J. Austral. Math. Soc. Ser. B 32(1990), 180–192

EXISTENCE OF NONOSCILLATORY SOLUTIONS OF FIRST ORDER NONLINEAR NEUTRAL EQUATIONS

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(Received September 1989; revised February 1990)

Abstract

Consider the nonlinear neutral equation

$$(x(t) - \sum_{i=1}^{m} p_i(t)x(h_i(t)))' + \sum_{j=1}^{n} f_j(t, x(g_j(t))) = Q(t)$$

where $p_i(t)$, $h_i(t)$, $g_j(t)$, $Q(t) \in C[t_0, \infty)$, $\lim_{t\to\infty} h_i(t) = \infty$, $\lim_{t\to\infty} g_j(t) = \infty$, $i \in I_m = \{1, 2, \ldots, m\}$, $j \in I_n = \{1, 2, \ldots, n\}$. We obtain a necessary and sufficient condition (2) for this equation to have a nonoscillatory solution $x(t)$ with $\lim_{t\to\infty} \inf|x(t)| > 0$ (Theorems 5 and 6) or to have a bounded nonoscillatory solution $x(t)$ with $\lim_{t\to\infty} \inf|x(t)| > 0$ (Theorem 7).

1. Introduction

Consider the first order nonlinear neutral equation

$$(x(t) - \sum_{i=1}^{m} p_i(t)x(h_i(t)))' + \sum_{j=1}^{n} f_j(t, x(g_j(t))) = Q(t), \qquad (t \geq t_0) \qquad (1)$$

where $p_i(t)$, $h_i(t)$, $g_j(t)$, $Q(t) \in C[t_0, \infty)$, $\lim_{t\to\infty} h_i(t) = \infty$, $\lim_{t\to\infty} g_j(t) = \infty$, $i \in I_m = \{1, 2, \ldots, m\}$, $j \in I_n = \{1, 2, \ldots, n\}$.

Recently, many authors have studied oscillations of neutral equations. But since Grove et al [3] published the existence theorem of nonoscillatory solution of first order linear neutral equation with variable coefficients in 1987, there have been few results about the existence of nonoscillatory solutions, especially for nonlinear neutral equations.

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https://doi.org/10.1017/S0334270000008419 Published online by Cambridge University Press

In this paper we study nonoscillations of (1) under very general conditions and obtain criterion theorems for the existence of nonoscillatory solutions of nonlinear neutral equations for the first time.

Neutral equations have applications in electric networks containing lossless transmission lines. Such networks arise in high speed computers where lossless transmission lines are used to interconnect switching circuits (see [1], [2], [4]).

A solution of (1) is called oscillatory if it has arbitrary large zeros and nonoscillatory if it is eventually positive or eventually negative.

2. Main results

We denote by C_B all bounded continuous functions of $C[t_0, \infty)$. Define a distance in C_B by

$$R(x, y) = \sup_{t \geq t_0} |x(t) - y(t)|, \quad \text{for } x, y \in C_B.$$

Then C_B becomes a complete metric space and every closed subset of C_B is also a complete subspace.

We suppose that $f_j(t, x)$, $j \in I_n$, $Q(t)$ satisfy the following conditions:
(a) $|f_j(t, x)| \leq |f_j(t, y)|$ when $|x| \leq |y|$;
(b) for each closed interval $L = [c, d](0 < c < d)$, there are positive functions $L_j(t)(j \in I_n)$ such that

$$|f_j(t, x) - f_j(t, y)| \leq L_j(t)|x - y| \text{ when } x, y \in L$$

where $L_j(t) \in C[t_0, \infty)$ are generally dependent on L and $\int^{\infty} L_j(t)\,dt < \infty$, $j \in I_n$;
(c) $\int^{\infty} |Q(t)|dt < \infty$.

THEOREM 1. *Assume that either*

$$\mathrm{A(i)}: p_i(t) \geq 0, \quad i \in I_m, \qquad \sum_{i=1}^{m} p_i(t) \leq 1 - r, \quad 0 < r < 1, \quad t \geq t_0,$$

or

$$\mathrm{A(ii)}: p_i(t) \leq 0, \quad i \in I_m, \qquad -\sum_{i=1}^{m} p_i(t) \leq 1 - r, \quad 0 < r < 1, \quad t \geq t_0,$$

holds and that $f_j(t, x)(j \in I_n)$ *and* $Q(t)$ *satisfy condition* (a)–(c). *If*

$$\sum_{j=1}^{n} \int^{\infty} |f_j(t, d)|\,dt < \infty \quad \textit{for some } d \neq 0 \tag{2}$$

https://doi.org/10.1017/S0334270000008419 Published online by Cambridge University Press

then (1) *has a bounded nonoscillatory solution* $x(t)$ *with* $\lim_{t\to\infty}\inf|x(t)| > 0$.

