $\overline{u_{w_\mathfrak{s} w_h}} = \frac{w_\mathfrak{s} - w_h}{\pi^{\rho_\mathfrak{t}}} \mod \pi$, and $w_\mathfrak{s}$ is any rational centre of $\mathfrak{s}$.

In particular, $\Gamma^h_\mathfrak{t} \subset \mathbb{G}_{\mathfrak{t},w_h} \times \mathbb{A}^1_k$ given by $\overline{g|_F} = 0$ is the open subscheme $U^h_M \cap \{Z = 0\}$ of $\mathring{X}_F$, and the points in S_M belong to all irreducible components of $\bar{X}_F$.

Proof. From [1, §3.5] and the equation of C^{w_h}, the polynomial $\mathcal{F}^h_M(X,Y,0)$ reduces modulo π to $X^{a_1}Y^b + X^{a_2}\overline{g_h|_L}(X)$, for some $b = 1, 2$ and $a \in \mathbb{Z}$. Lemma 4.9 shows that $b = D_\mathfrak{t}$. By Lemma 4.3, $a_1 = 2\tilde{m}_{12}$, $a_2 = |\mathfrak{t}_h|\tilde{m}_{11} + (\epsilon_\mathfrak{t} - |\mathfrak{t}_h|\rho_\mathfrak{t})\tilde{m}_{13}$, where $\mathfrak{t}_h \in \Sigma^{w_h}_C \cup \{\varnothing\}$, $\mathfrak{t}_h < \mathfrak{t}$. Then $a_1 = 0$ and $a_2 = \frac{|\mathfrak{t}_h|}{b_\mathfrak{t}} - c\epsilon_\mathfrak{t}$ by Lemma 5.2.

If $\mathfrak{t}$ has one or no child, or $D_\mathfrak{t} = 1$, then $\overline{g_h|_L} = -\overline{f^W_{\mathfrak{t},w_h}}$ by (7). On the other hand, if $D_\mathfrak{t} = 2$ and $\mathfrak{t}$ has two or more children in Σ^{rat}_C, then $b_\mathfrak{t} = 1$, and so $c = 0$. Therefore the equality (7) concludes the proof of the first part of the statement also in this case. Finally, the last part of the Lemma follows from Lemma 5.8. □

Let c as in the previous Lemma and define $\tilde{\mathfrak{t}}^W := \{\mathfrak{s} \in \hat{\mathfrak{t}}^W \mid \frac{|\mathfrak{s}|}{b_\mathfrak{t}} - c\epsilon_\mathfrak{t} \notin 2\mathbb{Z}\}$.

Proposition 5.18. *Let $L = L^{w_h}_\mathfrak{t}$ and $M = M_{L,0}$. The dense open subscheme $\Gamma_\mathfrak{t} \cap U^h_M$ of $\Gamma_\mathfrak{t}$ is isomorphic to the closed subscheme of $\mathbb{G}_{\mathfrak{t},w_h} \times \mathbb{A}^1_k$ given by*

$$Y^{D_\mathfrak{t}} = \prod_{\mathfrak{s}\in\tilde{\mathfrak{t}}^W} (X - \overline{u_{w_\mathfrak{s} w_h}}) \cdot \overline{f^W_{\mathfrak{t},w_h}}(X),$$

where $\overline{u_{w_\mathfrak{s} w_h}} = \frac{w_\mathfrak{s} - w_h}{\pi^{\rho_\mathfrak{t}}} \mod \pi$, and $w_\mathfrak{s}$ is any rational centre of $\mathfrak{s}$.

Proof. The proposition follows from Lemma 5.17 and the definition of $\mathbb{G}_{\mathfrak{t},w_h}$. □

We conclude this subsection describing how the glueing morphism (4) restricts to the special fibre. Suppose $\mathfrak{t} \supseteq \mathfrak{s}_l \wedge \mathfrak{s}_h$ for $l \neq h$ and let M be a matrix associated to $\mathfrak{t}$. Consider the glueing map $U_M^h \to U_M^l$ explicitly defined in Section 5.3.

Suppose first $M = M_{V,j}$ with $V = V_{\mathfrak{t}}^{w_l}$. By Lemma 5.10 the glueing morphism restricts to the identity on $X_V \times \mathbb{P}_V$.

Suppose $M = M_{L,i}$ with $L = L_{\mathfrak{t}}^{w_l}$. Note that $\tilde{m}_{12} = 0$ from Lemma 5.2. Recall the open subscheme $\Gamma_{\mathfrak{t}}^h$ of $X_{F_{\mathfrak{t}}^{w_h}}$ defined in Lemma 5.17. Then, Lemma 5.7 implies that the glueing map restricts to an isomorphism $\Gamma_{\mathfrak{t}}^h \mapsto \Gamma_{\mathfrak{t}}^l$ induced by the ring homomorphism sending $X \mapsto X + \overline{u_{w_h w_l}}$, where $\overline{u_{w_h w_l}} = \frac{w_h - w_l}{\pi^{\rho_{\mathfrak{t}}}}$ mod π. Similarly, it restricts to an isomorphism $X_{L_{\mathfrak{t}}^{w_h}}^W \times \mathbb{P}_{L_{\mathfrak{t}}^{w_h}} \to X_{L_{\mathfrak{t}}^{w_l}}^W \times \mathbb{P}_{L_{\mathfrak{t}}^{w_l}}$, where $\mathbb{P}_{L_{\mathfrak{t}}^{w_h}} \to \mathbb{P}_{L_{\mathfrak{t}}^{w_l}}$ is the identity and $X_{L_{\mathfrak{t}}^{w_h}}^W \to X_{L_{\mathfrak{t}}^{w_l}}^W$ is induced by the ring homomorphism sending $X \mapsto X + \overline{u_{w_h w_l}}$.

5.7. Regularity

Let $w_h \in W$. We want to show that if $\Sigma = \Sigma_C^{\min}$, and C has an almost rational cluster picture and is y-regular, then $\mathring{\mathcal{C}}_\Delta^{w_h}$ is a regular scheme.

Lemma 5.19. *Consider the model $\mathcal{C}_\Delta^{w_h}/O_K$ and let $f_h(x) = f(x + w_h)$. Suppose $\Sigma = \{\mathfrak{s}_1, \dots, \mathfrak{s}_m\} = \Sigma_C^{\min}$, and C has an almost rational cluster picture and is y-regular. If P is a singular point of $\mathcal{C}_\Delta^{w_h}$ then*

$$P \in \operatorname{Spec} \frac{O_K[X^{\pm 1}, Y, Z]}{(\mathcal{F}_M^h(X,Y,Z), X + u_{hl}, Y^{\tilde{m}_{23}} Z^{\tilde{m}_{33}}, \pi)} \subset \mathcal{C}_\Delta^{w_h} \cap X_M,$$

for some $l \neq h$, where $M = M_{L_{\mathfrak{s}_h \wedge \mathfrak{s}_l}^{w_h}, i}$ for $i = 0, \dots, r_{L_{\mathfrak{s}_h \wedge \mathfrak{s}_l}^{w_h}}$.

Proof. Denote by $m_\alpha(X) \in O_K[X]$ a lift of the minimal polynomial in $k[X]$ of $\bar{\alpha} \in \bar{k}$. By Lemma 5.9, we only need to show that if $P \in \mathcal{C}_\Delta^{w_h}$ is a singular point then

$$P \in \operatorname{Spec} \frac{O_K[X^{\pm 1}, Y, Z]}{(\mathcal{F}_{M_{L,i}}^h(X,Y,Z), m_\alpha(X), Y^{\tilde{m}_{23}} Z^{\tilde{m}_{33}}, \pi)}, \tag{8}$$

for some v-edge $L = L_{\mathfrak{t}}^{w_h}$ of Δ^{w_h}, and some multiple root $\bar{\alpha}$ of $\overline{f_h|_L}$. We study the polynomial $\mathcal{F}_M^h$ to varying of the matrix M, using [1, §4.5]. Let $g_h(x,y) = y^2 - f_h(x)$. Let $L = L_{\mathfrak{t}}^{w_h}$ and $M = M_{L,i}$. Note that $\overline{g_h|_L} = -\overline{f_h|_L}$. We have $\mathcal{F}_M^h(X, 0, Z) = \overline{g_h|_L}(X)$ for any i. On the other hand, $\mathcal{F}_M^h(X, Y, 0) = \overline{g_h|_L}(X)$ if $i > 0$ and $\mathcal{F}_M^h(X, Y, 0) = \overline{g_h|_F}(X, Y)$ if $i = 0$. From the description given in Lemma 5.17, we conclude that for these matrices M the points in (8) are the only possibly singular points of $\mathcal{C}_\Delta^{w_h} \cap X_M$. In particular, this proves that for any v-face F of Δ^{w_h}, the points in X_F are non-singular in $\mathcal{C}_\Delta^{w_h}$.

Let $V = V_{\mathfrak{t}}^{w_h}$ or $V = V_0^{w_h}$ and $M = M_{V,j}$. Since C is y-regular, $p \nmid \deg(\overline{g_h|_V})$ by Lemma 4.9. By [1, §4.5] and the fact that the points in X_F are non-singular for all v-faces F, we conclude that $\mathcal{C}_\Delta^{w_h}$ has no singular point on X_M for these matrices M, as required. □

Proposition 5.20. *Suppose $\Sigma = \Sigma_C^{\min}$, and C has an almost rational cluster picture and is y-regular, then $\mathcal{C}$ is a regular scheme.*

Proof. Lemmas 5.19 and 5.8 show that $\mathring{\mathcal{C}}_\Delta^{w_h}$ is regular for every h. Thus their glueing $\mathcal{C}$ is regular as well. □

5.8. Separatedness

It remains to prove that $\mathcal{C}$ is a proper scheme. We first show it is separated. Clearly it suffices to prove that $\mathcal{X}/O_K$ is separated. Since the schemes X_Δ^h are separated, then the open subschemes $\mathring{X}_\Delta^h$ are separated as well by [9, Proposition 3.3.9]. Consider the open cover $\{V_M^h\}_{h,M}$ of $\mathcal{X}$. Let $h, l = 1, \dots, m$ and let M_h and

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

M_l be matrices associated to proper clusters $\mathfrak{t}_h \in \Sigma_C^{w_h}$ and $\mathfrak{t}_l \in \Sigma_C^{w_l}$ respectively. By [9, Proposition 3.3.6] we want to show

(i) $V^h_{M_h} \cap V^l_{M_l}$ is affine,
(ii) The canonical homomorphism

$$\mathcal{O}_{\mathcal{X}}(V^h_{M_h}) \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{X}}(V^l_{M_l}) \longrightarrow \mathcal{O}_{\mathcal{X}}(V^h_{M_h} \cap V^l_{M_l})$$

is surjective.

The definition of the glueing map (4) implies (i). If $h = l$, or $\mathfrak{s}_l \subseteq \mathfrak{t}_h$, or $\mathfrak{s}_h \subseteq \mathfrak{t}_l$, then (ii) follows from the separatedness of $\mathring{X}^h_\Delta$ and $\mathring{X}^l_\Delta$. So assume $l \neq h$, and $\mathfrak{t}_h, \mathfrak{t}_l \subsetneq \mathfrak{s}_h \wedge \mathfrak{s}_l$. Consider the Moebius transformation

$$\psi_l\colon \quad x \mapsto \frac{x}{xw_{hl}^{-1}+1}, \qquad y \mapsto \frac{y}{(xw_{hl}^{-1}+1)^{g+1}}.$$

It sends the curve C^{w_l} to the isomorphic hyperelliptic curve

$$C_l^h : y^2 = (xw_{hl}^{-1}+1)^{2g+2} f\left(x(xw_{hl}^{-1}+1)^{-1} + w_l\right).$$

As

$$\begin{aligned} f_l^h(x) &:= (xw_{hl}^{-1}+1)^{2g+2} f\left(x(xw_{hl}^{-1}+1)^{-1} + w_l\right) \\ &= c_f w_{hl}^{|\mathfrak{R}|}(xw_{hl}^{-1}+1)^{2g+2-|\mathfrak{R}|} \prod_{r \in \mathfrak{R} \smallsetminus \{w_h\}} \frac{r-w_h}{w_{lh}} \left(xw_{hl}^{-1} + \frac{r-w_l}{r-w_h}\right), \end{aligned}$$

every cluster $\mathfrak{s} \in \Sigma_C^{w_l}$ such that $\mathfrak{s} \subsetneq \mathfrak{s}_h \wedge \mathfrak{s}_l$, corresponds to a unique cluster $\mathfrak{s}^h \in \Sigma^0_{C_l^h}$ of same size, same radius and rational centre 0. Moreover,

$$\epsilon_{\mathfrak{s}^h} = v(c_{f_l^h}) + \sum_{r' \in \mathfrak{s}^h} \rho_{\mathfrak{s}^h} + \sum_{r' \notin \mathfrak{s}^h} v(r') = \epsilon_{\mathfrak{s}}.$$

Call $\mathfrak{t}_l^h$ the cluster in $\Sigma^0_{C_l^h}$ corresponding to $\mathfrak{t}_l$. Let Δ^{lh} and Δ_v^{lh} be the Newton polytopes attached to $y^2 - f_l^h(x)$ and let X_Δ^{lh} be the associated toric scheme (defined in [1, §4.2]). Since $\mathfrak{t}_l \subsetneq \mathfrak{s}_h \wedge \mathfrak{s}_l$, the v-faces $F_{\mathfrak{t}_l}$ of Δ^{w_l} and $F_{\mathfrak{t}_l^h}$ of Δ^{lh} are identical by Lemma 4.3. Furthermore, note that if $\mathfrak{t}_l < \mathfrak{s}_h \wedge \mathfrak{s}_l$, then $\rho_{P(\mathfrak{t}_l^h)} \leq \rho_{hl} = \rho_{P(\mathfrak{t}_l)}$ and so $s_2^{V^0} \leq s_2^V$, where $V^0 = V^0_{\mathfrak{t}_l^h}$ and $V = V^{w_l}_{\mathfrak{t}_l}$. Therefore the matrix $M := M_l$ is also associated to $\mathfrak{t}_l^h$.

