BULL. AUSTRAL. MATH. SOC. 35J20
VOL. 53 (1996) [235–247]

SEMILINEAR PROBLEMS ON THE HALF SPACE WITH A HOLE

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In this article we prove that there is a positive solution in $H_0^1(\Omega)$ of the equation $-\Delta u + \lambda u = |u|^{p-2}u$ in Ω where Ω is the half space with a hole, $\lambda > 0$ and $2 < p < 2N/(N-2)$.

1. INTRODUCTION

In this article we use the following notation:

$\mathbf{R}^N$: the N-dimensional Euclidean space, $N \geqslant 3$,
$\mathbf{R}^N_+ = \{(x', x_N) \in \mathbf{R}^{N-1} \times \mathbf{R} \mid 0 < x_N < \infty\}$: the upper half space,
$\mathbf{R}^N_- = \{(x', x_N) \in \mathbf{R}^{N-1} \times \mathbf{R} \mid -\infty < x_N < 0\}$: the lower half space,
Ω_r an unbounded smooth domain such that $\overline{\Omega}_r \subset \mathbf{R}^N_+$, $a_r = (a, r) \notin \overline{\Omega}_r$, and its complement $\overline{\Omega}_r^c$ is contained in a ball $B_\rho(a_r)$ centred at a_r with radius ρ : the upper half space with a hole.
D: One of $\mathbf{R}^N, \mathbf{R}^N_+$ and Ω_r.
For $\lambda > 0$ and $2 < p < 2N/(N-2)$, consider the semilinear elliptic equation:

$$(1_D) \qquad \begin{cases} -\Delta u + \lambda u = |u|^{p-2}u & \text{in} \quad \mathrm{D} \\ u \in H_0^1(\mathrm{D}), \end{cases}$$

$H_0^1(\mathrm{D})$: the usual Sobolev space on D under the norm

$$\|u\|_D^2 = \int_D \left(|\nabla u|^2 + \lambda u^2\right).$$

For $u \in H_0^1(\mathrm{D})$, define

$$f_D(u) = \int_D \left(|\nabla u|^2 + \lambda u^2\right),$$

$$M_D = \left\{u \in H_0^1(D) \,\middle|\, \int_D |u|^p = 1\right\},$$

$$\alpha_D = \inf\{f_D(u) \mid u \in M_D\},$$

$$F_D(u) = \frac{1}{2}\int_D \left(|\nabla u|^2 + \lambda u^2\right) - \frac{1}{p}\int_D |u|^p.$$

Received 1st May, 1995
I would like to express my gratitude to the referee for his valuable suggestions.

https://doi.org/10.1017/S0004972700016956 Published online by Cambridge University Press

Write $\|\cdot\|$, f, M, α, F for $\|\cdot\|_{\mathbf{R}^N}$, $f_{\mathbf{R}^N}$, $M_{\mathbf{R}^N}$, $\alpha_{\mathbf{R}^N}$, $F_{\mathbf{R}^N}$, respectively.

The motivation to study our problem is as follows: by applying the compactness of the embedding $H_r^1(\mathbf{R}^N) \hookrightarrow L^p(\mathbf{R}^N)$, where $H_r^1(\mathbf{R}^N)$ consists of the radially symmetric functions in $H^1(\mathbf{R}^N)$, Berestycki-Lions [4] proved that α is achieved, and hence concluded that there is a positive solution of equation $(1_{\mathbf{R}^N})$. Gidas-Ni-Nirenberg [9] proved that every positive solution u of equation $(1_{\mathbf{R}^N})$ is radially symmetric with respect to some point in $\mathbf{R}^N$ satisfying

$$\text{(1-1)} \qquad \begin{cases} u(r)r^{(N-1)/2}e^{\sqrt{\lambda}r} = \gamma + o(1) & \text{as} \quad r \to \infty \\ u'(r)r^{(N-1)/2}e^{\sqrt{\lambda}r} = -\sqrt{\lambda}\gamma + o(1) & \text{as} \quad r \to \infty \end{cases}$$

where $\gamma > 0$ a constant. Kwong [11] proved that the positive solution of $(1_{\mathbf{R}^N})$ is unique up to translations. Throughout this article denote by $\overline{u}$ the unique solution of equation $(1_{\mathbf{R}^N})$ which attains its maximum at 0, $\int_{\mathbf{R}^N} |\overline{u}|^p = 1, \|\overline{u}\|^2 = \alpha$, and satisfies (1-1).