Proof. Set $S = \{x(t) \in C[t_0, \infty) : c \le x(t) \le |d|,\ t \ge t_0\}$ where $0 < c < r|d|$. It is easy to see that S is a complete metric space. Define a mapping as follows

$$(Mx)(t) = \begin{cases} c_1 + \sum_{i=1}^m p_i(t)x(h_i(t)) \\ \quad + \sum_{j=1}^n \int_t^\infty f_j(s, x(g_j(s)))\,ds - \int_t^\infty Q(s)\,ds, & t \ge T \\ (Mx)(T), & t_0 \le t < T. \end{cases}$$

where c_1 and T satisfy the following conditions:

When A(i) holds, $c < c_1 < r|d|$ and T is sufficiently large such that $h_i(t) \ge t_0$ $(i \in I_m)$ and $g_j(t) \ge t_0$ $(j \in I_n)$ as $t \ge T \ge t_0$ and further

$$\sum_{j=1}^n \int_T^\infty |f_j(s, d)|\,ds + \int_T^\infty |Q(s)|\,ds \le \min\{c_1 - c, r|d| - c_1\}, \tag{3}$$

$$\sum_{j=1}^n \int_T^\infty L_j(s)\,ds \le \frac{r}{2}. \tag{4}$$

When A(ii) holds, $c + (1 - r)|d| < c_1 < \frac{1}{2}(c + (2 - r)|d|)$ and T is sufficiently large such that $h_i(t) \ge t_0$ for $i \in I_m$, $g_j(t) \ge t_0$ for $j \in I_n$ as $t \ge T \ge t_0$, (4) holds and

$$\sum_{j=1}^n \int_T^\infty |f_j(s, d)|\,ds + \int_T^\infty |Q(s)|\,ds \le c_1 - c - (1 - r)|d|. \tag{5}$$

We need to prove

(i) $M(S) \subset S$. If A(i) holds, for any $x \in S$ and by (3), we have when $t \ge T$

$$\begin{aligned}(Mx)(t) &= c_1 + \sum_{i=1}^m p_i(t)x(h_i(t)) + \sum_{j=1}^n \int_t^\infty f_j(x, s(g_j(s)))\,ds - \int_t^\infty Q(s)\,ds \\ &\ge c_1 - \sum_{j=1}^n \int_t^\infty |f_j(s, x(g_j(s)))|\,ds - \int_t^\infty |Q(s)|\,ds \\ &\ge c_1 - \sum_{j=1}^n \int_T^\infty |f_j(s, d)|\,ds - \int_T^\infty |Q(s)|\,ds \\ &\ge c_1 - (c_1 - c) = c, \\ (Mx)(t) &\le c_1 + |d|\sum_{i=1}^m p_i(t) + \sum_{j=1}^n \int_T^\infty |f_j(s, d)|\,ds + \int_T^\infty |Q(s)|\,ds \\ &\le c_1 + (1 - r)|d| + (r|d| - c_1) = |d|.\end{aligned}$$

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Since $(Mx)(t) = (Mx)(T)$ as $t_0 \le t < T$, then $c \le (Mx)(t) \le |d|$ as $t_0 \le t < T$.

If A(ii) holds, by (5), we have when $t \ge T$

$$(Mx)(t) \ge c_1 + |d| \sum_{i=1}^{m} p_i(t) - \sum_{j=1}^{n} \int_T^\infty |f_j(s, d)|\, ds - \int_T^\infty |Q(s)|\, ds$$
$$\ge c_1 - (1-r)|d| - (c_1 - c - (1-r)|d|) = c,$$
$$(Mx)(t) \le c_1 + \sum_{j=1}^{n} \int_T^\infty |f_j(s, d)|\, ds + \int_T^\infty |Q(s)|\, ds$$
$$\le c_1 + (c_1 - c - (1-r)|d|) \le |d|.$$

Since $(Mx)(t) = (Mx)(T)$ as $t_0 \le t < T$, then $c \le (Mx)(t) \le |d|$ as $t_0 \le t < T$.