For every $o = 1, \dots, m$, with $o \neq l$, define

$$w_{hlo} = \begin{cases} \dfrac{w_{hl}w_{lo}}{w_{ho}} & \text{if } o \neq h, \\ w_{hl} & \text{if } o = h, \end{cases}$$

and write $w_{hlo} = u_{hlo}\pi^{\rho_{hlo}}$, where $u_{hlo} \in \mathcal{O}_K^\times$ and $\rho_{hlo} \in \mathbb{Z}$, i.e.

$$u_{hlo} = \begin{cases} \dfrac{u_{hl}u_{lo}}{u_{ho}} & \text{if } o \neq h, \\ u_{hl} & \text{if } o = h, \end{cases} \quad \text{and} \quad \rho_{hlo} = \begin{cases} \rho_{hl} + \rho_{lo} - \rho_{ho} & \text{if } o \neq h, \\ \rho_{hl} & \text{if } o = h. \end{cases}$$

Define

$$\tilde{T}_M^{hlo}(X,Y,Z) := \begin{cases} 1 + u_{hlo}X^{\rho_{hlo}\tilde{m}_{13}-\tilde{m}_{11}}Y^{\rho_{hlo}\tilde{m}_{23}-\tilde{m}_{21}}Z^{\rho_{hlo}\tilde{m}_{33}-\tilde{m}_{31}} & \text{if } \mathfrak{t}_l \supseteq \mathfrak{s}_o, \\ u_{hlo}^{-1}X^{\tilde{m}_{11}-\rho_{hlo}\tilde{m}_{13}}Y^{\tilde{m}_{21}-\rho_{hlo}\tilde{m}_{23}}Z^{\tilde{m}_{31}-\rho_{hlo}\tilde{m}_{33}} + 1 & \text{if } \mathfrak{t}_l \not\supseteq \mathfrak{s}_o. \end{cases}$$

We want to show $\tilde{T}_M^{hlo}(X,Y,Z) \in R$. If $o = h$ then

$$\tilde{T}_M^{hlo}(X,Y,Z) = T_M^{hl}(X,Y,Z) \in R.$$

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

So assume $o \neq h$. If $\mathfrak{s}_o \subseteq \mathfrak{t}_l$, then it follows from Lemma 5.4 as $\mathfrak{s}_l \wedge \mathfrak{s}_o \subsetneq \mathfrak{s}_l \wedge \mathfrak{s}_h$ and so $\rho_{hlo} = \rho_{lo}$. On the other hand, if $\mathfrak{s}_o \not\subseteq \mathfrak{t}_l$, then it follows from Lemma 5.5 as $\tilde{m}_{23}, \tilde{m}_{33} > 0$ and $\rho_{hlo} \leq \max\{\rho_{hl}, \rho_{lo}\}$. Let

$$\tilde{T}^{hl}_M(X,Y,Z) := \prod_{o\neq l} \tilde{T}^{hlo}_M(X,Y,Z).$$

The Moebius transformation

$$K[x^{\pm 1}, y^{\pm 1}, \textstyle\prod_{o\neq l}(x+w_{lo})^{-1}] \xrightarrow{\psi_l} K[x^{\pm 1}, y^{\pm 1}, \textstyle\prod_{o\neq l}(x+w_{hlo})^{-1}]$$

considered above induces an isomorphism

$$R[T^l_M(X,Y,Z)^{-1}] \xrightarrow{M^{-1}\circ\psi_l\circ M} R[\tilde{T}^{hl}_M(X,Y,Z)^{-1}],$$

sending

$$\begin{aligned} X &\mapsto X \cdot T^{hl}_M(X,Y,Z)^{-m_{11}-(g+1)m_{21}}, \\ Y &\mapsto Y \cdot T^{hl}_M(X,Y,Z)^{-m_{12}-(g+1)m_{22}}, \\ Z &\mapsto Z \cdot T^{hl}_M(X,Y,Z)^{-m_{13}-(g+1)m_{23}}. \end{aligned}$$

Then

$$\tilde{V}^{lh}_M := \operatorname{Spec} R[\tilde{T}^{hl}_M(X,Y,Z)^{-1}]$$

is an open subscheme of X^{lh}_Δ, isomorphic to V^l_M. We can clearly carry out similar constructions for t_h, M_h.

By comparing the Newton polytopes Δ^{lh}_v and Δ^{hl}_v, we see that the Moebius transformation $x \mapsto w_{hl}/(w_{lh}^{-1}x)$, $y \mapsto y/(w_{lh}^{-1}x)^{g+1}$ gives an isomorphism

$$\psi : K[x^{\pm 1}, y^{\pm 1}, \prod_{o\neq l}(x+w_{hlo})^{-1}] \longrightarrow K[x^{\pm 1}, y^{\pm 1}, \prod_{o\neq h}(x+w_{lho})^{-1}]$$

which induces a birational map $X^{hl}_\Delta \dashrightarrow X^{lh}_\Delta$, defined on the open set $\tilde{V}^{hl}_{M_h}$ of X^{hl}_Δ. In particular, there exists an open set $\tilde{V}^{lh}_{M_h}$ of X^{lh}_Δ, isomorphic to $V^h_{M_h}$ via the map induced by $\psi_h^{-1} \circ \psi$.

Recall the definition of ϕ in (1), which induces the glueing map between $V^l_{M_l}$ and $V^h_{M_h}$. Since the following diagram

$$\begin{array}{ccc} K[x^{\pm 1}, y^{\pm 1}, \prod_{o\neq l}(x+w_{lo})^{-1}] & \xrightarrow{\phi} & K[x^{\pm 1}, y^{\pm 1}, \prod_{o\neq h}(x+w_{ho})^{-1}] \\ \downarrow{\scriptstyle \psi_l} & & \downarrow{\scriptstyle \psi_h} \\ K[x^{\pm 1}, y^{\pm 1}, \prod_{o\neq l}(x+w_{hlo})^{-1}] & \xrightarrow{\psi} & K[x^{\pm 1}, y^{\pm 1}, \prod_{o\neq h}(x+w_{lho})^{-1}] \end{array}$$

is commutative, then the surjectivity of

$$\mathcal{O}_{\mathcal{X}}(V^h_{M_h}) \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{X}}(V^l_{M_l}) \longrightarrow \mathcal{O}_{\mathcal{X}}(V^h_{M_h} \cap V^l_{M_l})$$

follows from the separatedness of X^{lh}_Δ.

5.9. Properness

By [2, IV.15.7.10], it remains to show that $\mathcal{C}_s$ is proper. From [9, Exercise 3.3.11], we only need to prove that the 1-dimensional subscheme $\Gamma_{\mathfrak{t}}$ is proper for every $\mathfrak{t} = \mathfrak{s}_h \wedge \mathfrak{s}_l$. Indeed every other component is entirely contained in a model $\mathcal{C}^{w_h}_\Delta$, which is proper (see Section 5.5). Let $\mathfrak{t} = \mathfrak{s}_h \wedge \mathfrak{s}_l$ for some $h, l = 1, \dots, m$, with $h \neq l$. For any $o = 1, \dots, m$ such that $\mathfrak{s}_o \subset \mathfrak{t}$, let $\mathfrak{t}_o$ be the unique child of $\mathfrak{t}$ with $\mathfrak{s}_o \subseteq \mathfrak{t}_o < \mathfrak{t}$. Then $\Gamma_{\mathfrak{t}}$ is equal to the glueing of the schemes

$$\operatorname{Spec} \frac{R[T^o_M(X,Y,Z)^{-1}]}{(\mathcal{F}^o_M(X,Y,Z), Z, \pi)}, \quad M = M_{L^{w_o}_{\mathfrak{t}},0}, M_{V^{w_o}_{\mathfrak{t}},0},$$

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

and

$$\operatorname{Spec} \frac{R[T_M^o(X,Y,Z)^{-1}]}{(\mathcal{F}_M^o(X,Y,Z),Y,\pi)}, \qquad M = M_{V_{\mathfrak{t}_o}^{w_o}, r_{V_{\mathfrak{t}_o}^{w_o}}},$$

for all o such that $\mathfrak{s}_o \subset \mathfrak{t}$, through the isomorphism (4) and the glueing maps in the definition of $\mathcal{C}_\Delta^{w_o}$. In particular, for any o as above there exists a natural birational map $s_o : \Gamma_{\mathfrak{t}} \dashrightarrow \bar{X}_{F_{\mathfrak{t}}^{w_o}}$ which is defined as the identity morphism on the dense open $X_{F_{\mathfrak{t}}^{w_o}} = \Gamma_{\mathfrak{t}} \cap \mathring{\mathcal{C}}_\Delta^{w_o}$.

Let D/k be a normal curve, let $P \in D$ and let $D \setminus \{P\} \xrightarrow{g} \Gamma_{\mathfrak{t}}$ be a non-constant morphism of curves. We want to show that g extends to D. For every o as above, $\bar{X}_{F_{\mathfrak{t}}^{w_o}}$ is proper, so the birational map

$$g_o := s_o \circ g : D \setminus \{P\} \dashrightarrow \bar{X}_{F_{\mathfrak{t}}^{w_o}}$$

extends to a morphism $\bar{g}_o : D \longrightarrow \bar{X}_{F_{\mathfrak{t}}^{w_o}}$. If

$$P_o := \bar{g}_o(P) \in \left(\bar{X}_{F_{\mathfrak{t}}^{w_o}} \cap \mathring{\mathcal{C}}_\Delta^{w_o}\right) = s_o\left(\Gamma_{\mathfrak{t}} \cap \mathring{\mathcal{C}}_\Delta^{w_o}\right)$$

for some o such that $\mathfrak{s}_o \subset \mathfrak{t}$ (we will later show this is always the case), then there exists an open neighbourhood U of P_o such that $U \subseteq \left(\bar{X}_{F_{\mathfrak{t}}^{w_o}} \cap \mathring{\mathcal{C}}_\Delta^{w_o}\right)$ and so $s_o|^U_{s_o^{-1}(U)}$ is an isomorphism. Since $P \in \bar{g}_o^{-1}(U)$, the map

$$\bar{g}_o^{-1}(U) \xrightarrow{\bar{g}_o|^U_{\bar{g}_o^{-1}(U)}} U \xrightarrow{\left(s_o|^U_{s_o^{-1}(U)}\right)^{-1}} s_o^{-1}(U) \hookrightarrow \Gamma_{\mathfrak{t}},$$

induces an extension $D \longrightarrow \Gamma_{\mathfrak{t}}$ of g.

Suppose that $P_o \notin \bar{X}_{F_{\mathfrak{t}}^{w_o}} \cap \mathring{\mathcal{C}}_\Delta^{w_o}$ for any o such that $\mathfrak{s}_o \subset \mathfrak{t}$. From Section 5.5 we have

$$P_o \in S_M = \operatorname{Spec} \frac{R}{(\mathcal{F}_M^o(X,Y,Z), \prod_l (X + u_{ol}), Z, \pi)}, \tag{9}$$

where $M = M_{L_{\mathfrak{t}}^{w_o},0}$, and the product runs over all $l \neq o$ such that $\mathfrak{t} = \mathfrak{s}_o \wedge \mathfrak{s}_l$. In particular P_o is a point of each irreducible component of $\bar{X}_{F_{\mathfrak{t}}^{w_o}}$ by Lemma 5.17. Let $h \neq o$ such that $X + u_{oh}$ vanishes at P_o. Let ξ be the generic point of D and let $\xi_o = g_o(\xi)$, $\xi_h = g_h(\xi)$ be generic points of $\bar{X}_{F_{\mathfrak{t}}^{w_o}}$ and $\bar{X}_{F_{\mathfrak{t}}^{w_h}}$ respectively. Then the birational maps s_o and s_h give

$$\begin{tikzcd}
 & & \bar{X}_{F_{\mathfrak{t}}^{w_o}} \\
D \setminus \{P\} \arrow[r, "g"] & \Gamma_{\mathfrak{t}} \arrow[ur, dashed, "s_o"] \arrow[dr, dashed, "s_h"'] & \\
 & & \bar{X}_{F_{\mathfrak{t}}^{w_h}}
\end{tikzcd}
\implies
\begin{tikzcd}
 & k\left(\overline{\xi_o}\right) \arrow[dl, "\phi_{g_o}"'] \arrow[dd, "\simeq"] \\
k(D) & \\
 & k\left(\overline{\xi_h}\right) \arrow[ul, "\phi_{g_h}"]
\end{tikzcd}$$

where we denote by ϕ_{g_o} and ϕ_{g_h} the homomorphisms between function fields induced by g_o and g_h. The vertical isomorphism is induced by the map

$$\frac{R[T_M^o(X,Y,Z)^{-1}]}{(\mathcal{F}_M^o(X,Y,Z),Z)} \longrightarrow \frac{R[T_M^h(X,Y,Z)^{-1}]}{(\mathcal{F}_M^h(X,Y,Z),Z)}$$

which sends (see Section 5.3 and Lemma 5.7)

$$X + u_{oh} \mapsto X \cdot T_M^{ho}(X,Y,Z)^{m_{11}} + u_{oh} = X\left(1 + u_{ho}X^{-1}\right) + u_{oh} = X.$$

But the rational function $X + u_{oh}$ vanishes at P_o, while X does not vanish at P_h by (9). This gives a contradiction, as $\bar{g}_o(P) = P_o$ and $\bar{g}_h(P) = P_h$.

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

5.10. Genus

Suppose $\Sigma = \{\mathfrak{s}_1, \ldots, \mathfrak{s}_m\} = \Sigma_C^{\min}$, and C has an almost rational cluster picture and is y-regular. In the previous subsections we proved that $\mathcal{C}/O_K$ is a proper regular model of C. Let $\mathfrak{t} \in \Sigma_C^{w_h}$ be a proper cluster.

Proposition 5.21. *Let $\mathfrak{t} \in \Sigma_C^{w_h}$. Then $\Gamma_\mathfrak{t}$ is isomorphic to the smooth projective 1-dimensional scheme given by*

$$Y^{D_\mathfrak{t}} = \prod_{\mathfrak{s} \in \tilde{\mathfrak{t}}^W} (X - \overline{u_{w_\mathfrak{s} w_h}}) \overline{f_{\mathfrak{t}, w_h}}(X)$$

where $\overline{u_{w_\mathfrak{s} w_h}} = \frac{w_\mathfrak{s} - w_h}{\pi^{\rho_\mathfrak{t}}} \mod \pi$, and $w_\mathfrak{s}$ is any rational centre of $\mathfrak{s}$.

In particular,

(1) *if $D_\mathfrak{t} = 1$, then $\Gamma_\mathfrak{t} \simeq \mathbb{P}_k^1$;*
(2) *if $D_\mathfrak{t} = 2$ and $\mathfrak{t}$ is übereven, then $\Gamma_\mathfrak{t}$ is the disjoint union of two $\mathbb{P}^1$s over some quadratic extension of k;*
(3) *in all other cases, $\Gamma_\mathfrak{t}$ is a hyperelliptic curve of genus $g(\mathfrak{t})$.*

Proof. The first part of the proposition follows from Proposition 5.18.

For the second part of the statement note that if $D_\mathfrak{t} = 1$ then the result follows. Suppose $D_\mathfrak{t} = 2$. Then $p \neq 2$ as C is y-regular. Note that since $\Sigma = \Sigma_C^{\min}$, the proper clusters in Σ^W correspond to the proper clusters in Σ_C^{rat}. Recall the definition of $\tilde{\mathfrak{t}}$ given in Definition 4.13. Let $h(X) = \prod_{\mathfrak{s} \in \tilde{\mathfrak{t}}^W} (X - \overline{u_{w_\mathfrak{s} w_h}}) \overline{f_{\mathfrak{t}, w_h}}(X)$.

Suppose $\mathfrak{t}$ is übereven. Then all its children are (proper) rational cluster by Lemma 3.30 since they are even and $p \neq 2$. In particular $b_\mathfrak{t} = 1$ by Lemma 3.18 and so $\epsilon_\mathfrak{t} \in 2\mathbb{Z}$ and $\tilde{\mathfrak{t}} = \tilde{\mathfrak{t}}^W = \varnothing$ since it equals the set of odd rational children. Moreover, $\mathfrak{t} = \bigcup_{\mathfrak{s} < \mathfrak{t},\, \mathfrak{s} \text{ proper}} \mathfrak{s}$, and so $\overline{f_{\mathfrak{t}, w_h}} \in k$. Thus $h(X) \in k$.