Esteban-Lions [8] used the infinitesimal $\frac{\partial}{\partial y}$ of the translation operators to derive an important integral identity for the equation $-\Delta u = f(u)$ in an unbounded domain with boundary Γ:

$$\int_\Gamma n_i(x)\, |\nabla u|^2\, ds = 0 \quad \text{for} \quad 1 \leqslant i \leqslant N.$$

Let $\Omega_1 = \{(x', x_N) \in \mathbf{R}^{N-1} \times \mathbf{R} \mid |x'| < 1, 0 < x_N\}$ be an upper half strip. Two of its consequences are that there does not exist any nontrivial solution neither in $H_0^1(\mathbf{R}_+^N)$ of equation $\left(1_{\mathbf{R}_+^N}\right)$ nor in $H_0^1(\Omega_1)$ of equation (1_{Ω_1}). Such a surprising result attracted mathematicians to study the equations on the half space $\mathbf{R}_+^N$ and on Ω_1. Ai-Zhu [1] proved that there are positive solutions of the equation

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2}\, u & \text{in } \mathbf{R}_+^N \\ u > 0 & \text{in } \mathbf{R}_+^N \\ u(x', 0) = f(x') & \text{on } \partial\mathbf{R}_+^N, \end{cases}$$

where $f \geqslant 0, f \not\equiv 0$ in $H^{1/2}(\mathbf{R}^{N-1}) \cap L^\infty(\mathbf{R}^{N-1})$. In 1992, the author gave a talk in the second nonlinear France-Taiwan PDE Conference held in Paris. We proved that if r is large, $\Omega_2 = \Omega_1 \cup B_r(0)$ the upper half strip adding a big ball, then there is a positive solution in $H_0^1(\Omega_2)$ of equation (1_{Ω_2}) (see Lien-Tzeng-Wang [12, Example 5.6, p.1296]). In my talk, Berestycki asked the following problem: is there any positive solution of the equation on the upper half strip with a hole? We have only partial result for the Berestycki problem. However in this article, we try to answer a related problem affirmatively:

Theorem A. *There is $\rho_0 > 0$ and $r_0 > 0$ such that if $0 < \rho \leqslant \rho_0$ and $r \geqslant r_0$ then there is a positive solution of equation (1_{Ω_r}).*

To prove Theorem A we use a higher energy process through a barycentre function. Such a process was first used by Coron [7], then by Benci-Cerami [3], Grossi [10] and many others. In this article we adapt several tools from Benci-Cerami [3] and Grossi [10].

2. Existence of solutions

For $c \in \mathbf{R}$, a $(PS)_c$-sequence in $H_0^1(\Omega_r)$ for F is a sequence $\{u_n\}$ such that

$$\begin{aligned} F(u_n) &\longrightarrow c, \\ F'(u_n) &\longrightarrow 0 \quad \text{strongly in } H^{-1}(\Omega_r). \end{aligned}$$

We state a classical and interesting known decomposition theorem for a $(PS)_c$-sequence. For the convenience of the readers we sketch its proof.

Theorem 1. *Let $\{u_n\}$ be a $(PS)_c$-sequence in $H_0^1(\Omega_r)$ for F_{Ω_r}. Then there are a nonnegative integer k, k sequences $\{y_n^i\}$ of points of the form $(x_n', m_n + 1/2)$ for integers m_n, $i = 1, 2, \cdots, k$, u^0 in $H_0^1(\Omega_r)$ solving equation (1_{Ω_r}) and nontrivial functions $u^1, \cdots, u^k$ in $H^1(\mathbf{R}^N)$ solving equation $(1_{\mathbf{R}^N})$. Moreover there is a subsequence $\{u_n\}$ satisfying*

(1) $u_n(x) = u^0(x) + u^1(x - x_n^1) + \cdots + u^k(x - x_n^k) + o(1)$ strongly, where $x_n^i = y_n^1 + \cdots + y_n^i \to \infty$, $i = 1, 2, \cdots, k$.