(ii) M is a compression mapping on S. For $x, y \in S$ and when $t \ge T$, we have

$$|(Mx)(t) - (My)(t)| = \left| \sum_{i=1}^{m} p_i(t)(x(h_i(t)) - y(h_i(t))) + \sum_{i=1}^{m} \int_T^\infty (f_j(s, x(g_j(s))) - f_j(s, y(g_j(s))))\, ds \right|$$
$$\le \left(\sum_{i=1}^{m} |p_i(t)| \right) \sup_{t \ge t_0} |x(t) - y(t)| + \sum_{j=1}^{n} \int_t^\infty L_j(s) |x(g_j(s)) - y(g_j(s))|\, ds$$
$$\le \left(\sum_{i=1}^{m} |p_i(t)| + \sum_{j=1}^{n} \int_T^\infty L_j(s)\, ds \right) R(x, y)$$
$$\le \left((1-r) + \frac{r}{2} \right) R(x, y) = \left(1 - \frac{r}{2} \right) R(x, y)$$

When $t_0 \le t < T$, we have

$$|(Mx)(t) - (My)(t)| = |(Mx)(T) - (My)(T)| \le \left(1 - \frac{r}{2} \right) R(x, y)$$

then

$$R(Mx, My) = \sup_{t \ge t_0} |(Mx)(t) - (My)(t)| \le \left(1 - \frac{r}{2} \right) R(x, y).$$

According to the Banach fixed point theorem, M has a fixed point x^* in S. Obviously, $x^*(t)$ is a bounded nonoscillatory solution of (1) and satisfies $\lim_{t\to\infty} \inf |x^*(t)| > 0$. The proof is complete.

Theorem 2. *Assume that either*

B(i) : *There is some integer* $k(1 \le k \le m)$ *such that* $h_k(t)$ *is strictly increasing,* $h_k(t) \le t$, $h_i(H(t)) \ge t (i \ne k)$, $p_k(t) > 1$, $p_i(t) \le 0 \quad (i \ne k)$,

$$\left| \frac{1}{p_k(t)} + \sum_{i \ne k} \frac{-p_i(t)}{p_k(t)} \right| \le 1 - r, \qquad 0 < r < 1, \ t \ge t_0 \tag{6}$$

and when $t \ge t_0$ *and* $H(h_i(t)) \ge t_0 \quad (i \ne k)$,

$$\left| \frac{1}{p_k(H(t))} + \sum_{i \ne k} \frac{-p_i(t)}{p_k(H(h_i(t)))} \right| \le 1 - r, \qquad (0 < r < 1) \tag{7}$$

where $H(t)$ *is the inverse function of* $h_k(t)$; *or*

B(ii) : *There is some integer* $k \ (1 \le k \le m)$ *such that* $h_k(t)$ *is strictly increasing,* $h_k(t) \le t$, $h_i(H(t)) \ge t \quad (i \ne k)$, $p_k(t) < -1$, $p_i(t) \le 0 \quad (i \ne k)$ *and* (6) *and* (7) *hold*; *is true. Further assume that* $f_j(t, x) \quad (j \in I_n)$ *and* $Q(t)$ *satisfy condition* (a)–(c). *If* (2) *holds, then* (1) *has a bounded nonoscillatory solution* $x(t)$, *and when* $p_k(t)$ *is bounded,* $\lim_{t \to \infty} \inf |x(t)| > 0$.

Proof. If B(i) holds, set

$$S = \left\{ x(t) \in C[t_0, \infty) : \begin{array}{ll} \dfrac{c}{p_k(H(t))} \le x(t) \le \dfrac{|d|}{p_k(H(t))}, & t \ge T \\ \dfrac{c}{p_k(H(T))} \le x(t) \le \dfrac{|d|}{p_k(H(T))}, & t_0 \le t < T \end{array} \right\}$$

where $0 < c < r|d|$. Choose a positive number c_1 such that $c < c_1 < r|d|$. T is sufficiently large such that $g_j(t) \ge t_0$, $(j \in I_n)$ as $t \ge T \ge t_0$ and

$$\sum_{j=1}^{n} \int_T^\infty |f_j(s, d)|\, ds + \int_T^\infty |Q(s)|\, ds \le \min\{c_1 - c, r|d| - c_1\} \tag{8}$$

$$\sum_{j=1}^{n} \int_T^\infty L_j(s)\, ds \le \frac{r}{2}. \tag{9}$$

Define a mapping as follows:

$$(Mx)(t) = \begin{cases} \dfrac{c_1}{p_k(H(t))} + \dfrac{x(H(t))}{p_k(H(t))} + \sum_{i \ne k} \dfrac{-p_i(H(t))}{p_k(H(t))} x(h_i(H(t))) \\ \quad + \dfrac{-1}{p_k(H(t))} \sum_{j=1}^{n} \int_{H(t)}^\infty f_j(s, x(g_j(s)))\, ds \\ \quad + \dfrac{1}{p_k(H(t))} \displaystyle\int_{H(t)}^\infty Q(s)\, ds, & t \ge T, \\ (Mx)(T), & t_0 \le t < T. \end{cases}$$