Now suppose $h(X) \in k$. Then $\tilde{\mathfrak{t}}^W = \varnothing$ and $\mathfrak{t} = \bigcup_{\mathfrak{s} < \mathfrak{t}} \mathfrak{s}$, where $\mathfrak{s}$ runs through all children $\mathfrak{s} \in \Sigma^W$ of $\mathfrak{t}$. The non-proper clusters in Σ^W are of the form $\{w_l\}$ for some $l = 1, \ldots, m$. If $\{w_l\} < \mathfrak{t}$, then $\mathfrak{t} = \mathfrak{s}_l$, but in that case $\mathfrak{t}$ would not equal the union of its children in Σ^W. Hence $\mathfrak{t}$ has no non-proper children. It follows that $\tilde{\mathfrak{t}} = \tilde{\mathfrak{t}}^W$ and $\mathfrak{t}$ equals the union of its proper rational children. In particular, $\mathfrak{t}$ has two or more children in Σ_C^{rat}, so $b_\mathfrak{t} = 1$, by Lemma 3.18. But then $\tilde{\mathfrak{t}}$ is the set of odd children of $\mathfrak{t}$ as $\epsilon_\mathfrak{t} \in 2\mathbb{Z}$, and so all rational children of $\mathfrak{t}$ are even.

It only remains to prove that if $h(x) \notin k$, then the genus of $\Gamma_\mathfrak{t}$ is $g(\mathfrak{t})$. Since $h(X)$ is a separable polynomial, we need to show that

$$\deg h = \frac{|\mathfrak{t}| - \sum_{\mathfrak{s} \in \Sigma_C^{\mathrm{rat}},\, \mathfrak{s} < \mathfrak{t}} |\mathfrak{s}|}{b_\mathfrak{t}} + \left|\tilde{\mathfrak{t}}\right|.$$

It suffices to prove that if $\mathfrak{s} \in \Sigma_C^{\mathrm{rat}}$ is a non-proper rational child of $\mathfrak{t}$ different from $\{w_h\}$, then $b_\mathfrak{t} = 1$ and $\mathfrak{s} \in \tilde{\mathfrak{t}}$. Suppose $\mathfrak{s} = \{r\}$ is such a rational cluster. Since $r \in \mathfrak{t}$, we have $v(r - w_h) \geq \rho_\mathfrak{t}$. Suppose $v(r - w_h) > \rho_\mathfrak{t}$. Then $\mathfrak{s} \in \Sigma_C^{w_h}$, as $\mathfrak{s} < \mathfrak{t}$ and $r \neq w_h$. But this contradicts our choice of W. Then $\rho_\mathfrak{t} = v(r - w_h) \in \mathbb{Z}$ and so $b_\mathfrak{t} = 1$. It follows that $\tilde{\mathfrak{t}}$ is the set of odd children of $\mathfrak{t}$. Thus $\mathfrak{s} \in \tilde{\mathfrak{t}}$. □

5.11. Minimal regular NC model

Suppose the base extended curve $C_{K^{nr}}$ is y-regular and has an almost rational cluster picture. Consider the model $\mathcal{C}/O_{K^{nr}}$ constructed before with $\Sigma = \Sigma_{C_{K^{nr}}}^{\min}$. We want to see what components of $\mathcal{C}_s$ should be blown down to obtain the minimal regular model with normal crossings. Recall [1, §5]. Let $\Sigma_{K^{nr}} = \Sigma_{C_{K^{nr}}}^{\mathrm{rat}}$ and fix a proper cluster $\mathfrak{t} \in \Sigma_{C_{K^{nr}}}^{w_h}$.

Suppose first $\mathfrak{t} \neq \mathfrak{s}_h \wedge \mathfrak{s}_l$ for all $l = 1, \ldots, m$ with $l \neq h$. Equivalently, $\mathfrak{t}$ has at most one proper child in $\Sigma_{K^{nr}}$. Then $\Gamma_\mathfrak{t} \simeq \bar{X}_{F_\mathfrak{t}^{w_h}}$ and can be seen entirely in $\mathring{\mathcal{C}}_\Delta^{w_h}$. In particular, if $\Gamma_\mathfrak{t}$ can be blown down then $F_\mathfrak{t}^{w_h}$ is a removable or contractible v-face (see [1, Theorem 5.7]). By Lemma 4.3, we find

- $F_{\mathfrak{t}}^{w_h}$ is removable if and only if $\mathfrak{t} = \mathfrak{R}$ with a child in $\Sigma_{K^{nr}}$ of size $2g+1$.
- $F_{\mathfrak{t}}^{w_h}$ is contractible if and only if either $|\mathfrak{t}| = 2$ and $\frac{\epsilon_{\mathfrak{t}}}{2} - \rho_{\mathfrak{t}} \in \mathbb{Z}$ or $\mathfrak{t}$ has a proper rational child $\mathfrak{s} \in \Sigma_{K^{nr}}$, of size $2g$, and $\frac{\epsilon_{\mathfrak{t}}}{2} - g\rho_{\mathfrak{t}} \in \mathbb{Z}$.

Recall Definition 4.20. Note that $F_{\mathfrak{t}}^{w_h}$ is removable if and only if $\mathfrak{t}$ is removable. In this case, $F_{\mathfrak{t}}^{w_h}$ can be ignored for the construction of $\mathcal{C}_\Delta^{w_h}$ (for any h since $\mathfrak{t} = \mathfrak{R}$), and so $\mathfrak{t}$ can be ignored for the construction of $\mathcal{C}$.

Assume now $F_{\mathfrak{t}}^{w_h}$ contractible. We want to understand when $\Gamma_{\mathfrak{t}}$ can be blown down. First consider the case $|\mathfrak{t}| = 2$ and $\frac{\epsilon_{\mathfrak{t}}}{2} - \rho_{\mathfrak{t}} \in \mathbb{Z}$. Then $\Gamma_{\mathfrak{t}}$ intersects other components of $\mathcal{C}_s$ in 2 points (as $V_{\mathfrak{t}}^{w_h}$ gives two chains of $\mathbb{P}^1$s and the v-edges $V_0^{w_h}$ and $L_{\mathfrak{t}}^{w_h}$ give no component in $\mathcal{C}_{\Delta,s}^{w_h}$). To have self-intersection -1, $\Gamma_{\mathfrak{t}}$ has to have multiplicity > 1. It follows from Lemma 5.16 that $\rho_{\mathfrak{t}} \notin \mathbb{Z}$, as $\frac{\epsilon_{\mathfrak{t}}}{2} - \rho_{\mathfrak{t}} \in \mathbb{Z}$. Moreover, by Lemma 3.12, one has $\rho_{\mathfrak{t}} \in \frac{1}{2}\mathbb{Z}$. Therefore $\epsilon_{\mathfrak{t}}$ is odd and the multiplicity of $\Gamma_{\mathfrak{t}}$ is 2. Let $r := r_{V_{\mathfrak{t}}^{w_h}}$ and consider

$$\gamma_{\mathfrak{t}} s_{\mathfrak{t}} = \frac{n_0}{d_0} > \frac{n_1}{d_1} > \ldots > \frac{n_r}{d_r} > \frac{n_{r+1}}{d_{r+1}} = \gamma_{\mathfrak{t}} \left(s_{\mathfrak{t}} - \rho_{\mathfrak{t}} + \rho_{P(\mathfrak{t})}\right)$$

given by $V_{\mathfrak{t}}^{w_h}$. If $\Gamma_{\mathfrak{t}}$ can be blown down then $d_1 = 1$. Since $\gamma_{\mathfrak{t}} s_{\mathfrak{t}} = -\frac{\epsilon_{\mathfrak{t}}}{2} + 2\rho_{\mathfrak{t}}$, we have $d_0 = 2$. In particular $d_1 = 1$ if and only if $\rho_{\mathfrak{t}} - \rho_{P(\mathfrak{t})} = \frac{n_0}{d_0} - \frac{n_{r+1}}{d_{r+1}} \geq \frac{1}{2}$ (see also [1, Remark 3.15]). Thus if $|\mathfrak{t}| = 2$, then $\Gamma_{\mathfrak{t}}$ can be blown down if and only if $\rho_{\mathfrak{t}} \notin \mathbb{Z}$, $\epsilon_{\mathfrak{t}}$ odd, $\rho_{P(\mathfrak{t})} \leq \rho_{\mathfrak{t}} - \frac{1}{2}$. Note that this is case (1) of Definition 4.20.

Second consider the case $|\mathfrak{t}| = 2g+2$ with a proper rational child $\mathfrak{s}$ of size $2g$ and $\frac{\epsilon_{\mathfrak{t}}}{2} - g\rho_{\mathfrak{t}} \in \mathbb{Z}$. The argument is very similar to the previous one. If $\Gamma_{\mathfrak{t}}$ can be blown down then it must have multiplicity > 1 and this implies $\rho_{\mathfrak{t}} \notin \mathbb{Z}$ again by Lemma 5.16. From Lemma 3.12 it follows that $(|\mathfrak{t}| - |\mathfrak{s}|)\rho_{\mathfrak{t}} \in \mathbb{Z}$, so $\rho_{\mathfrak{t}} \in \frac{1}{2}\mathbb{Z}$. Then $m_{\mathfrak{t}} = 2$ and

$$\frac{v(c_f)}{2} = \frac{\epsilon_{\mathfrak{t}}}{2} - (g+1)\rho_{\mathfrak{t}} \in \tfrac{1}{2}\mathbb{Z} \setminus \mathbb{Z},$$

so $v(c_f)$ odd. Let $r := r_{V_{\mathfrak{s}}^{w_h}}$ and consider

$$\gamma_{\mathfrak{s}} s_{\mathfrak{s}} = \frac{n_0}{d_0} > \frac{n_1}{d_1} > \ldots > \frac{n_r}{d_r} > \frac{n_{r+1}}{d_{r+1}} = \gamma_{\mathfrak{s}}(s_{\mathfrak{s}} - \rho_{\mathfrak{s}} + \rho_{\mathfrak{t}})$$

given by $V_{\mathfrak{s}}^{w_h}$. If $\Gamma_{\mathfrak{t}}$ can be blown down then $d_r = 1$. Recall that $\epsilon_{\mathfrak{s}} - |\mathfrak{s}|\rho_{\mathfrak{s}} = \epsilon_{\mathfrak{t}} - |\mathfrak{s}|\rho_{\mathfrak{t}}$. Then $\gamma_{\mathfrak{s}}(s_{\mathfrak{s}} - \rho_{\mathfrak{s}} + \rho_{\mathfrak{t}}) = -\frac{\epsilon_{\mathfrak{t}}}{2} + (g+1)\rho_{\mathfrak{t}}$, so $d_{r+1} = 2$. In particular $d_r = 1$ if and only if $\rho_{\mathfrak{s}} - \rho_{\mathfrak{t}} = \frac{n_0}{d_0} - \frac{n_{r+1}}{d_{r+1}} \geq \frac{1}{2}$. Thus if $\mathfrak{t}$ has size $2g+2$ and has a unique proper rational child $\mathfrak{s} \in \Sigma_{K^{nr}}$, then $\Gamma_{\mathfrak{t}}$ can be blown down if and only if $|\mathfrak{s}| = 2g$, $\rho_{\mathfrak{t}} \notin \mathbb{Z}$, $v(c_f)$ odd, $\rho_{\mathfrak{s}} \geq \rho_{\mathfrak{t}} + \frac{1}{2}$. This is case (2) of Definition 4.20.

Finally, if $|\mathfrak{t}| = 2g+1$, $\mathfrak{t}$ has a proper child $\mathfrak{s} \in \Sigma_{K^{nr}}$ of size $2g$ and $\frac{\epsilon_{\mathfrak{t}}}{2} - g\rho_{\mathfrak{t}} \in \mathbb{Z}$, then $\rho_{\mathfrak{t}} \in \mathbb{Z}$, as $(|\mathfrak{t}| - |\mathfrak{s}|)\rho_{\mathfrak{t}} \in \mathbb{Z}$. It follows that $\epsilon_{\mathfrak{t}} \in \mathbb{Z}$ and so $m_{\mathfrak{t}} = 1$. This implies the self-intersection of $\Gamma_{\mathfrak{t}}$ is not -1, since it intersects the rest of $\mathcal{C}_{\mathfrak{t}}$ in at least two points as before. Hence in this case $\Gamma_{\mathfrak{t}}$ can never be blown down.

Now assume there exists $l \neq h$ such that $\mathfrak{t} = \mathfrak{s}_h \wedge \mathfrak{s}_l$. Then $\mathfrak{t}$ is not minimal. Let $\mathfrak{t}_h, \mathfrak{t}_l \in \Sigma_{K^{nr}}$ be such that $\mathfrak{s}_h \subseteq \mathfrak{t}_h < \mathfrak{t}$ and $\mathfrak{s}_l \subseteq \mathfrak{t}_l < \mathfrak{t}$. Suppose $\Gamma_{\mathfrak{t}}$ irreducible. If $|\mathfrak{t}| \leq 2g$ (or, equivalently, $\mathfrak{t}$ is not the largest non-removable cluster), then $\Gamma_{\mathfrak{t}}$ intersects at least other 3 components of $\mathcal{C}_s$ (given by $\mathfrak{t}_h$, $\mathfrak{t}_l$, and $P(\mathfrak{t})$). So it cannot be contracted to obtain a model with normal crossings. A similar argument holds if there exists $o \neq l$ such that $\mathfrak{s}_o \wedge \mathfrak{s}_h = \mathfrak{t}$: at least 3 components (given by $\mathfrak{t}_h$, $\mathfrak{t}_l$ and $\mathfrak{t}_o$) intersect $\Gamma_{\mathfrak{t}}$, so blowing down $\Gamma_{\mathfrak{t}}$ would make the model lose normal crossings. Assume then $|\mathfrak{t}| > 2g$ and $\mathfrak{s}_o \wedge \mathfrak{s}_h \neq \mathfrak{t}$ for all $o \neq l$. Then $\Gamma_{\mathfrak{t}}$ intersects at least other 2 components of $\mathcal{C}_s$ given by $V_{\mathfrak{t}_h}^{w_h}$ and $V_{\mathfrak{t}_l}^{w_l}$. Firstly, if $\Gamma_{\mathfrak{t}}$ can be blown down, then $m_{\mathfrak{t}} > 1$. But $\rho_{\mathfrak{t}} = \rho_{hl} \in \mathbb{Z}$. Then $m_{\mathfrak{t}}$ is at most 2. If $m_{\mathfrak{t}} = 2$ then $D_{\mathfrak{t}} = 1$, that implies $\epsilon_{\mathfrak{t}}$ odd and $\Gamma_{\mathfrak{t}} \simeq \mathbb{P}^1$ by Proposition 5.21. It also follows $s_{\mathfrak{t}} \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. If $\mathfrak{t}$ is odd then this implies that $V_{\mathfrak{t}}^{w_h}$ gives a $\mathbb{P}^1$ intersecting $\Gamma_{\mathfrak{t}}$. Since that would be a third component intersecting $\Gamma_{\mathfrak{t}}$, the cluster $\mathfrak{t}$ has to be even. Hence $\mathfrak{t} = \mathfrak{R}$ and $|\mathfrak{t}| = 2g+2$. Then $\epsilon_{\mathfrak{t}}$ is odd if and only if $v(c_f)$ is odd, as $\rho_{\mathfrak{t}} \in \mathbb{Z}$. Now, $L_{\mathfrak{t}}^{w_h}$ gives some $\mathbb{P}^1$s intersecting $\bar{X}_{F_{\mathfrak{t}}^{w_h}} \subset \mathcal{C}_{\Delta,s}^{w_h}$. All these $\mathbb{P}^1$s are not in $\mathring{\mathcal{C}}_{\Delta,s}^{w_h}$ (and so in $\mathcal{C}_s$) if and only if $\mathfrak{t}_h \cup \mathfrak{t}_l = \mathfrak{t}$. In particular, $\mathfrak{t}_h$ and $\mathfrak{t}_l$ are either both even or both odd. If $\mathfrak{t}_h$ is even, then $\gamma_{\mathfrak{t}_h} = 2$, and so the component given by $V_{\mathfrak{t}_h}^{w_h}$ has

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

multiplicity at least 2. The self-intersection of $\Gamma_{\mathfrak{t}}$ could not be -1 in this case. Assume $\mathfrak{t}_h$ is odd. Let $r := r_{V_{\mathfrak{t}_h}^{w_h}}$ and consider

$$\gamma_{\mathfrak{t}_h} s_{\mathfrak{t}_h} = \frac{n_0}{d_0} > \frac{n_1}{d_1} > \ldots > \frac{n_r}{d_r} > \frac{n_{r+1}}{d_{r+1}} = \gamma_{\mathfrak{t}_h}\left(s_{\mathfrak{t}_h} - \frac{\rho_{\mathfrak{t}_h} - \rho_{\mathfrak{t}}}{2}\right)$$

given by $V_{\mathfrak{t}_h}^{w_h}$. We want $d_r = 1$. Since

$$\gamma_{\mathfrak{t}_h}\left(s_{\mathfrak{t}_h} - \frac{\rho_{\mathfrak{t}_h} - \rho_{\mathfrak{t}}}{2}\right) = -\frac{\epsilon_{\mathfrak{t}}}{2} + \frac{|\mathfrak{t}_h| - 1}{2}\rho_{\mathfrak{t}} \in \tfrac{1}{2}\mathbb{Z} \smallsetminus \mathbb{Z},$$

we have $d_{r+1} = 2$. As before $d_r = 1$ if and only if $\frac{\rho_{\mathfrak{t}_h} - \rho_{\mathfrak{t}}}{2} = \frac{n_0}{d_0} - \frac{n_{r+1}}{d_{r+1}} \geq \frac{1}{2}$ and similarly for $\mathfrak{t}_l$. Thus if $\mathfrak{t}$ has two or more rational children and $\Gamma_{\mathfrak{t}}$ is irreducible then it can be blown down if and only if $v(c_f)$ is odd and $\mathfrak{t} = \mathfrak{R}$ is union of its 2 odd rational children $\mathfrak{t}_h$ and $\mathfrak{t}_l$, satisfying $\rho_{\mathfrak{t}_h} \geq \rho_{\mathfrak{t}} + 1$, $\rho_{\mathfrak{t}_l} \geq \rho_{\mathfrak{t}} + 1$. This is case (3) of Definition 4.20.