(2) $\|u_n\|_{\Omega_r}^2 = \|u^0\|_{\Omega_r}^2 + \|u^1\|^2 + \cdots + \|u^k\|^2 + o(1)$,

(3) $F_{\Omega_r}(u_n) = F_{\Omega_r}(u^0) + F(u^1) + \cdots + F(u^k) + o(1)$.

If $u_n \geqslant 0$ for $n = 1, 2, \cdots$, then $u^1, \cdots, u^k$ can be chosen as positive solutions, and $u^0 \geqslant 0$.

Proof: Note that each function in $H_0^1(\Omega_r)$, by extending it to be 0 outside Ω_r, can be considered as a function in $H^1(\mathbf{R}^N)$. Since

$$\begin{aligned} F_{\Omega_r}(u_n) &= \frac{1}{2}\|u_n\|_{\Omega_r}^2 - \frac{1}{p}\|u_n\|_{L^p(\Omega_r)}^p = c + o(1), \\ F_{\Omega_r}'(u_n) &= \|u_n\|_{\Omega_r}^2 - \|u_n\|_{L^p(\Omega_r)}^p = o\left(\|u_n\|_{\Omega_r}\right), \end{aligned}$$

we see that $\{u_n\}$ is bounded in $H_0^1(\Omega_r)$. Take a subsequence $\{u_n\}$ and u^0 in $H_0^1(\Omega_r)$ such that $u_n \rightharpoonup u^0$ weakly in $H_0^1(\Omega_r)$, almost everywhere in Ω_r, and strongly in $L_{loc}^p(\Omega_r)$. Let $\varphi_n^1 = u_n - u^0$. By the Brezis-Lieb Lemma (see [5]) and the Vitali

Lemma, we have

$$-\Delta u^0 + \lambda u^0 = |u^0|^{p-2} u^0 \quad \text{in} \quad \Omega_r$$
$$\|\varphi_n^1\|^2_{\Omega_r} = \|u_n\|^2_{\Omega_r} - \|u^0\|^2_{\Omega_r} + o(1)$$
$$\|\varphi_n^1\|^p_{L^p(\Omega_r)} = \|u_n\|^p_{L^p(\Omega_r)} - \|u^0\|^p_{L^p(\Omega_r)} + o(1).$$
$$F_{\Omega_r}(\varphi_n^1) = F_{\Omega_r}(u_n) - F_{\Omega_r}(u^0) + o(1)$$
$$F'_{\Omega_r}(\varphi_n^1) = o(1) \qquad \text{strongly}.$$

CASE 1. If $\varphi_n^1 \to 0$ strongly, then

$$u_n(x) = u^0(x) + o(1) \qquad \text{strongly},$$
$$\|u_n\|^2_{\Omega_r} = \|u^0\|^2_{\Omega_r} + o(1),$$
$$F_{\Omega_r}(u_n) = F_{\Omega_r}(u^0) + o(1).$$

□

In order to prove the second case, we need the following lemma in which the proof follows from Bahri-Lions [**2**]:

Decompose $\mathbf{R}^N$ into nonoverlapping countable cubes Q_i with centres $(x', m + 1/2)$ for integers m and side length 1. Define the concentration function h_k of $|u_k|^2$ by

$$h_k = \sup_{|i|=0,1,2,\cdots} \int_{Q_i} |u_k|^2$$

LEMMA 2. *If $\{u_k\}$ is a bounded $(PS)_c$ sequence in $H^1(\mathbf{R}^N)$ such that $h_k \to 0$ as $k \to \infty$, then $u_k \to 0$ strongly in $H^1(\mathbf{R}^N)$.*