We need to prove

(i) $M(S) \subset S$. For any $x \in S$ and from (7), (8), and (9), we have when $t \geq T$,

$$
\begin{aligned}
(Mx)(t) &\geq \frac{1}{p_k(H(t))}\left(c_1 - \sum_{j=1}^{n}\int_{H(t)}^{\infty}|f_j(s, x(g_j(s)))|\,ds - \int_{H(t)}^{\infty}|Q(s)|\,ds\right) \\
&\geq \frac{1}{p_k(H(t))}\left(c_1 - \sum_{j=1}^{n}\int_{T}^{\infty}|f_j(s, d)|\,ds - \int_{T}^{\infty}|Q(s)|\,ds\right) \\
&\geq \frac{1}{p_k(H(t))}(c_1 - (c_1 - c)) = \frac{c}{p_k(H(t))}, \\
(Mx)(t) &\leq \frac{1}{p_k(H(t))}\left(c_1 + \frac{|d|}{p_k(H(H(t)))} + \sum_{i\neq k}\frac{-p_i(H(t))|d|}{p_k(H(h_i(H(t))))}\right. \\
&\qquad \left. + \sum_{j=1}^{n}\int_{T}^{\infty}|f_j(s, d)|\,ds + \int_{T}^{\infty}|Q(s)|\,ds\right) \\
&\leq \frac{1}{p_k(H(t))}(c_1 + (1-r)|d| + (r|d| - c_1)) = \frac{|d|}{p_k(H(t))}.
\end{aligned}
$$

Since $(Mx)(t) = (Mx)(T)$ as $t_0 \leq t < T$, then

$$
\frac{c}{p_k(H(T))} \leq (Mx)(t) \leq \frac{|d|}{p_k(H(T))} \quad \text{when } t_0 \leq t < T
$$

(ii) M is a compression mapping on S. For $x, y \in S$ and from (6) and (9), we have when $t \geq T$,

$$
\begin{aligned}
|(Mx)(t) - (My)(t)| = \Bigg| &\frac{x(H(t)) - y(H(t))}{p_k(H(t))} \\
&+ \sum_{i\neq k}\frac{-p_i(H(t))}{p_k(H(t))}(x(h_i(H(t))) - y(h_i(H(t)))) \\
&- \frac{1}{p_k(H(t))}\sum_{j=1}^{n}\int_{H(t)}^{\infty}(f_j(s, x(g_j(s))) \\
&\qquad - f_j(s, y(g_j(s))))\,ds\Bigg|
\end{aligned}
$$

(continues)

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$$\leq \left|\frac{1}{p_k(H(t))}+\sum_{i\neq k}\frac{-p_i(H(t))}{p_k(H(t))}\right| R(x\,,y)$$
$$+\sum_{j=1}^{n}\int_{H(t)}^{\infty} L_j(s)|x(g_j(s))-y(g_j(s))|\,ds$$
$$\leq (1-r)R(x\,,y)+\left(\sum_{j=1}^{n}\int_{T}^{\infty} L_j(s)\,ds\right) R(x\,,y)$$
$$\leq ((1-r)+r/2)\,R(x\,,y)=(1-r/2)\,R(x\,,y)$$

When $t_0\leq t<T$, we have

$$|(Mx)(t)-(My)(t)|=|(Mx)(T)-(My)(T)|\leq (1-r/2)\,R(x\,,y)$$

Hence $R(Mx\,,My)\leq(1-r/2)R(x\,,y)$. According to the Banach fixed point theorem, M has a fixed x^* in S. Obviously, when t is sufficiently large, $x^*(t)$ satisfies

$$x^*(h_k(t))=\frac{c_1}{p_k(t)}+\frac{x^*(t)}{p_k(t)}+\sum_{i\neq k}\frac{-p_i(t)}{p_k(t)}x^*(h_i(t))$$
$$-\frac{1}{p_k(t)}\sum_{j=1}^{n}\int_t^{\infty} f_j(s\,,x^*(g_j(s)))\,ds+\frac{1}{p_k(t)}\int_t^{\infty}Q(s)\,ds$$

and then

$$\left(x^*(t)-\sum_{i=1}^{m}p_i(t)x^*(h_i(t))\right)'+\sum_{j=1}^{n}f_j(t\,,x^*(g_j(t)))=Q(t)\,.$$