Suppose now $\Gamma_{\mathfrak{t}}$ reducible. By Proposition 5.21 the cluster $\mathfrak{t}$ is übereven, $\epsilon_{\mathfrak{t}}$ is even and $\Gamma_{\mathfrak{t}}$ is the disjoint union of $\Gamma_{\mathfrak{t}}^- \simeq \mathbb{P}^1$ and $\Gamma_{\mathfrak{t}}^+ \simeq \mathbb{P}^1$. As before, both $\Gamma_{\mathfrak{t}}^-$ and $\Gamma_{\mathfrak{t}}^+$ intersect at least other two components (given by the proper children of $\mathfrak{t}$). But then neither $\Gamma_{\mathfrak{t}}^-$ nor $\Gamma_{\mathfrak{t}}^+$ has self-intersection -1, as $m_{\mathfrak{t}} = 1$.

We have showed that, for a rational cluster $\mathfrak{t} \in \Sigma_{K^{nr}}$, an irreducible component of $\Gamma_{\mathfrak{t}}$ can be blown down if and only if $\mathfrak{t}$ is contractible. Moreover, in this case, $\Gamma_{\mathfrak{t}}$ is irreducible. It remains to show that after blowing down all components $\Gamma_{\mathfrak{t}}$ where $\mathfrak{t}$ is a contractible cluster, no other component can be blown down. First note that if $\mathfrak{t}$ is a contractible cluster, then $m_{\mathfrak{t}} = 2$ and $\Gamma_{\mathfrak{t}}$ intersects one or two other components of multiplicity 1 at two points in total. If it intersects only one component, then after the blowing down, the latter will have a node and will not be isomorphic to $\mathbb{P}^1$. If $\Gamma_{\mathfrak{t}}$ intersects two components and those intersect something else in $\mathcal{C}_s$, then they will not have self-intersection -1 also when $\Gamma_{\mathfrak{t}}$ is blown down. Therefore suppose that one of those two does not intersect any other component of $\mathcal{C}_s$. If we are in case (1) or case (2), it is easy to see that this never happens. Indeed, in those cases, $\Gamma_{\mathfrak{t}}$ intersects non-open-ended chains of $\mathbb{P}^1$s. Then without loss of generality assume to be in case (3) and that $\Gamma_{\mathfrak{t}_h}$ is the component that can be blown down once $\Gamma_{\mathfrak{t}}$ has been contracted. This implies $\mathfrak{s}_h = \mathfrak{t}_h$ and $\rho_{\mathfrak{s}_h} = \rho_{\mathfrak{t}} + 1$. Then $b_{\mathfrak{s}_h} = 1$ and $\epsilon_{\mathfrak{s}_h} = \epsilon_{\mathfrak{t}} + |\mathfrak{s}_h|$. Since both $\epsilon_{\mathfrak{t}}$ and $\mathfrak{s}_h$ are odd, we have $\epsilon_{\mathfrak{s}_h} \in 2\mathbb{Z}$. So $D_{\mathfrak{s}_h} = 2$ and $\tilde{\mathfrak{s}}_h$ is the set of rational children of $\mathfrak{s}_h$. Hence $g(\mathfrak{s}_h) = \left\lfloor \frac{|\mathfrak{s}_h| - 1}{2} \right\rfloor \geq 1$ since $|\mathfrak{s}_h| \geq 3$. But then $\Gamma_{\mathfrak{s}_h}$ cannot be blown down.

5.12. Galois action

Consider the base extended hyperelliptic curve $C_{K^{nr}}/K^{nr}$. The rational clusters of $C_{K^{nr}}$ and their corresponding rational centres are then over K^{nr}. Denote $\Sigma_{K^{nr}} = \Sigma_{C_{K^{nr}}}^{\mathrm{rat}}$. For any proper cluster $\mathfrak{s} \in \Sigma_{K^{nr}}$, let $G_{\mathfrak{s}} = \mathrm{Stab}_{G_K}(\mathfrak{s})$, $K_{\mathfrak{s}} = (K^{\mathrm{s}})^{G_{\mathfrak{s}}}$ and $k_{\mathfrak{s}}$ be the residue field of $K_{\mathfrak{s}}$. Let $\Sigma_{C_{K^{nr}}}^{\min} = \{\mathfrak{s}_1, \ldots, \mathfrak{s}_m\}$ be the set of rationally minimal clusters of $C_{K^{nr}}$. Fix a set $W = \{w_1, \ldots, w_m\} \subset K^{nr}$ of corresponding rational centres. By Lemma A.1, we can assume this choice to be G_K-equivariant, that is for any $\sigma \in G_K$, one has $\sigma(w_l) = w_h$ if and only if $\sigma(\mathfrak{s}_l) = \mathfrak{s}_h$. We can also require that $w_h \in \mathfrak{s}_h$ if $\mathfrak{s}_h \cap K_{\mathfrak{s}_h} \neq \varnothing$. Similarly, for any proper cluster $\mathfrak{t} \in \Sigma_{K^{nr}} \setminus \Sigma_{C_{K^{nr}}}^{\min}$, fix a rational centre $w_{\mathfrak{t}}$ in such a way that $w_{\sigma(\mathfrak{t})} = \sigma(w_{\mathfrak{t}})$ for any $\sigma \in G_K$. Set $w_{\mathfrak{s}_o} := w_o$ for any $o = 1, \ldots, m$.

Lemma 5.22. *With the choices above, for any $h = 1, \ldots, m$, the set of proper clusters in $\Sigma_{C_{K^{nr}}}^{w_h}$ coincides with $\mathring{\Sigma}_{C_{K^{nr}}}^{w_h}$.*

Proof. Suppose by contradiction that there exists a non-proper cluster $\{r\} = \mathfrak{s} \in \Sigma_{C_{K^{nr}}}^{w_h}$, with $r \neq w_h$. Note that $r \in \mathfrak{s}_h$ and so $\mathfrak{s} < \mathfrak{s}_h$. Recall that since $\mathfrak{s}$ is a cluster centred at w_h, it is cut out by the disc $\mathcal{D} = \{x \in \bar{K} \mid v(x - w_h) \geq \rho_{\mathfrak{s}}^{w_h}\}$, with $\rho_{\mathfrak{s}}^{w_h} = v(r - w_h) > \rho_{\mathfrak{s}_h}$. This implies that $w_h \notin \mathfrak{R}$, otherwise $w_h \in \mathfrak{s}$

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

and $|\mathfrak{s}| \geq 2$. In particular, $w_h \notin \mathfrak{s}_h$. For our choice of w_h, it follows that $\mathfrak{s}_h \cap K_{\mathfrak{s}_h} = \varnothing$. Therefore $r \notin K_{\mathfrak{s}_h}$ and so there exists $\sigma \in G_{\mathfrak{s}_h}$ such that $\sigma(r) \neq r$. Since $w_h \in K_{\mathfrak{s}_h}$ we have

$$v(\sigma(r) - w_h) = v(\sigma(r - w_h)) = v(r - w_h) = \rho_{\mathfrak{s}}^{w_h}.$$

But then $\sigma(r) \in \mathfrak{s}$, and so $|\mathfrak{s}| \geq 2$, a contradiction. □

Assume that $C_{K^{nr}}$ is y-regular and has an almost rational cluster picture. By the previous lemma, from the set of rational centres W we can construct the proper regular model $\mathcal{C}/O_{K^{nr}}$ of $C_{K^{nr}}$ as previously presented in this section. In this subsection we show how the Galois group $\mathrm{Gal}(K^{nr}/K)$ acts on the $O_{K^{nr}}$-scheme $\mathcal{C}$. Moreover, we describe the induced action of G_k on the special fibre $\mathcal{C}_s/k^s$, and give defining equations for the principal components of $\mathcal{C}_s$ compatibly with the action.

For any $l = 1, \ldots, m$, recall the notation $f_l(x) = f(x + w_l) \in K^{nr}[x]$ and $C^{w_l}/K^{nr} : y^2 = f_l(x)$. Fix $\sigma \in G_K$. Let $l, h = 1, \ldots, m$ such that $\sigma(\mathfrak{s}_l) = \mathfrak{s}_h$. Then $\sigma(f_l) = f_h$. Now, let $\mathfrak{t} \in \Sigma_{C_{K^{nr}}}^{w_l}$ be a proper cluster. Then $\sigma(\mathfrak{t}) \in \Sigma_{C_{K^{nr}}}^{w_h}$ and $\rho_{\mathfrak{t}} = \rho_{\sigma(\mathfrak{t})}$. It follows that most of the quantities attached to $\mathfrak{t}$, such as those of Definition 4.6, are the same for $\sigma(\mathfrak{t})$, for example $\epsilon_{\mathfrak{t}} = \epsilon_{\sigma(\mathfrak{t})}$. In particular, if M is a matrix associated to $\mathfrak{t}$ then M is associated to $\sigma(\mathfrak{t})$ as well. So $\sigma(\mathcal{F}_M^l) = \mathcal{F}_M^h$. Finally, as $\sigma(\prod_{o \neq l} (x + w_{lo})^{-1}) = \prod_{o \neq h} (x + w_{ho})^{-1}$ we also have $\sigma(T_M^l) = T_M^h$.

Hence the natural K^{nr}-isomorphism $C^{w_h} \xrightarrow{\sigma} C^{w_l}$ induces $O_{K^{nr}}$-isomorphisms of schemes

$$\mathcal{C}_{\Delta}^{w_h} \xrightarrow{\sigma} \mathcal{C}_{\Delta}^{w_l}, \qquad \mathring{\mathcal{C}}_{\Delta}^{w_h} \xrightarrow{\sigma} \mathring{\mathcal{C}}_{\Delta}^{w_l}, \qquad U_M^h \xrightarrow{\sigma} U_M^l. \tag{10}$$

Via the glueing morphisms (4), these maps describe the action of G_K on $\mathcal{C}$.

We want to study the action of G_k on the special fibre of $\mathcal{C}$ more in detail. Let $\sigma \in \mathrm{Gal}(K^{nr}/K)$ and let $\bar{\sigma} \in G_k$ corresponding to σ via the canonical isomorphism $\mathrm{Gal}(K^{nr}/K) \simeq G_k$. Let l, h and $\mathfrak{t}$ as above. In Section 5.6 we described closed 1-dimensional subschemes composing $\mathring{\mathcal{C}}_{\Delta,s}^{w_l}$ and the morphisms induced by the glueing maps. Recall the polynomials introduced in Definition 5.12. From (10) we get

$$\bar{\sigma}(\overline{g_{\mathfrak{s}_l,w_l}^0}) = \overline{g_{\mathfrak{s}_h,w_h}^0}, \quad \bar{\sigma}(\overline{g_{\mathfrak{t},w_l}}) = \overline{g_{\sigma(\mathfrak{t}),w_h}}, \quad \bar{\sigma}(\overline{g_l|_{L_{\mathfrak{t}}^{w_l}}}) = \overline{g_h|_{L_{\sigma(\mathfrak{t})}^{w_h}}}.$$

From the equality (7) we obtain $\bar{\sigma}(f_{\mathfrak{t},w_l}) = f_{\sigma(\mathfrak{t}),w_h}$. Note that the previous relations can also be recovered directly from the definitions.

Lemma 5.23. *Let $w_{\mathfrak{t}}$ be the rational centre of $\mathfrak{t}$ fixed above. Then*

(i) $\overline{g_{\mathfrak{t},w_{\mathfrak{t}}}}, \overline{f_{\mathfrak{t},w_{\mathfrak{t}}}} \in k_{\mathfrak{t}}[X]$;
(ii) $\overline{g_{\mathfrak{t},w_{\mathfrak{t}}}} = \overline{g_{\mathfrak{t},w_l}}$ *and* $\overline{f_{\mathfrak{t},w_{\mathfrak{t}}}}(X) = \overline{f_{\mathfrak{t},w_l}}(X + \overline{u_{w_{\mathfrak{t}}w_l}})$ *where* $\overline{u_{w_{\mathfrak{t}}w_l}} = \frac{w_{\mathfrak{t}} - w_l}{\pi^{\rho_{\mathfrak{t}}}} \mod \pi$;

Proof. For any rational centre w of $\mathfrak{t}$, let $u_{\mathfrak{t},w} = c_f \prod_{r \in \mathfrak{R} \setminus \mathfrak{t}} (w - r)$ as in Definition 5.12. Note that $u_{\mathfrak{t},w}/\pi^{v(u_{\mathfrak{t},w})}$ is independent of w since

$$v((w_{\mathfrak{t}} - r) - (w - r)) = v(w_{\mathfrak{t}} - w) \geq \rho_{\mathfrak{t}} > v(w_{\mathfrak{t}} - r)$$

for any $r \in \mathfrak{R} \setminus \mathfrak{t}$. Then $\overline{g_{\mathfrak{t},w_{\mathfrak{t}}}} = \overline{g_{\mathfrak{t},w_l}}$. If $\bar{\sigma} \in \mathrm{Gal}(k^s/k_{\mathfrak{t}})$, i.e. $\sigma \in \mathrm{Gal}(K^{nr}/K_{\mathfrak{t}})$, then

$$\bar{\sigma}(\overline{g_{\mathfrak{t},w_{\mathfrak{t}}}}) = \bar{\sigma}(\overline{g_{\mathfrak{t},w_l}}) = \overline{g_{\mathfrak{t},w_h}} = \overline{g_{\mathfrak{t},w_{\mathfrak{t}}}}.$$

In particular $\overline{g_{\mathfrak{t},w_{\mathfrak{t}}}} \in k_{\mathfrak{t}}[X]$.

Since $u_{\mathfrak{t},w}/\pi^{v(u_{\mathfrak{t},w})}$ is independent of w we also have

$$\overline{f_{\mathfrak{t},w_{\mathfrak{t}}}}(X^{b_{\mathfrak{t}}}) = \overline{f_{\mathfrak{t},w_l}}((X + \overline{u_{w_{\mathfrak{t}}w_l}})^{b_{\mathfrak{t}}}).$$

Suppose $\rho_{\mathfrak{t}} \in \mathbb{Z}$. Then $b_{\mathfrak{t}} = 1$ and so the equality above implies $\overline{f_{\mathfrak{t},w_{\mathfrak{t}}}}(X) = \overline{f_{\mathfrak{t},w_l}}(X + \overline{u_{w_{\mathfrak{t}}w_l}})$. Suppose $\rho \notin \mathbb{Z}$. Then $v(w - w_{\mathfrak{t}}) > \rho_{\mathfrak{t}}$ for any rational centre w of $\mathfrak{t}$ as $v(w - w_{\mathfrak{t}}) \in \mathbb{Z}$ and $v(w - w_{\mathfrak{t}}) \geq \rho_{\mathfrak{t}}$. Hence $\overline{u_{w_{\mathfrak{t}}w_l}} = 0$. Thus $\overline{f_{\mathfrak{t},w_{\mathfrak{t}}}}(X^{b_{\mathfrak{t}}}) = \overline{f_{\mathfrak{t},w_l}}(X^{b_{\mathfrak{t}}})$, which implies $\overline{f_{\mathfrak{t},w_{\mathfrak{t}}}}(X) = \overline{f_{\mathfrak{t},w_l}}(X) = \overline{f_{\mathfrak{t},w_l}}(X + \overline{u_{w_{\mathfrak{t}}w_l}})$. If $\bar{\sigma} \in \mathrm{Gal}(k^s/k_{\mathfrak{t}})$, i.e. $\sigma \in$

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

$\mathrm{Gal}(K^{nr}/K_{\mathfrak{t}})$, then

$$\bar{\sigma}(\overline{f_{\mathfrak{t},w_{\mathfrak{t}}}})(X)=\bar{\sigma}(\overline{f_{\mathfrak{t},w_l}})(X+\bar{\sigma}(\overline{u_{w_{\mathfrak{t}}w_l}}))=\overline{f_{\mathfrak{t},w_h}}(X+\overline{u_{w_{\mathfrak{t}}w_h}})=\overline{f_{\mathfrak{t},w_{\mathfrak{t}}}}(X),$$

and so $\overline{f_{\mathfrak{t},w_{\mathfrak{t}}}}\in k_{\mathfrak{t}}[X]$. □

Remark 5.24. *Note that the polynomials $\overline{f_{\mathfrak{t},w_{\mathfrak{t}}}}$, $\overline{g_{\mathfrak{t},w_{\mathfrak{t}}}}$ and $\overline{g^0_{\mathfrak{s}_h,w_h}}$ equal the polynomials $\overline{f_{\mathfrak{t}}}$, $\overline{g_{\mathfrak{t}}}$ and $\overline{g^0_{\mathfrak{s}_h}}$ given in Definition 4.22.*

Let $V=V_{\mathfrak{t}}^{w_l}$ and consider the subscheme $X_V\times\mathbb{P}_V$ of $\mathcal{C}_s$ given by V, where $\mathbb{P}_V$ is a chain of $\mathbb{P}^1$s and $X_V:\{\overline{g_{\mathfrak{t},w_l}}=0\}$ over $\mathbb{G}_{m,k^s}$. If $\mathfrak{s}_h\subset\mathfrak{t}$, then the glueing map $U_M^h\to U_M^l$ induces the identity $\phi_V^{hl}:X_{V_{\mathfrak{t}}^{w_h}}\xrightarrow{=}X_{V_{\mathfrak{t}}^{w_l}}$. Define $X_{\mathfrak{t}}\subseteq\mathbb{G}_{m,k^s}$ given by $g_{\mathfrak{t},w_{\mathfrak{t}}}=0$. By Lemma 5.23, $\phi_V^o:X_{\mathfrak{t}}\xrightarrow{\simeq}X_{V_{\mathfrak{t}}^{w_o}}$, for $o=h,l$, and this isomorphism is compatible with the Galois action and the glueing maps, that is $\sigma\circ\phi_V^h=\phi_V^l\circ\sigma$ and $\phi_V^{hl}\circ\phi_V^h=\phi_V^l$ as morphisms on $X_{\mathfrak{t}}$.

When $V_0=V_0^{w_l}$ we can consider the subscheme $X_{V_0}\times\mathbb{P}_{V_0}$ given by V_0, where $\mathbb{P}_{V_0}$ is a chain of $\mathbb{P}^1$s and $X_{V_0}:\{\overline{g_{\mathfrak{s}_l,w_l}}=0\}$ over $\mathbb{G}_{m,k^s}$. Since $X_{V_0}\times\mathbb{P}_{V_0}$ is never glued to any other component there is no need to choose a different model for it.

Let $L=L_{\mathfrak{t}}^{w_l}$ and consider the subscheme $X_L^W\times\mathbb{P}_L$ given by L, where $\mathbb{P}_L$ is a chain of $\mathbb{P}^1$s and $X_L^W:\{\overline{f_{\mathfrak{t},w_l}}=0\}$ over $\mathbb{A}^1_{k^s}$. If $\mathfrak{s}_h\subset\mathfrak{t}$, then the isomorphism $\phi_L^{hl}:X^W_{L_{\mathfrak{t}}^{w_h}}\xrightarrow{\simeq}X^W_{L_{\mathfrak{t}}^{w_l}}$ given by the glueing map $U_M^h\to U_M^l$ is induced by the ring isomorphism $k^s[X]\to k^s[X]$, sending $X\mapsto X+\overline{u_{w_hw_l}}$, where $\overline{u_{w_hw_l}}=\frac{w_h-w_l}{\pi^{\rho_{\mathfrak{t}}}}\mod\pi$. Define $X_{\mathfrak{t}}^W\subseteq\mathbb{A}^1_{k^s}$ given by $\overline{f_{\mathfrak{t},w_{\mathfrak{t}}}}=0$. By Lemma 5.23, the map $X\mapsto X+\overline{u_{w_{\mathfrak{t}}w_l}}$ induces an isomorphism $\phi_L^o:X_{\mathfrak{t}}^W\xrightarrow{\simeq}X^W_{L_{\mathfrak{t}}^{w_o}}$, for $o=h,l$, compatible with the Galois action and the glueing morphisms, that is $\sigma\circ\phi_L^h=\phi_L^l\circ\sigma$ and $\phi_L^{hl}\circ\phi_L^h=\phi_L^l$ as morphisms on $X_{\mathfrak{t}}^W$.

Recall the definitions of $\hat{\mathfrak{t}}^W$ and $\mathbb{G}_{\mathfrak{t},w_l}\subseteq\mathbb{A}^1_{k^s}$ given in Definition 5.15 and the definition of $\mathfrak{t}$ given in Definition 4.22. Note that by Lemma 5.22,

$$\hat{\mathfrak{t}}^W=\{\mathfrak{s}\in\Sigma_{K^{nr}}\cup\{\varnothing\}\mid\mathfrak{s}<\mathfrak{t}\}\setminus\{\{r\}\in\Sigma_{K^{nr}}\mid r\notin W\}.$$

Fix $c=0,\dots,b_{\mathfrak{t}}-1$ such that $1/b_{\mathfrak{t}}-c\rho_{\mathfrak{t}}\in\mathbb{Z}$. For any rational centre $w\in K^{nr}$ of $\mathfrak{t}$ define $\hat{f}_{\mathfrak{t},w}\in k^s[X,Y]$ by

$$\hat{f}_{\mathfrak{t},w}(X)=\prod_{\mathfrak{s}\in\hat{\mathfrak{t}}^W}(X-\overline{u_{w_{\mathfrak{s}}w}})^{\frac{|\mathfrak{s}|}{b_{\mathfrak{t}}}-c\epsilon_{\mathfrak{t}}}\overline{f_{\mathfrak{t},w}}(X),$$

where $\overline{u_{w_{\mathfrak{s}}w}}=\frac{w_{\mathfrak{s}}-w}{\pi^{\rho_{\mathfrak{t}}}}\mod\pi$ ($w_{\mathfrak{s}}=w_l$ if $\mathfrak{s}=\varnothing$). Let $L=L_{\mathfrak{t}}^{w_l}$, $F=F_{\mathfrak{t}}^{w_l}$ and $M=M_{L,0}$. It follows from Lemma 5.17 that the scheme $\Gamma_{\mathfrak{t}}^{w_l}=X_F\cap U_M^l$ is given by $Y^{D_{\mathfrak{t}}}=\hat{f}_{\mathfrak{t},w_l}(X)$ as a subscheme of $\mathbb{G}_{\mathfrak{t},w_l}\times\mathbb{A}^1_{k^s}$. We then obtain $\bar{\sigma}(\hat{f}_{\mathfrak{t},w_l})=\hat{f}_{\sigma(\mathfrak{t}),w_h}$ from the action (10) of σ on U_M^l.

Lemma 5.25. *With the notation above,*

(i) $\hat{f}_{\mathfrak{t},w_{\mathfrak{t}}}\in k_{\mathfrak{t}}[X]$;
(ii) $\hat{f}_{\mathfrak{t},w_{\mathfrak{t}}}(X)=\hat{f}_{\mathfrak{t},w_l}(X+\overline{u_{w_{\mathfrak{t}}w_l}})$ *where* $\overline{u_{w_{\mathfrak{t}}w_l}}=\frac{w_{\mathfrak{t}}-w_l}{\pi^{\rho_{\mathfrak{t}}}}\mod\pi$;

Proof. If $\mathfrak{s}\in\mathfrak{t}$, then $\sigma(\mathfrak{s})\in(\sigma(\mathfrak{t}))$ and $\bar{\sigma}(\overline{u_{w_{\mathfrak{s}}w}})=\overline{u_{w_{\sigma(\mathfrak{s})}\sigma(w)}}$ for any rational centre w of $\mathfrak{t}$. Hence $\hat{f}_{\mathfrak{t},w_{\mathfrak{t}}}\in k_{\mathfrak{t}}[X]$ and $\bar{\sigma}(\hat{f}_{\mathfrak{t},w_l})=\hat{f}_{\sigma(\mathfrak{t}),w_h}$ by Lemma 5.23(i),(iii). Since $\overline{u_{w_{\mathfrak{s}}w_{\mathfrak{t}}}}=\overline{u_{w_{\mathfrak{s}}w_l}}-\overline{u_{w_{\mathfrak{t}}w_l}}$, Lemma 5.23(ii) implies $\hat{f}_{\mathfrak{t},w_{\mathfrak{t}}}(X)=\hat{f}_{\mathfrak{t},w_l}(X+\overline{u_{w_{\mathfrak{t}}w_l}})$. □

Define $\Gamma_{\mathfrak{t}}^{w_{\mathfrak{t}}}\subset\mathbb{G}_{\mathfrak{t},w_{\mathfrak{t}}}\times\mathbb{A}^1_{k^s}$ given by $Y^{D_{\mathfrak{t}}}=\hat{f}_{\mathfrak{t},w_{\mathfrak{t}}}$. Suppose $\mathfrak{s}_h\subset\mathfrak{t}$, and let $\phi_{\mathfrak{t}}^{hl}:\Gamma_{\mathfrak{t}}^{w_h}\simeq\Gamma_{\mathfrak{t}}^{w_l}$ be the isomorphism coming from the glueing map $U_M^h\to U_M^l$ induced by the ring homomorphism $X\mapsto X+\overline{u_{w_hw_l}}$. By Lemma 5.25, the map $X\mapsto X+\overline{u_{w_{\mathfrak{t}}w_l}}$ induces an isomorphism $\phi_{\mathfrak{t}}^o:\Gamma_{\mathfrak{t}}^{w_{\mathfrak{t}}}\simeq\Gamma_{\mathfrak{t}}^{w_o}$, for $o=h,l$, which is compatible with the Galois action and the glueing maps, that is $\sigma\circ\phi_{\mathfrak{t}}^h=\phi_{\mathfrak{t}}^l\circ\sigma$ and $\phi_{\mathfrak{t}}^{hl}\circ\phi_{\mathfrak{t}}^h=\phi_{\mathfrak{t}}^l$ as morphisms on $\Gamma_{\mathfrak{t}}^{w_{\mathfrak{t}}}$. Therefore $\Gamma_{\mathfrak{t}}$ is isomorphic to the smooth completion of $\Gamma_{\mathfrak{t}}^{w_{\mathfrak{t}}}$, and so it is given by $Y^{D_{\mathfrak{t}}} = \tilde{f}_{\mathfrak{t}}(X)$, where $\tilde{f}_{\mathfrak{t}}(X) = \prod_{\mathfrak{s}\in\mathfrak{t}} (X - \overline{u_{w_{\mathfrak{s}} w_{\mathfrak{t}}}}) \overline{f_{\mathfrak{t},w_{\mathfrak{t}}}}(X)$ is the polynomial given in Definition 4.22.

6. Integral differentials

Let C be a hyperelliptic curve of genus $g \geq 1$ defined over K by a Weierstrass equation $y^2 = f(x)$. It is well-known that the K-vector space of global sections of the sheaf of differentials of C, namely $H^0(C, \Omega^1_{C/K})$, is spanned by the basis

$$\underline{\omega} = \left\{ \frac{dx}{2y}, x\frac{dx}{2y}, \ldots, x^{g-1}\frac{dx}{2y} \right\}.$$

Let $\mathcal{C}$ be a regular model of C over O_K and consider its canonical (or dualising) sheaf $\omega_{\mathcal{C}/O_K}$. The free O_K-module of its global sections $H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K})$ can be viewed as an O_K-lattice in $H^0(C, \Omega^1_{C/K})$ (see [9, Corollary 9.2.25(a)]). The aim of this section is to present a basis of $H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K})$ as an O_K-linear combination of the elements in $\underline{\omega}$ under the assumptions of Theorem 4.23. Note that by [9, Corollary 9.2.25(b)] the problem is independent of the choice of model $\mathcal{C}$ but it does depend on the choice of the equation $y^2 = f(x)$ since the basis $\underline{\omega}$ does. Throughout this section let C and $\mathcal{C}/O_K$ be as above.

If C is Δ_v-regular, [1, Theorem 8.12] gives an O_K-basis of $H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K})$, as required. We rephrase it in terms of rational cluster invariants, by using results of Section 3 and Lemma 4.12.

Theorem 6.1. *Suppose C has an almost rational cluster picture and is y-regular, and all proper clusters $\mathfrak{s} \in \Sigma_C$ have same rational centre $w \in K$. Let $\mathfrak{s}_1 \subset \cdots \subset \mathfrak{s}_n = \mathfrak{R}$ be the proper clusters in Σ_C^{rat}. For every $j = 0, \ldots, g-1$, define*

$$i_j := \min\{i \in \{1, \ldots, n\} \mid 2(j+1) < |\mathfrak{s}_i|\}$$

and

$$e_j := \tfrac{1}{2}\epsilon_{\mathfrak{s}_{i_j}} - (j+1)\rho_{\mathfrak{s}_{i_j}}.$$

Then the differentials

$$\mu_j = \pi^{\lfloor e_j \rfloor}(x-w)^j \frac{dx}{2y} \qquad j = 0, \ldots, g-1,$$

form an O_K-basis of $H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K})$.