Hence $x^*(t)$ is a bounded nonoscillatory solution of (1) and when $p_k(t)$ is bounded, $\lim_{t\to\infty}\inf|x^*(t)|>0$. If B(ii) holds, set

$$S=\left\{x(t)\in C[t_0\,,\infty):\begin{array}{ll}\dfrac{-c}{p_k(H(t))}\leq x(t)\leq\dfrac{-|d|}{p_k(H(t))}\,, & t\geq T\\ \dfrac{-c}{p_k(H(T))}\leq x(t)\leq\dfrac{-|d|}{p_k(H(T))}\,, & t_0\leq t<T\end{array}\right\}$$

where $0<c<r|d|$. Choose a positive number c_1 such that $c+(1-r)|d|<c_1<r(c+(2-r)|d|)/2$. T is sufficiently large such that when $t\geq T\geq t_0$, $g_j(t)\geq t_0(j\in I_n)$ and

$$\sum_{j=1}^{n}\int_T^{\infty}|f_j(s\,,d)|\,ds+\int_T^{\infty}|Q(s)|\,ds\leq c_1-c-(1-r)|d|\,, \qquad (10)$$

$$\sum_{j=1}^{n}\int_T^{\infty}L_j(s)\,ds\leq\frac{r}{2} \qquad (11)$$

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Define a mapping as follows:

$$(Mx)(t) = \begin{cases} \frac{-c_1}{p_k(H(t))} + \frac{x(H(t))}{p_k(H(t))} + \sum_{i \neq k} \frac{-p_i(H(t))}{p_k(H(t))} x(h_i(H(t))) \\ \qquad + \frac{-1}{p_k(H(t))} \sum_{j=1}^{n} \int_{H(t)}^{\infty} f_j(s, x(g_j(s)))\, ds \\ \qquad + \frac{1}{p_k(H(t))} \int_{H(t)}^{\infty} Q(s)\, ds, \qquad t \geq T \\ (Mx)(T), \qquad t_0 \leq t < T. \end{cases}$$

Using (6), (7), (10), and (11) and by a proof similar to that of B(i), we can complete our proof.

REMARK 1. If $p_k(t)$ is a constant, it is easy to see that (6) is identical with (7) and Theorem 2 is still true when we omit the conditions $h_k(t) \leq t$ and $h_i(H(t)) \geq t$ $(i \neq k)$.

The following discussion is about the nonoscillation of (1) when $p_i(t)$ are oscillatory functions.

THEOREM 3. *Assume that*

$$C: \qquad \sum_{i=1}^{m} |p_i(t)| \leq 1 - r, \qquad t \geq t_0, \ \left(\frac{1}{2} < r < 1\right)$$

holds and $f_j(t, x) (j \in I_n)$, $Q(t)$ *satisfy conditions* (a)–(c). *If* (2) *holds, then* (1) *has a bounded nonoscillatory solution* $x(t)$ *with* $\lim_{t\to\infty} \inf |x(t)| > 0$.

PROOF. Set $S = \{x(t) \in C[t_0, \infty) : c \leq x(t) \leq |d|, \ t \geq t_0\}$ where $0 < c < (2r-1)|d|$. Define a mapping as follows:

$$(Mx)(t) = \begin{cases} c_1 + \sum_{i=1}^{m} p_i(t) x(h_i(t)) \\ \quad + \sum_{j=1}^{n} \int_t^{\infty} f_j(s, x(g_j(s)))\, ds - \int_t^{\infty} Q(s)\, ds, \qquad t \geq T, \\ (Mx)(T), \qquad t_0 \leq t < T; \end{cases}$$

where c_1 satisfies $c + (1-r)|d| < c_1 < r|d|$ and T is sufficiently large such that when $t \geq T \geq t_0$, $h_i(t) \geq t_0$ $(i \neq k)$, $g_j(t) \geq t_0$ $(j \in I_n)$ and

$$\sum_{j=1}^{n} \int_T^{\infty} |f_j(s, d)|\, sd + \int_T^{\infty} |Q(s)|\, ds$$
$$\leq \min\{c_1 - c - (1-r)|d|, \ r|d| - c_1\}, \tag{12}$$

$$\sum_{j=1}^{n} \int_T^{\infty} L_j(s)\, ds \leq \frac{r}{2}. \tag{13}$$

The rest of the proof is similar to that of Theorem 1 and 2.

Theorem 4. *Assume that*

D: *There is some integer* k $(1 \le k \le m)$ *such that* $h_k(t)$ *is strictly increasing,* $h_k(t) \le t$, $h_i(H(t)) \ge t$ $(i \ne k)$, $|p_k(t)| > 1$,

$$\left|\frac{1}{p_k(t)}\right| + \sum_{i\ne k}\left|\frac{p_i(t)}{p_k(t)}\right| \le 1 - r, \qquad t \ge t_0,\ \frac{1}{2} < r < 1, \tag{14}$$

and when $t \ge t_0$ *and* $H(h_i(t)) \ge t_0 (i \ne k)$,

$$\left|\frac{1}{p_k(H(t))}\right| + \sum_{i\ne k}\left|\frac{p_i(t)}{p_k(H(h_i(t)))}\right| \le 1 - r, \qquad \left(\frac{1}{2} < r < 1\right), \tag{15}$$

where $H(t)$ *is the inverse function of* $h_k(t)$;
holds and $f_j(t, x)$ $(j \in I_n)$ *and* $Q(t)$ *satisfy* (a)–(c). *If* (2) *holds, then* (1) *has a bounded nonoscillatory solution* $x(t)$, *and when* $p_k(t)$ *is bounded,* $\lim_{t\to\infty} \inf |x(t)| > 0$.