Proof. Let $C^w: y^2 = f(x+w)$ be the hyperelliptic curve isomorphic to C through the change of variable $y \mapsto y$, $x \mapsto x + w$. By Corollary 3.25 and Lemma 4.12, the curve C^w is Δ_v-regular. Since Σ_C^{rat} consists of the proper clusters in Σ_C^w, Lemma 4.3 and [1, Theorem 8.12] implies that

$$\mu_j = \pi^{\lfloor e_j \rfloor} x^j \frac{dx}{2y} \qquad j = 0, \ldots, g-1,$$

form an O_K-basis of $H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K})$ as a lattice in $H^0(C^w, \Omega^1_{C^w/K})$ (that is if $\mathcal{C}$ is regarded as a model of C^w). Changing variables concludes the proof. □

Suppose now C has an almost rational cluster picture and is y-regular. Let $\Sigma_C^{\min}$ be the set of rationally minimal clusters and let $W = \{w_{\mathfrak{s}} \mid \mathfrak{s} \in \Sigma_C^{\min}\}$ be a corresponding set of rational centres, such that all clusters in $\Sigma_C^{w_{\mathfrak{s}}}$ are proper. For every proper cluster $\mathfrak{t} \in \Sigma_C^{\mathrm{rat}}$, choose a minimal cluster $\mathfrak{s} \subseteq \mathfrak{t}$ and set $w_{\mathfrak{t}} := w_{\mathfrak{s}}$. Consider the regular model $\mathcal{C}/O_K$ of C of Theorem 4.18, constructed in Section 5 by glueing the open subschemes $\mathring{\mathcal{C}}_\Delta^w$ of $\mathcal{C}_\Delta^w$ for $w \in W$. We want to describe the canonical morphism $C \to \mathcal{C}$. Write $W = \{w_1, \ldots, w_m\}$ as in Section 5. For any $h = 1, \ldots, m$, let $\mathfrak{t} \in \Sigma_C^{w_h}$ be a proper cluster and let M be a matrix associated to $\mathfrak{t}$. Let $C^{w_h}: y^2 = f(x + w_h)$ and

$$y^2 - f(x+w_h) \overset{M}{=} Y^{n_Y} Z^{n_Z} \mathcal{F}_M^h(X, Y, Z).$$

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

Then, on the affine chart X_M the projection $C \to \mathcal{C}_\Delta^{w_h}$ is induced by

$$\frac{R}{\left(\mathcal{F}_M^h(X,Y,Z)\right)} \xrightarrow{M} \frac{K[(x')^{\pm 1},(y')^{\pm 1}]}{((y')^2 - f(x'+w_h))} \xrightarrow{\simeq} \frac{K[x^{\pm 1},y^{\pm 1}]}{(y^2 - f(x))},$$

where $(X,Y,Z) = (x',y',\pi) \bullet M$ and $(x',y') = (x-w_h,y)$. In particular it follows that $(X,Y,Z) = (x-w_h,y,z) \bullet M$ and so

$$\begin{pmatrix} x-w_h \\ y \\ \pi \end{pmatrix} = \begin{pmatrix} X^{\tilde{m}_{11}} Y^{\tilde{m}_{21}} Z^{\tilde{m}_{31}} \\ X^{\tilde{m}_{12}} Y^{\tilde{m}_{22}} Z^{\tilde{m}_{32}} \\ X^{\tilde{m}_{13}} Y^{\tilde{m}_{23}} Z^{\tilde{m}_{33}} \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \bullet M^{-1}.$$

For a proper cluster $\mathfrak{t} \in \Sigma_C^{\mathrm{rat}}$ recall the definitions of $\Gamma_\mathfrak{t}$ and $m_\mathfrak{t}$.

Proposition 6.2. *Let* $\mathfrak{t} \in \Sigma_C^{\mathrm{rat}}$ *be a proper cluster. Then*[5]

$$\begin{aligned} \operatorname{ord}_{\Gamma_\mathfrak{t}}(x-w_\mathfrak{s}) &= m_\mathfrak{t}\rho_\mathfrak{t}, \\ \operatorname{ord}_{\Gamma_\mathfrak{t}} \tfrac{dx}{2y} &= -m_\mathfrak{t}\left(\tfrac{1}{2}\epsilon_\mathfrak{t} - \rho_\mathfrak{t} - 1\right) - 1. \end{aligned}$$

for every proper cluster $\mathfrak{s} \in \Sigma_C^{\mathrm{rat}}$, $\mathfrak{s} \subseteq \mathfrak{t}$.

Proof. Let $g(x,y) = y^2 - f(x)$. Let $W = \{w_1,\ldots,w_m\}$ as above. Let $h \in \{1,\ldots,m\}$ such that $w_h = w_\mathfrak{s}$. Let $F = F_\mathfrak{t}^{w_h}$, $V = V_\mathfrak{t}^{w_h}$, $M = M_{V,0}$ and let X,Y,Z be the transformed variables $(X,Y,Z) = (x-w_\mathfrak{s},y,\pi) \bullet M$. Define $\mathcal{H}(X,Y,Z) = \pi - X^{\tilde{m}_{13}} Y^{\tilde{m}_{23}} Z^{\tilde{m}_{33}}$, and $\mathcal{G}(X,Y,Z) = g((X,Y,Z) \bullet M^{-1})$. We have

$$\mathcal{F}_M^h(X,Y,Z) = Y^{-n_Y} Z^{-n_Z} \mathcal{G}(X,Y,Z),$$

where $n_Z = m_\mathfrak{t}\epsilon_\mathfrak{t}$, since $\operatorname{ord}_Z(y^2) = m_\mathfrak{t}\epsilon_\mathfrak{t}$ for Lemma 5.2. Write $\mathcal{F} = \mathcal{F}_M^h$.

Note that $d(x-w_\mathfrak{s}) = dx$ and $(g_{w_\mathfrak{s}})'_x(x-w_\mathfrak{s}) = g'_x(x)$, where $g_{w_\mathfrak{s}}(x,y) = g(x+w_\mathfrak{s},y)$. Then, by [1, 8.7],

$$\begin{cases} (x-w_\mathfrak{s})g'_x = m_{11}X\mathcal{G}'_X + m_{12}Y\mathcal{G}'_Y + m_{13}Z\mathcal{G}'_Z \\ yg'_y = m_{21}X\mathcal{G}'_X + m_{22}Y\mathcal{G}'_Y + m_{23}Z\mathcal{G}'_Z \end{cases}$$

from which it follows that

$$\begin{aligned} m_{11}yg'_y - m_{21}(x-w_\mathfrak{s})g'_x &= (m_{11}m_{22} - m_{21}m_{12})Y\mathcal{G}'_Y - (m_{21}m_{13} - m_{11}m_{23})Z\mathcal{G}'_Z \\ &= \tilde{m}_{33}Y\mathcal{G}'_Y - \tilde{m}_{23}Z\mathcal{G}'_Z. \end{aligned}$$

We will show later that this quantity is non-zero. Moreover,

$$\tilde{m}_{33}Y\mathcal{G}'_Y - \tilde{m}_{23}Z\mathcal{G}'_Z = Y^{n_Y}Z^{n_Z}\left(\tilde{m}_{33}Y\mathcal{F}'_Y - \tilde{m}_{23}Z\mathcal{F}'_Z + (n_Y+n_Z)\mathcal{F}\right).$$

Recall that $X = (x-w_\mathfrak{s})^{m_{11}} y^{m_{21}} \pi^{m_{31}}$. Then $\frac{dX}{X} = m_{11}\frac{dx}{x-w_\mathfrak{s}} + m_{21}\frac{dy}{y}$. Furthermore, as $0 = dg = g'_x dx + g'_y dy$ in $\Omega_{C/K}$, we have

$$\frac{dX}{X} = \left(\frac{m_{11}}{x-w_\mathfrak{s}} - \frac{m_{21}}{y}\frac{g'_x}{g'_y}\right)dx = \frac{dx}{(x-w_\mathfrak{s})yg'_y}\left(m_{11}yg'_y - m_{21}(x-w_\mathfrak{s})g'_x\right).$$

Therefore

$$\frac{dx}{2(x-w_\mathfrak{s})y^2} = \frac{dX}{XY^{n_Y}Z^{n_Z}\left(\tilde{m}_{33}Y\mathcal{F}'_Y - \tilde{m}_{23}Z\mathcal{F}'_Z + (n_Y+n_Z)\mathcal{F}\right)}. \tag{11}$$

Let $S = \operatorname{Spec} O_K$. Considering X^{-1} as an independent variable, the scheme

$$U = \operatorname{Spec} \frac{O_K[Y,Z,X^{-1},X]}{(\mathcal{F},\mathcal{H},X\cdot X^{-1}-1)}$$

[5] *If* $\Gamma_\mathfrak{t}$ *is reducible, say* $\Gamma_\mathfrak{t} = \Gamma_\mathfrak{t}^- \cup \Gamma_\mathfrak{t}^+$, *with* $\operatorname{ord}_{\Gamma_\mathfrak{t}}(\cdot)$ *we mean* $\min\{\operatorname{ord}_{\Gamma_\mathfrak{t}^-}(\cdot), \operatorname{ord}_{\Gamma_\mathfrak{t}^+}(\cdot)\}$

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

defines a complete intersection in $\mathbb{A}^4_S$. Furthermore, U is an open subscheme of $\mathcal{C}_\Delta^{w_h} \cap X_M$ that restricted to $\mathbb{A}^4_S \setminus \{T^h_M(X,Y,Z)=0\}$ equals U^h_M. In particular, U is integral. From Section 5.5 it follows that $U_\mathfrak{t} = U \cap \{Z=0\}$ is a dense open subset of X_F. Recall that X_F is an open subscheme of $\Gamma_\mathfrak{t}$. Hence it suffices to prove the proposition for $U_\mathfrak{t}$ instead of $\Gamma_\mathfrak{t}$ ([9, Lemma 9.2.17(a)]). Since X and Y are units and Z vanishes to order 1 on $U_\mathfrak{t}$, Lemma 5.2 implies that

$$\mathrm{ord}_{U_\mathfrak{t}}(x-w_\mathfrak{s}) = \tilde{m}_{31} = m_\mathfrak{t}\rho_\mathfrak{t}, \quad \mathrm{ord}_{U_\mathfrak{t}} y = \tilde{m}_{32} = m_\mathfrak{t}\tfrac{\epsilon_\mathfrak{t}}{2}, \quad \mathrm{ord}_{U_\mathfrak{t}}\pi = \tilde{m}_{33} = m_\mathfrak{t}. \tag{12}$$

Recall that U is integral and that U_η is isomorphic to an open subscheme of C. Then U_η is smooth. Hence, by [9, Corollary 6.4.14(b)], the sheaf $\omega_{\mathcal{C}/O_K}$ is generated on U by $\mathcal{E}^{-1}dX$ where

$$\mathcal{E} := \begin{vmatrix} \mathcal{F}'_Y & \mathcal{F}'_Z & \mathcal{F}'_{X^{-1}} \\ H'_Y & H'_Z & \mathcal{F}'_{X^{-1}} \\ 0 & 0 & X \end{vmatrix} = -\pi X Y^{-1} Z^{-1}\left(\tilde{m}_{33} Y \mathcal{F}'_Y - \tilde{m}_{23} Z \mathcal{F}'_Z\right), \tag{13}$$

if $\mathcal{E}$ is non-zero. Suppose it is; we are going to prove it later. Thus, as $\mathcal{F}=0$ on U, we have

$$\begin{aligned} \mathrm{ord}_{U_\mathfrak{t}} \frac{dx}{2(x-w_\mathfrak{s})y^2} &= \mathrm{ord}_{U_\mathfrak{t}} \frac{dX}{XY^{n_Y}Z^{n_Z}\left(\tilde{m}_{33}Y\mathcal{F}'_Y - \tilde{m}_{23}Z\mathcal{F}'_Z\right)} && \text{from (11)} \\ &= \mathrm{ord}_{U_\mathfrak{t}}\left(\pi Y^{-n_Y-1}Z^{-n_Z-1}\mathcal{E}^{-1}dX\right) && \text{from (13)} \\ &= m_\mathfrak{t} - n_Z - 1 = m_\mathfrak{t}\left(-\epsilon_\mathfrak{t}+1\right) - 1 && \text{from (12).} \end{aligned}$$

Then it follows from (12) that

$$\mathrm{ord}_{U_\mathfrak{t}} \frac{dx}{2y} = m_\mathfrak{t}\left(\rho_\mathfrak{t} + \tfrac{1}{2}\epsilon_\mathfrak{t}\right) + \mathrm{ord}_{U_\mathfrak{t}} \frac{dx}{2(x-w_\mathfrak{s})y^2} = m_\mathfrak{t}\left(-\tfrac{1}{2}\epsilon_\mathfrak{t} + \rho_\mathfrak{t} + 1\right) - 1.$$

It remains to show that $\mathcal{E}$ does not equal 0 on U. Suppose it does. Then from the computations above, it follows that $m_{11}yg'_y - m_{21}(x-w_\mathfrak{s})g'_x = 0$ in $K(C)$. Since m_{21} equals either 1 or 2 by Lemma 5.2, it follows that there exists a non-zero $c \in K$, such that

$$m_{11}yg'_y - m_{21}(x-w_\mathfrak{s})g'_x + cg = 0$$

($c \in K$ from degree analysis). Then $cf(x) = m_{21}(x-w_\mathfrak{s})f'(x)$. Note that m_{21} is non-zero as $\mathrm{char}(K) \neq 2$. But then a contradiction follows since f is a separable polynomial of degree ≥ 3. □

In the following two theorems we describe a basis of integral differentials of C. We use Definitions/Notations 3.1, 3.3, 3.2, 3.8, 3.9, 3.26, 4.6, 4.10 in the statements.

Theorem 6.3. *Let C/K be a hyperelliptic curve of genus $g \geq 1$ defined by the Weierstrass equation $y^2 = f(x)$ and let $\mathcal{C}/O_K$ be a regular model of C. Suppose C has an almost rational cluster picture and is y-regular. For $i = 0, \ldots, g-1$ inductively*

(i) *define* $e_i := \max_{\mathfrak{t} \in \Sigma_C^{\mathrm{rat}}} \left\{ \frac{\epsilon_\mathfrak{t}}{2} - \rho_\mathfrak{t} - \sum_{j=0}^{i-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{t}} \right\}$;

(ii) *choose a maximal element $\mathfrak{s}_i$ of* $\left\{ \mathfrak{t} \in \Sigma_C^{\mathrm{rat}} \mid e_i = \frac{\epsilon_\mathfrak{t}}{2} - \rho_\mathfrak{t} - \sum_{j=0}^{i-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{t}} \right\}$ *freely.*

Then the differentials

$$\mu_i = \pi^{\lfloor e_i \rfloor} \prod_{j=0}^{i-1} (x - w_{\mathfrak{s}_j}) \frac{dx}{2y}, \qquad i = 0, \ldots, g-1,$$

form an O_K-basis of $H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K})$.

Proof. Since $H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K})$ is independent of the choice of regular model, we consider $\mathcal{C}$ to be the model described in Theorem 4.18 and constructed in Section 5.

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

We first show that the differentials μ_i are global sections of $\omega_{\mathcal{C}/O_K}$. It suffices to prove they are regular along all components $\Gamma_{\mathfrak{t}}$, where $\mathfrak{t}\in\Sigma_C^{\mathrm{rat}}$ proper. Indeed for the construction of $\mathcal{C}$ and the definition of the e_i's, the differentials μ_i are regular along all other components of $\mathcal{C}_s$ by Theorem 6.1. Fix $i=1,\dots,g-1$ and let $j=0,\dots,i-1$. Let $\mathfrak{t}\in\Sigma_C^{\mathrm{rat}}$ be a proper cluster. If $\mathfrak{s}_j\subseteq\mathfrak{t}$ then

$$\mathrm{ord}_{\Gamma_{\mathfrak{t}}}(x-w_{\mathfrak{s}_j})=m_{\mathfrak{t}}\rho_{\mathfrak{t}}=m_{\mathfrak{t}}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}},$$

by Proposition 6.2. If $\mathfrak{t}\subsetneq\mathfrak{s}_j$ then $w_{\mathfrak{t}}$ is a rational centre of $\mathfrak{s}_j$. Hence

$$v(w_{\mathfrak{t}}-w_{\mathfrak{s}_j})\geq\min_{r\in\mathfrak{t}}\min\{v(r-w_{\mathfrak{t}}),v(r-w_{\mathfrak{s}_j})\}\geq\min\{\rho_{\mathfrak{t}},\rho_{\mathfrak{s}_j}\}=\rho_{\mathfrak{s}_j}=\rho_{\mathfrak{s}_j\wedge\mathfrak{t}}.$$

Therefore Proposition 6.2 implies

$$\begin{aligned}\mathrm{ord}_{\Gamma_{\mathfrak{t}}}(x-w_{\mathfrak{s}_j})&\geq\min\{\mathrm{ord}_{\Gamma_{\mathfrak{t}}}(x-w_{\mathfrak{t}}),\mathrm{ord}_{\Gamma_{\mathfrak{t}}}(w_{\mathfrak{t}}-w_{\mathfrak{s}_j})\}\\&\geq\min\{m_{\mathfrak{t}}\rho_{\mathfrak{t}},m_{\mathfrak{t}}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}}\}=m_{\mathfrak{t}}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}}.\end{aligned}$$

If $\mathfrak{s}_j\not\subseteq\mathfrak{t}$ and $\mathfrak{t}\not\subseteq\mathfrak{s}_j$ then from Lemma 3.18 it follows that

$$\mathrm{ord}_{\Gamma_{\mathfrak{t}}}(x-w_{\mathfrak{s}_j})=\min\{m_{\mathfrak{t}}\rho_{\mathfrak{t}},m_{\mathfrak{t}}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}}\}=m_{\mathfrak{t}}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}}.$$

as $\rho_{\mathfrak{t}}>\rho_{\mathfrak{s}_j\wedge\mathfrak{t}}$. Thus we have proved that

$$\mathrm{ord}_{\Gamma_{\mathfrak{t}}}(x-w_{\mathfrak{s}_j})\geq m_{\mathfrak{t}}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}},\qquad\text{where the equality holds if }\mathfrak{t}\not\subset\mathfrak{s}_j.\tag{14}$$

Hence it follows from Proposition 6.2 that

$$\mathrm{ord}_{\Gamma_{\mathfrak{t}}}\mu_i\geq m_{\mathfrak{t}}\left(\lfloor e_i\rfloor+\sum_{j=0}^{i-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}}-\frac{\epsilon_{\mathfrak{t}}}{2}+\rho_t+1\right)-1.$$

But

$$\lfloor e_i\rfloor\geq\left\lfloor\frac{\epsilon_{\mathfrak{t}}}{2}-\rho_{\mathfrak{t}}-\sum_{j=0}^{i-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}}\right\rfloor>\frac{\epsilon_{\mathfrak{t}}}{2}-\rho_{\mathfrak{t}}-\sum_{j=0}^{i-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}}-1,$$

then $\mathrm{ord}_{\Gamma_{\mathfrak{t}}}\mu_i>-1$, that implies $\mathrm{ord}_{\Gamma_{\mathfrak{t}}}\mu_i\geq 0$, as required.

Now we need to show that the differentials μ_i span $H^0(\mathcal{C},\omega_{\mathcal{C}/O_K})$, that is the lattice they span is saturated in the global sections of $\omega_{\mathcal{C}/O_K}$. Suppose not. Then there exist $I\subseteq\{0,\dots,g-1\}$ and $u_i\in O_K^\times$ for $i\in I$ such that the differential $\frac{1}{\pi}\sum_{i\in I}u_i\mu_i$ is regular along $\Gamma_{\mathfrak{t}}$, for every proper cluster $\mathfrak{t}\in\Sigma_C^{\mathrm{rat}}$. First we want to show that for any $i_1,i_2=0,\dots,g-1$ with $i_1<i_2$, one has $\mathfrak{s}_{i_2}\not\subset\mathfrak{s}_{i_1}$. Suppose by contradiction that $\mathfrak{s}_{i_2}\subsetneq\mathfrak{s}_{i_1}$. Then

$$\begin{aligned}e_{i_2}&\geq\frac{\epsilon_{\mathfrak{s}_{i_1}}}{2}-\rho_{\mathfrak{s}_{i_1}}-\sum_{j=0}^{i_2-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_1}}=e_{i_1}-\rho_{\mathfrak{s}_{i_1}}-\sum_{j=i_1+1}^{i_2-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_1}}\geq e_{i_1}-\rho_{\mathfrak{s}_{i_1}}-\sum_{j=i_1+1}^{i_2-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_2}}\\&\geq\frac{\epsilon_{\mathfrak{s}_{i_2}}}{2}-\rho_{\mathfrak{s}_{i_2}}-\sum_{j=0}^{i_1-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_2}}-\rho_{\mathfrak{s}_{i_1}}-\sum_{j=i_1+1}^{i_2-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_2}}=\frac{\epsilon_{\mathfrak{s}_{i_2}}}{2}-\sum_{j=0}^{i_2}\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_2}}=e_{i_2}.\end{aligned}$$

Therefore

$$\max_{\mathfrak{t}\in\Sigma_C^{\mathrm{rat}}}\left\{\frac{\epsilon_{\mathfrak{t}}}{2}-\rho_{\mathfrak{t}}-\sum_{j=0}^{i_2-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}}\right\}=e_{i_2}=\frac{\epsilon_{\mathfrak{s}_{i_1}}}{2}-\rho_{\mathfrak{s}_{i_1}}-\sum_{j=0}^{i_2-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_1}},$$

and this means that $\mathfrak{s}_{i_1}$ is a possible choice for the i_2th cluster $\mathfrak{s}_{i_2}$. But $\mathfrak{s}_{i_2}\subsetneq\mathfrak{s}_{i_1}$, so the i_2th cluster should have been $\mathfrak{s}_{i_1}$, a contradiction.

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

Let $I_0 \subseteq I$ be the set of indices i such that $\gamma_i := e_i - \lfloor e_i \rfloor$ is maximal. Let $i_0 = \min I_0$ and let $\Gamma_0 = \Gamma_{\mathfrak{s}_{i_0}}$. Since $\mathfrak{s}_{i_0} \not\subset \mathfrak{s}_j$, for all $j = 0, \dots, i_0 - 1$, from (14) it follows that

$$m := \operatorname{ord}_{\Gamma_0} \frac{1}{\pi}\mu_{i_0} = -m_{\mathfrak{s}_{i_0}}\gamma_{i_0} + m_{\mathfrak{s}_{i_0}}\left(e_{i_0} - \frac{\epsilon_{\mathfrak{s}_{i_0}}}{2} + \rho_{\mathfrak{s}_{i_0}} + \sum_{j=0}^{i_0-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{s}_{i_0}}\right) - 1$$
$$= -m_{\mathfrak{s}_{i_0}}\gamma_{i_0} - 1 < 0.$$

Furthermore,

$$\operatorname{ord}_{\Gamma_0} \frac{1}{\pi}\mu_i \geq -m_{\mathfrak{s}_{i_0}}\gamma_i + m_{\mathfrak{s}_{i_0}}\left(e_i - \frac{\epsilon_{\mathfrak{s}_{i_0}}}{2} + \rho_{\mathfrak{s}_{i_0}} + \sum_{j=0}^{i-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{s}_{i_0}}\right) - 1$$
$$\geq -m_{\mathfrak{s}_{i_0}}\gamma_i - 1 \geq -m_{\mathfrak{s}_{i_0}}\gamma_{i_0} - 1 = m,$$

for all $i \in I$. Let $J := \{i \in I \mid \operatorname{ord}_{\Gamma_0} \frac{1}{\pi}\mu_i = m\}$. Then $J \neq \varnothing$ since $i_0 \in J$. The order of the differential $\frac{1}{\pi}\sum_{i\in J} u_i\mu_i$ along Γ_0 must be $> m$. Let $i \in I$. From the computations above $i \in J$ if and only if

(i) $\operatorname{ord}_{\Gamma_0}(x - w_{\mathfrak{s}_j}) = m_{\mathfrak{s}_{i_0}}\rho_{\mathfrak{s}_{i_0}\wedge\mathfrak{s}_j}$ for all $j = 0, \dots, i-1$. Equivalently, if $\mathfrak{s}_j \supsetneq \mathfrak{s}_{i_0}$ for some $j < i$, then $v(w_{\mathfrak{s}_{i_0}} - w_{\mathfrak{s}_j}) = \rho_{\mathfrak{s}_{i_0}\wedge\mathfrak{s}_j}$.
(ii) $e_i = \frac{\epsilon_{\mathfrak{s}_{i_0}}}{2} - \rho_{\mathfrak{s}_{i_0}} - \sum_{j=0}^{i-1} \rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_0}}$. In particular, if $\mathfrak{s}_i \subseteq \mathfrak{s}_{i_0}$, then $\mathfrak{s}_i = \mathfrak{s}_{i_0}$.
(iii) $\gamma_i = \gamma_{i_0}$. Equivalently, $i \in I_0$.

Therefore $J \subseteq I_0$, $i_0 = \min J$ and

$$\lfloor e_i \rfloor - \lfloor e_{i_0} \rfloor = e_i - \gamma_i - e_{i_0} + \gamma_{i_0} = e_i - e_{i_0} = -\sum_{j=i_0}^{i-1} \rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_0}},$$

for all $i \in J$. Hence

$$\frac{1}{\pi}\sum_{i\in J} u_i\mu_i = \frac{1}{\pi}\mu_{i_0}\left(\sum_{i\in J} \frac{u_i}{\pi^{\sum_{j=i_0}^{i-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_0}}}} \prod_{j=i_0}^{i-1}(x - w_{\mathfrak{s}_j})\right),$$

and since $\operatorname{ord}_{\Gamma_0}\frac{1}{\pi}\mu_{i_0} = m < 0$ we must have

$$\operatorname{ord}_{\Gamma_0}\left(\sum_{i\in J} \frac{u_i}{\pi^{\sum_{j=i_0}^{i-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_0}}}} \prod_{j=i_0}^{i-1}(x - w_{\mathfrak{s}_j})\right) > 0. \tag{15}$$

For any $j < i \in J$, with $i_0 \leq j$ we have $\mathfrak{s}_j \not\subset \mathfrak{s}_{i_0}$. Therefore either $\mathfrak{s}_j = \mathfrak{s}_{i_0}$ or $\mathfrak{s}_j \wedge \mathfrak{s}_{i_0} \supsetneq \mathfrak{s}_{i_0}$. In the latter case,

$$\operatorname{ord}_{\Gamma_0}(x - w_{\mathfrak{s}_{i_0}}) = m_{\mathfrak{s}_{i_0}}\rho_{\mathfrak{s}_{i_0}} > m_{\mathfrak{s}_{i_0}}\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_0}} = \operatorname{ord}_{\Gamma_0}(x - w_{\mathfrak{s}_j}).$$

It follows from (15) that

$$\operatorname{ord}_{\Gamma_0}\left(\sum_{i\in J} v_i \frac{(x - w_{\mathfrak{s}_{i_0}})^{\beta_i}}{\pi^{\beta_i\rho_{\mathfrak{s}_{i_0}}}}\right) > 0,$$

where $J_i = \{j \in I \mid i_0 \leq j < i \text{ and } \mathfrak{s}_j \neq \mathfrak{s}_{i_0}\}$, $v_i = u_i \prod_{j\in J_i} \frac{w_{\mathfrak{s}_{i_0}} - w_{\mathfrak{s}_j}}{\pi^{\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_0}}}} \in O_K^\times$, and $\beta_i = |\{i_0, \dots, i-1\} \setminus J_i|$.

To find a contradiction, we will use the explicit description of a dense open affine subset of Γ_0. Let $W = \{w_1, \dots, w_m\}$ be the set of rational centres of the rationally minimal clusters for C fixed at the beginning of the section. Let $w_h \in W$ such that $w_h = w_{\mathfrak{s}_{i_0}}$, and let $L = L_{\mathfrak{s}_{i_0}}^{w_h}$, $M = M_{L,0}$, and consider

$$U_M^h \cap \{Z = 0\} = \operatorname{Spec} \frac{R[T_M^h(X, Y, Z)^{-1}]}{(\mathcal{F}_M^h(X, Y, Z), Z)} \subset \Gamma_{\mathfrak{t}},$$

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

dense open subscheme of $\Gamma_{\mathfrak{t}}$. From Lemma 5.2,

$$\sum_{i\in J} v_i \frac{(x-w_h)^{\beta_i}}{\pi^{\beta_i\rho_{\mathfrak{s}_{i_0}}}} = \sum_{i\in J} v_i X^{\beta_i/b_{\mathfrak{s}_{i_0}}},$$

which is a unit since the polynomial $\mathcal{F}_M^h(X,Y,Z)$ in $\{Z=0\}$ is of the form $Y^2-G(X)$ or $Y-G(X)$ for some non-constant $G(X)\in K[X]$ (for more details see Lemma 5.17). This gives a contradiction and concludes the proof. □

Assume now $C_{K^{nr}}$ has an almost rational cluster picture and is y-regular as in Theorem 4.23. Since $|\Sigma_C|$ is finite, there exists a finite unramified extension F/K such that C_F has an almost rational cluster picture and is y-regular. Denote by O_F the ring of integers of F. Let $\Sigma_F=\Sigma_{C_F}^{\mathrm{rat}}$. Fix a rational centre $w_{\mathfrak{s}}\in F$ for every rationally minimal cluster $\mathfrak{s}\in\Sigma_F$. For all non-minimal clusters $\mathfrak{t}\in\Sigma_F$ choose a rational centre $w_{\mathfrak{t}}=w_{\mathfrak{s}}$ for some rationally minimal cluster $\mathfrak{s}\subseteq\mathfrak{t}$. In this setting the next theorem gives a basis of integral differentials of C.

Theorem 6.4. *Let C/K be a hyperelliptic curve of genus $g\geq 1$ defined by the Weierstrass equation $y^2=f(x)$ and let $\mathcal{C}/O_K$ be a regular model of C. Suppose there exists a finite unramified extension F/K such that C_F has an almost rational cluster picture and is y-regular. For $i=0,\ldots,g-1$ inductively*

(i) *define* $e_i := \max_{\mathfrak{t}\in\Sigma_F}\left\{\frac{\epsilon_{\mathfrak{t}}}{2}-\rho_{\mathfrak{t}}-\sum_{j=0}^{i-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}}\right\}$;

(ii) *choose a maximal element $\mathfrak{s}_i$ of* $\left\{\mathfrak{t}\in\Sigma_F \mid e_i=\frac{\epsilon_{\mathfrak{t}}}{2}-\rho_{\mathfrak{t}}-\sum_{j=0}^{i-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}}\right\}$ *freely.*

Then the differentials

$$\mu_i=\pi^{\lfloor e_i\rfloor}\cdot\mathrm{Tr}_{F/K}\left(\beta\prod_{j=0}^{i-1}(x-w_{\mathfrak{s}_j})\right)\frac{dx}{2y},\qquad i=0,\ldots,g-1,$$

form an O_K-basis of $H^0(\mathcal{C},\omega_{\mathcal{C}/O_K})$.