Proof. Set

$$S = \left\{ x(t) \in C[t_0, \infty) : \begin{array}{ll} \dfrac{c}{|p_k(H(t))|} \le x(t) \le \dfrac{|d|}{|p_k(H(t))|}, & t \ge T \\ \dfrac{c}{|p_k(H(T))|} \le x(t) \le \dfrac{|d|}{|p_k(H(T))|}, & t_0 \le t < T \end{array} \right\}$$

where $0 < c < (2r-1)|d|$. Choose c_1 such that $c + (1-r)|d| < c_1 < r|d|$. T is sufficiently large such that when $t \ge T \ge t_0$, $g_j(t) \ge t_0$ $(j \in I_n)$, (12) and (13) hold. Define a mapping as follows:

$$(Mx)(t) = \begin{cases} \dfrac{c_1}{|p_k(H(t))|} + \dfrac{x(H(t))}{p_k(H(t))} + \sum_{i\ne k} \dfrac{-p_i(H(t))}{p_k(H(t))} x(h_i(H(t))) \\ \qquad + \dfrac{-1}{p_k(H(t))} \sum_{j=1}^{n} \int_{H(t)}^{\infty} f_j(s, x(g_j(s)))\, ds \\ \qquad + \dfrac{1}{p_k(H(t))} \int_{H(t)}^{\infty} Q(s)\, ds, \qquad t \ge T \\ (Mx)(T), \qquad t_0 \le t < T. \end{cases}$$

The rest of the proof is similar to that of Theorem 1 and 2.

Remark 2. If $p_k(t)$ is constant, (14) is identical with (15) and Theorem 4 is still true when we omit the conditions $h_k(t) \le t$ and $h_i(H(t)) \ge t$ $(i \ne k)$.
Consider the linear equation

$$\left(x(t) - \sum_{i=1}^{m} p_i(t) x(h_i(t))\right)' + \sum_{j=1}^{n} q_j(t) x(g_j(t)) = Q(t), \qquad t \ge t_0 \tag{1$'$}$$

where $p_i(t), q_j(t), h_i(t), g_j(t), Q(t) \in C[t_0, \infty)$, $\lim_{t\to\infty} h_i(t) = \infty$ $(i \in I_m)$, $\lim_{t\to\infty} g_j(t) = \infty$ $(j \in I_n)$. From Theorem 1–4, we have

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Corollary 1. *Assume that one of* A(i), A(ii), B(i), B(ii), C *and* D *holds and that* $p_k(t)$ *of* B(i), B(ii) *and* D *is bounded. Further assume that* $Q(t)$ *satisfies* (c). *If*

$$\sum_{j=1}^{n} \int^{\infty} |q_j(s)|\, ds < \infty \tag{2$'$}$$

then $(1)'$ *has a bounded nonoscillatory solution* $x(t)$ *with* $\lim_{t\to\infty} \inf |x(t)| > 0$.

Lemma 1. *Assume that* A(ii) *or* B(ii) *holds and* $Q(t)$ *satisfies* (c). *Further assume that* $f_j(t, x)$ $(j \in I_n)$ *satisfy* (a) *and*

$$x f_j(t, x) \geq 0 \qquad (x \neq 0), \ \ j \in I_n. \tag{d}$$

If (1) *has a nonoscillatory solution* $x(t)$ *with* $\lim_{t\to\infty} \inf |x(t)| > 0$, *then* (2) *holds.*

Proof. Without loss of generality, assume that $x(t) \geq d > 0$ $(t \geq t_0)$. Set

$$y(t) = x(t) - \sum_{i=1}^{m} p_i(t) x(h_i(t)).$$

If (2) does not hold, we have for $t \geq T$,

$$y(t) - y(T) \leq \int_T^t Q(s)\, ds - \sum_{j=1}^{n} \int_T^t f_j(s, d)\, ds \to -\infty \qquad (t \to \infty).$$

Then $\lim_{t\to\infty} y(t) = -\infty$. This is a contradiction. The proof is complete.

Lemma 2. *Assume that* A(i) *or* C *holds,* $h_i(t) \leq t$ $(i \in I_m)$ *and that* $Q(t)$ *satisfies* (c) *and* $f_j(t, x)$ $(j \in I_n)$ *satisfy* (a) *and* (d). *If* (1) *has a nonoscillatory solution* $x(t)$ *with* $\lim_{t\to\infty} \inf |x(t)| > 0$, *then* (2) *holds.*

Proof. Without loss of generality, assume that $x(t) \geq d > 0$ $(t \geq t_0)$. If (2) doesn't hold, we have $\lim_{t\to\infty} y(t) = -\infty$. Then $x(t)$ is unbounded. There is a sequence $t_k \to \infty$ $(k \to \infty)$ such that $x(t_k) = \max_{t \leq t_k} x(t)$. Then

$$y(t_k) = x(t_k) - \sum_{i=1}^{m} p_i(t_k) x(h_i(t_k)) \geq \left(1 - \sum_{i=1}^{m} |p_i(t_k)|\right) x(t_k) > 0$$

which is a contradiction. The proof is complete.

The following lemma is obvious. We omit the proof.

https://doi.org/10.1017/S0334270000008419 Published online by Cambridge University Press

LEMMA 3. *Assume that one of* A(i), A(ii), B(i), B(ii), C *and* D *holds and that* $p_k(t)$ *of* B(i), B(ii) *and* D *is bounded. Further assume that* $Q(t)$ *satisfies* (c) *and* $f_j(t, x)$ $(j \in I_n)$ *satisfy* (a) *and* (d). *If* (1) *has a bounded nonoscillatory solution* $x(t)$ *with* $\lim_{t\to\infty} \inf|x(t)| > 0$, *then* (2) *holds.*

From Lemma 1–3 and Theorem 1–4, we have immediately

THEOREM 5. *Assume that* A(ii) *or* B(ii) *holds and that* $p_k(t)$ *of* B(ii) *is bounded. Further assume that* $Q(t)$ *satisfies* (c) *and* $f_j(t, x)$ $(j \in I_n)$ *satisfy* (a), (b) *and* (d). *Then* (2) *is a necessary and sufficient condition for* (1) *to have a nonoscillatory solution* $x(t)$ *with* $\lim_{t\to\infty} \inf|x(t)| > 0$.

THEOREM 6. *Assume that* A(i) *or* C *holds and* $h_i(t) \le t$ $(i \in I_m)$. *Further assume that* $Q(t)$ *satisfies* (c) *and* $f_j(t, x)$ $(j \in I_n)$ *satisfy* (a), (b) *and* (d). *Then* (2) *is a necessary and sufficient condition for* (1) *to have a nonoscillatory solution* $x(t)$ *with* $\lim_{t\to\infty} \inf|x(t)| > 0$.

THEOREM 7. *Assume that one of* A(i), A(ii), B(i), B(ii), C *and* D *holds and that* $p_k(t)$ *of* B(i), B(ii) *and* D *is bounded. Further assume that* $Q(t)$ *satisfies* (c) *and* $f_j(t, x)$ $(j \in I_n)$ *satisfy* (a), (b) *and* (d). *Then* (2) *is a necessary and sufficient condition for* (1) *to have a bounded nonoscillatory solution* $x(t)$ *with* $\lim_{t\to\infty} \inf|x(t)| > 0$.

COROLLARY 2. *Let* $q_j(t) \ge 0$ $(j \in I_n)$. *Assume that* A(ii) *or* B(ii) *holds and that* $p_k(t)$ *of* B(ii) *is bounded. Further assume that* $Q(t)$ *satisfies* (c). *Then* $(2)'$ *is a necessary and sufficient condition for* $(1)'$ *to have a nonoscillatory solution* $x(t)$ *with* $\lim_{t\to\infty} \inf|x(t)| > 0$.

COROLLARY 3. *Let* $q_j(t) \ge 0$ $(j \in I_n)$. *Assume that* A(i) *or* C *holds and* $h_i(t) \le t$ $(i \in I_m)$. *Further assume that* $Q(t)$ *satisfies* (c). *Then* $(2)'$ *is a necessary and sufficient condition for* $(1)'$ *to have a nonoscillatory solution* $x(t)$ *with* $\lim_{t\to\infty} \inf|x(t)| > 0$.

COROLLARY 4. *Let* $q_j(t) \ge 0$ $(j \in I_n)$. *Assume that one of* A(i), A(ii), B(i), B(ii), C *and* D *holds and that* $p_k(t)$ *of* B(i), B(ii) *and* D *is bounded. Further assume that* $Q(t)$ *satisfies* (c). *Then* $(2)'$ *is a necessary and sufficient condition for* $(1)'$ *to have a bounded nonoscillatory solution* $x(t)$ *with* $\lim_{t\to\infty} \inf|x(t)| > 0$.