Proof. First note that without loss of generality we can suppose F/K Galois. Moreover, since F/K is unramified, $\mathrm{Gal}(F/K)\simeq\mathrm{Gal}(\mathfrak{f}/k)$, where $\mathfrak{f}$ is the residue field of F, and so the existence of β is guaranteed by the surjectivity of $\mathrm{Tr}_{\mathfrak{f}/k}$. Let $\mathcal{C}$ be the minimal regular model of C over O_K. By [9, Proposition 10.1.17], the base extended scheme $\mathcal{C}_{O_F}$ is the minimal regular model of C_F over O_F. Let $\mu_0^F,\ldots,\mu_{g-1}^F$ be the basis of integral differentials of C_F given by Theorem 6.3.

Suppose $\mu_0',\ldots,\mu_{g-1}'$ is a basis of integral differentials of C_F that, for any $\sigma\in\mathrm{Gal}(F/K)$ and any $j=0,\ldots,g-1$, satisfies

$$\sigma(\mu_j')=\mu_j'+\sum_{0\leq i<j}\lambda_{ij}\mu_i',\tag{16}$$

for some $\lambda_{ij}\in O_F$ (depending on σ). Note that $\mu_0^F,\ldots,\mu_{g-1}^F$ is in fact such a basis. We want to prove that, for any $j=0,\ldots,g-1$, the differentials

$$\mu_0',\ldots,\mu_{j-1}',\mathrm{Tr}_{F/K}(\beta\mu_j'),\mu_{j+1}',\ldots,\mu_{g-1}'\tag{17}$$

still form a basis of $H^0(\mathcal{C}_F,\omega_{\mathcal{C}_F/O_F})$ satisfying condition (16). From equation (16) it follows that

$$\mathrm{Tr}_{F/K}(\beta\mu_j')=\sum_{\sigma\in\mathrm{Gal}(F/K)}\sigma(\beta)\sigma(\mu_j')=\mathrm{Tr}_{F/K}(\beta)\mu_j'+\sum_{i<j}\lambda_{ij}'\mu_i',$$

for some $\lambda_{ij}'\in O_F$. Since $\mathrm{Tr}_{F/K}(\beta)\in O_K^\times$, the differentials in (17) form a basis of $H^0(\mathcal{C}_F,\omega_{\mathcal{C}_F/O_F})$ satisfying condition (16).

https://doi.org/10.1017/S001708952400003X Published online by Cambridge University Press

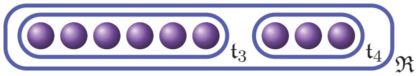
Since $\mu_0^F, \dots, \mu_{g-1}^F$ satisfies (16), by induction it follows that

$$\mathrm{Tr}_{F/K}(\beta\mu_0^F), \dots, \mathrm{Tr}_{F/K}(\beta\mu_{g-1}^F)$$

is a basis of $H^0(\mathcal{C}_F, \omega_{\mathcal{C}_F/O_F})$. Proposition B.2 concludes the proof. $\square$

We conclude this section with an application of Theorem 6.3.

Example 6.5. *Let p be a prime number and let $a \in \mathbb{Z}_p$, $b \in \mathbb{Z}_p^\times$ such that the polynomial $x^2 + ax + b$ is not a square modulo p. Let C be the hyperelliptic curve over $\mathbb{Q}_p$ of genus 4 described by the equation $y^2 = f(x)$, where $f(x) = (x^6 + ap^4x^3 + bp^8)((x-p)^3 - p^{11})$. We have already shown in Examples 3.32 and 4.25 that C satisfies the hypothesis of Theorem 6.3 and has rational cluster picture*



We choose rational centres for the minimal clusters $\mathfrak{t}_3$ and $\mathfrak{t}_4$: $w_{\mathfrak{t}_3} = 0$ and $w_{\mathfrak{t}_4} = p$. Since $\mathfrak{R} = \mathfrak{t}_3 \wedge \mathfrak{t}_4$, we can set either $w_{\mathfrak{R}} = w_{\mathfrak{t}_3}$ or $w_{\mathfrak{R}} = w_{\mathfrak{t}_4}$. Let us fix $w_{\mathfrak{R}} = w_{\mathfrak{t}_3} = 0$. Then to choose $\mathfrak{s}_0, \mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3$ as in Theorem 6.3 we draw the following table:

	$\rho_\mathfrak{t}$	$\epsilon_\mathfrak{t}$	$\frac{\epsilon_\mathfrak{t}}{2} - \rho_\mathfrak{t}$	$\frac{\epsilon_\mathfrak{t}}{2} - \rho_\mathfrak{t} - \rho_{\mathfrak{s}_0\wedge\mathfrak{t}}$	$\frac{\epsilon_\mathfrak{t}}{2} - \rho_\mathfrak{t} - \sum_{j=0}^{1}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}}$	$\frac{\epsilon_\mathfrak{t}}{2} - \rho_\mathfrak{t} - \sum_{j=0}^{2}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}}$
$\mathfrak{t}_3$	$\frac{4}{3}$	11	$\frac{25}{6}$	$\frac{19}{6}$	$\frac{11}{6}$	$\frac{1}{2}$
$\mathfrak{t}_4$	$\frac{11}{3}$	17	$\frac{29}{6}$	$\frac{7}{6}$	$\frac{1}{6}$	$-\frac{5}{6}$
$\mathfrak{R}$	1	9	$\frac{7}{2}$	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{1}{2}$

The numbers in red indicate that $\mathfrak{s}_0 = \mathfrak{t}_4$, $\mathfrak{s}_1 = \mathfrak{s}_2 = \mathfrak{t}_3$ and $\mathfrak{s}_3 = \mathfrak{R}$. Thus the differentials

$$\mu_0 = p^4 \cdot \frac{dx}{2y}, \quad \mu_1 = p^3 \cdot (x-p)\frac{dx}{2y}, \quad \mu_2 = p \cdot (x-p)x\frac{dx}{2y}, \quad \mu_3 = (x-p)x^2\frac{dx}{2y}$$

form a $\mathbb{Z}_p$-basis of $H^0(\mathcal{C}, \omega_{\mathcal{C}/\mathbb{Z}_p})$, for any regular model $\mathcal{C}/\mathbb{Z}_p$ of C.

Acknowledgements. The author would like to thank his supervisor Tim Dokchitser for the very useful conversations, corrections and general advice.

References

[1] T. Dokchitser, Models of curves over discrete valuation rings, *Duke Math. J.* **170**(11) (2021), 2519–2574.
[2] A. Grothendieck and J. A. Dieudonné, *Éléments de géométrie algébrique (French)* (Inst. Hautes Études Sci. Publ. Math., 1960–1967).
[3] J. Fernández, J. Guárdia, J. Montes and E. Nart, Residual ideals of MacLane valuations, *J. Algebra* **427** (2015), 30–75.
[4] E. V. Flynn, F. Leprévost, E. F. Schaefer, W. A. Stein, M. Stoll and J. Wetherell, Empirical evidence for the Birch and Swinnerton-Dyer conjectures for modular Jacobians of genus 2 curves, *Math. Comput.* **70**(236) (2001), 1675–1697.
[5] O. Faraggi and S. Nowell, Models of hyperelliptic curves with tame potentially semistable reduction, *Trans. Lond. Math. Soc.*, **7**(1) (2020), 49–95.
[6] I. Kausz, A discriminant and an upper bound for w^2 for hyperelliptic arithmetic surfaces, *Compos. Math.* **115** (1999), 37–69.
[7] K. Kodaira, On the structure of compact complex analytic surfaces I, II, *Am. J. Math.* **86**(1966) (1964), 751–798.
[8] S. Kunzweiler, Differential forms on hyperelliptic curves with semistable reduction, *Res. Number Theory* **6**(25) (2020), 1–17.
[9] Q. Liu, *Algebraic geometry and arithmetic curves* (Oxford Univ. Press, Oxford, 2002).

[10] Q. Liu, *Conducteur et discriminant minimal de courbes de genre 2* (French), *Compos. Math.* **94**(1) (1994), 51–79.
[11] Q. Liu, *Modéles minimaux des courbes de genre deux* (French), *J. Reine Angew. Math.* **453** (1994), 137–164.
[12] Q. Liu, Quelques erreurs dans la table de: the complete classification of fibers in pencils of curves (French) (1994). Available at https://www.math.u-bordeaux.fr/~qliu/articles/errata-NU.pdf
[13] Q. Liu and D. Lorenzini, Models of curves and finite covers, *Compos. Math.* **118**(1) (1999), 61–102.
[14] T. Dokchitser, V. Dokchitser, C. Maistret and A. Morgan, Arithmetic of hyperelliptic curves over local fields, *preprint, arXiv: 1808.02936*, to appear in Mathematische Annalen.
[15] S. MacLane, A construction for absolute values in polynomial rings, *Trans. Am. Math. Soc.* **40**(3) (1936), 363–395.
[16] S. Maugeais, *Relévement des revetements p-cycliques des courbes rationnelles semi-stables* (French), *Math. Ann.* **327** (2003), 365–393.
[17] J. Neukirch, *Algebraic number theory*, Grundlehren der mathematischen Wissenschaften, vol. **322** (Springer, Berlin, 1999).
[18] Y. Namikawa and K. Ueno, The complete classification of fibres in pencils of curves of genus two, *Manuscr. Math.* **9** (1973), 143–186.
[19] A. Néron, *Modèles minimaux des variétés abéliennes sur les corps locaux et globaux (French)*, vol. **21** (Publ. Math. IHES, 1964), 361–482.
[20] A. Obus and P. Srinivasan, Conductor-discriminant inequality for hyperelliptic curves in odd residue characteristic, *preprint, arXiv: 1910.02589* (2023).
[21] A. Obus and S. Wewers, Explicit resolution of weak wild arithmetic surface singularities, *J. Algebraic Geom.* **29** (2020), 691–728.
[22] T. Saito, Conductor, discriminant, and the Noether formula of arithmetic surfaces, *Duke Math. J.* **57**(1) (1988), 151–173.
[23] J. H. Silverman, *The arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. **106** (Springer, New York, 1986).
[24] J. H. Silverman, *Advanced topics in the arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. **151** (Springer, New York, 1994).
[25] R. van Bommel, Models of curves: the Birch and Swinnerton-Dyer conjecture and ordinary reduction, PhD thesis. Available at https://openaccess.leidenuniv.nl/handle/1887/66673

Appendix A: Rational centres over tame extensions

Let C/K be a hyperelliptic curve given by $y^2 = f(x)$.

Lemma A.1. *Let L/K be a field extension. Consider the base extended curve C_L/L and its associated cluster picture Σ_{C_L}. Let $\mathfrak{s} \in \Sigma_{C_L}$ be a proper cluster $G_\mathfrak{s} = \mathrm{Stab}_{G_K}(\mathfrak{s})$, and $K_\mathfrak{s} = (K^s)^{G_\mathfrak{s}}$. If $L/L \cap K_\mathfrak{s}$ is tamely ramified, then $\mathfrak{s}$ has a rational centre $w_\mathfrak{s} \in L \cap K_\mathfrak{s}$.*

Proof. Let $F_\mathfrak{s} = L \cap K_\mathfrak{s}$. Let $w_\mathfrak{s} \in L$ be a rational centre of $\mathfrak{s}$ and let $\rho_\mathfrak{s} = \max_{w\in L} \min_{r\in\mathfrak{s}} v(r-w)$ be its radius. Let $\mathcal{D} = \{x \in K^s \mid v(x - w_\mathfrak{s}) \geq \rho_\mathfrak{s}\}$ and define $G = \mathrm{Stab}_{G_K}(\mathcal{D})$. Since $\mathfrak{s} \subseteq \mathcal{D}$ we have $G_\mathfrak{s} \subseteq G$. Furthermore, $\mathrm{Gal}(K^s/L) \subseteq G$. Then $\mathrm{Gal}(K^s/F_\mathfrak{s}) \subseteq G$. Since $w_\mathfrak{s} \in \mathcal{D}$, for $\sigma \in \mathrm{Gal}(K^s/F_\mathfrak{s}) \subseteq G$ we have $\sigma(w_\mathfrak{s}) \in \mathcal{D}$. In particular, $v(r - \sigma(w_\mathfrak{s})) \geq \rho_\mathfrak{s}$ for any $r \in \mathfrak{s}$. Define

$$w = \frac{\mathrm{Tr}_{L/F_\mathfrak{s}}(w_\mathfrak{s})}{[L:F_\mathfrak{s}]} \in F_\mathfrak{s}.$$

If $m = [F_\mathfrak{s}[w_\mathfrak{s}]:F_\mathfrak{s}]$, then $w = \sum_{j=1}^m \sigma_j(w_\mathfrak{s})/m$, where $\sigma_1(w_\mathfrak{s}), \ldots, \sigma_m(w_\mathfrak{s})$ are the roots of the minimal polynomial of $w_\mathfrak{s}$ over $F_\mathfrak{s}$ (with $\sigma_j \in \mathrm{Gal}(K^s/F_\mathfrak{s})$). Since $L/F_\mathfrak{s}$ is tamely ramified, $p \nmid [L:F_\mathfrak{s}]$ and so $p \nmid m$. In particular, $v(m) = 0$ and so for any $r \in \mathfrak{s}$ we have

$$v(r-w) = v\left(m \cdot r - \sum_{j=1}^{m} \sigma_j(w_\mathfrak{s})\right) \geq \min_{j\in\{1,\ldots,m\}} v(r - \sigma_j(w_\mathfrak{s})) \geq \rho_\mathfrak{s}.$$

Then $w \in F_\mathfrak{s}$ is a rational centre of $\mathfrak{s}$. □

Appendix B: Dualising sheaf under base extensions

Let F/K be a finite Galois extension and let O_F be the ring of integers of F.

Lemma B.1. *Let M be a flat O_K-module and $M_F := M \otimes_{O_K} O_F$. Then*

$$M \simeq M_F^{\mathrm{Gal}(F/K)} = \{m \in M_F \mid \sigma(m) = m \text{ for every } \sigma \in \mathrm{Gal}(F/K)\}.$$

Proof. As M is flat, the functor $M \otimes_{O_K} -$ is (left) exact. From the isomorphism $O_K \simeq O_F^{\mathrm{Gal}(F/K)}$ it follows that

$$M \otimes_{O_K} O_K \simeq M \otimes_{O_K} O_F^{\mathrm{Gal}(F/K)},$$

that is $M \simeq M_F^{\mathrm{Gal}(F/K)}$, as required. □

Proposition B.2. *Let C be a smooth projective curve of genus $g \geq 1$ and let $\mathcal{C}$ be a regular model of C over O_K. Denote by C_F and $\mathcal{C}_{O_F}$ the base extended schemes. Then $H^0(\mathcal{C}_F, \omega_{\mathcal{C}_F/O_F}) \simeq H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K}) \otimes_{O_K} O_F$ and*

$$H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K}) \simeq H^0(\mathcal{C}_F, \omega_{\mathcal{C}_F/O_F})^{\mathrm{Gal}(F/K)}.$$

Proof. The Lemma follows from the following results: [9, Proposition 10.1.17], [9, Theorem 6.4.9(b)], [9, Exercise 6.4.6], [9, Corollary 5.2.27] and the previous lemma. □