3. Examples

EXAMPLE 1. Consider the equation

$$\left(x(t) \pm \frac{3}{5}x(h_1(t)) \pm \frac{1}{6+t^2}x(h_2(t))\right)' + \frac{1}{t^2}x^3(g(t)) = e^{-t} \qquad (t \geq t_0 > 0) \tag{16}$$

where $h_i(t)$ $(i = 1, 2)$, $g(t) \in C[t_0, \infty)$, $\lim_{t\to\infty} h_i(t) = \infty$ $(i = 1, 2)$, $\lim_{t\to\infty} g(t) = \infty$. Let $p_1(t) = \mp\frac{3}{5}$, $p_2(t) = \mp\frac{1}{6+t^2}$, $Q(t) = e^{-t}$, $f(t, x) = \frac{x^3}{t^2}$. Then

$$\mp(p_1(t) + p_2(t)) \leq \frac{4}{5}, \qquad \int^{\infty} |Q(t)|\, dt < \infty$$

$$|f(t, x)| = t^{-2}|x|^3 \leq t^{-2}|y|^3 = |f(t, y)| \quad \text{when } |x| \leq |y|.$$

When $x, y \in L = [c, d]$ $(0 < c < d)$,

$$|f(t, x) - f(t, y)| = t^{-2}|x^2 + xy + y^2|\,|x - y| \leq 3d^2t^{-2}|x - y|.$$

Let $L(t) = 3d^2t^{-2}$, then $\int^{\infty} L(t)\, dt < \infty$, and

$$\int^{\infty} |f(t, d)|\, dt = \int^{\infty} |d|^3 t^{-2}\, dt < \infty, \qquad (d \neq 0).$$

According to Theorem 1, (16) has a bounded nonoscillatory solution $x(t)$ with $\lim_{t\to\infty} \inf|x(t)| > 0$.

EXAMPLE 2. Consider the equation

$$(x(t) - 3(1+t^2)x(t-r_1) + t^2x(t-r_2))' + t^{-2}x^2(g(t)) = 0, \qquad t \geq t_0 > 0, \tag{17}$$

where $g(t) \in C[t_0, \infty)$, $\lim_{t\to\infty} g(t) = \infty$, $r_1 > 0$, $r_1 > r_2$. Let $p_1(t) = 3(1+t^2) > 1$, $p_2(t) = -t^2 < 0$, $h_1(t) = t - r_1 < t$, $h_2(t) = t - r_2$, $H(t) = t + r_1$ is the inverse function of $h_1(t)$, then $h_2(H(t)) = t + r_1 - r_2 > t$.

$$\left|\frac{1}{p_1(t)} + \frac{-p_2(t)}{p_1(t)}\right| = \frac{1}{3}$$

$$\left|\frac{1}{p_1(H(t))} + \frac{-p_2(t)}{p_1(H(h_2(t)))}\right| \leq \left|\frac{1}{3(1+t^2)} + \frac{t^2}{3(1+t^2)}\right| = \frac{1}{3}, \qquad t \geq t_0$$

Let $f(t, x) = t^{-2}x^2$, $L = [c, d]$ $(0 < c < d)$, then for $x, y \in L$,

$$|f(t, x) - f(t, y)| = t^{-2}|x + y|\,|x - y| \leq 2dt^{-2}|x - y|.$$

Set $L(t) = 2dt^{-2}$, then $\int^{\infty} L(t)dt < \infty$. It is easy to see that $f(t, x)$ also satisfies (a) and (2). According to Theorem 2, (17) has a bounded nonoscillatory solution.

EXAMPLE 3. Consider the equation

$$\left(x(t) + \frac{1}{3}\sin tx(h(t))\right)' + \frac{\cos t}{1+t^2}x(g(t)) = Q(t), \qquad t \geq 0, \qquad (18)$$

where $h(t), g(t), Q(t) \in C[0, \infty)$, $\lim_{t\to\infty} h(t) = \infty$, $\lim_{t\to\infty} g(t) = \infty$, and $Q(t)$ satisfies (c). According to Theorem 3, (18) has a bounded nonoscillatory solution $x(t)$ with $\lim_{t\to\infty} \inf |x(t)| > 0$.

EXAMPLE 4. Consider the equation

$$(x(t) - 5x(h_1(t)) + \cos tx(h_2(t)))' + \frac{\cos t}{1+t^2}|x(g(t))|^\alpha = 0, \qquad t \geq 0, \qquad (19)$$

where $h_1(t), h_2(t), g(t) \in C[t_0, \infty)$, $h_1(t)$ is strictly increasing, $\lim_{t\to\infty} h_i(t) = \infty$ $(i = 1, 2)$, $\lim_{t\to\infty} g(t) = \infty$, $0 < \alpha < 1$. According to Theorem 4 and Remark 2, (19) has a bounded nonoscillatory solution $x(t)$ with $\lim_{t\to\infty} \inf |x(t)| > 0$.

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