PROOF: For $2 < q < r < 2^* = 2N/(N-2)$, $q = (1-t)\cdot 2 + tr$, $t > 0$, $s = tr/2 \geqslant 1$. Now

$$\begin{aligned}
\int_{R^N} |u_k|^q &= \sum_i \int_{Q_i} |u_k|^{(1-t)\cdot 2} |u_k|^{tr} \\
&\leqslant \sum_i \left(\int_{Q_i} |u_k|^2\right)^{(1-t)} \left(\int_{Q_i} |u_k|^r\right)^t \\
&\leqslant (h_k)^{(1-t)} \sum_i \left(\int_{Q_i} |u_k|^r\right)^t \\
&\leqslant c(h_k)^{(1-t)} \sum_i \left(\int_{Q_i} |\nabla u_k|^2 + u_k^2\right)^{tr/2} \\
&\leqslant c(h_k)^{1-t} \left[\sum_i \int_{Q_i} \left(|\nabla u_k|^2 + u_k^2\right)\right]^{tr/2} \\
&\leqslant c(h_k)^{1-t} \left(\|u_k\|_{H^1(R^N)}\right)^{(tr)/2} \\
&\leqslant c(h_k)^{1-t} = o(1) \quad \text{as} \quad k \to \infty.
\end{aligned}$$

https://doi.org/10.1017/S0004972700016956 Published online by Cambridge University Press

By the $(PS)_c$ condition, we have

$$\|u_k\|^2_{H^1(R^N)} - \int_{R^N} |u_k|^{p+1} = \varepsilon_k \|u_k\|_{H^1(R^N)} = o(1)$$

where $\varepsilon_k = o(1)$. Since $\int_{R^N} |u_k|^{k+1} = o(1)$, we have

$$\|u_k\|_{H^1(R^N)} = o(1), \quad \text{as} \quad k \to \infty,$$

This completes the proof. □

CASE 2. If φ_n^1 does not converge to 0 strongly, then by Lemma 2 there is a subsequence $\{\varphi_n^1\}$ and $\delta > 0$ such that

$$\sup_{|i|=0,1,2,\cdots} \int_{Q_i} |u_k|^2 \geqslant \delta \text{ for } \quad n = 1, 2, \cdots.$$

where $\{Q_i\}$ are as in Lemma 2. For each n, find a Q_n^1 with centre y_n^1 of the form $(x_n', m_n + 1/2)$ such that

$$\|\varphi_n^1\|^2_{L^2(Q_n^1)} \geqslant \frac{\delta}{2}.$$

Take u^1 in $H^1(\mathbf{R}^N)$ and a subsequence $\{\varphi_n^1(x + y_n^1)\}$ satisfying $\varphi_n^1(x + y_n^1) \rightharpoonup u^1(x)$ weakly in $H^1(\mathbf{R}^N)$, almost everywhere in $\mathbf{R}^N$ and strongly in $L^p_{\ell oc}(\mathbf{R}^N)$. Since

$$\|u^1\|^2_{L^2(Q)} = \lim_{n\to\infty} \|\varphi_n^1(x + y_n^1)\|^2_{L^2(Q)} \geqslant \frac{\delta}{2},$$

where $Q = \{(x', x_N) \in \mathbf{R}^{N-1} \times \mathbf{R} \mid |x'| < 1/2, -1/2 < x_N < 1/2\}$, we have $u^1 \not\equiv 0$.

Let $\varphi_n^2(x) = \varphi_n^1(x + y_n^1) - u^1(x)$. Then $\varphi_n^2 \rightharpoonup 0$ weakly in $H^1(\mathbf{R}^N)$, almost everywhere in $\mathbf{R}^N$ and strongly in $L^p_{\ell oc}(\mathbf{R}^N)$. We obtain that u^1 solves $(1_{\mathbf{R}^N})$ and satisfies

$$\|u^1\|^2 \geqslant \alpha^{p/(p-2)} \tag{2-1}$$

and similar equalities as in Case 1 above. Continuing this process, by (2-1), we have to stop after a finite number of steps. This completes the proof. □

Let $\{u_n\} \subset M_{\Omega_r}$ satisfy $f_{\Omega_r}(u_n) = c + o(1)$. Set $v_n = c^{1/(p-2)} u_n$ for $n = 1, 2, \cdots$. Then we have

$$F_{\Omega_r}(v_n) = \left(\frac{1}{2} - \frac{1}{p}\right) c^{p/(p-2)} + o(1),$$
$$F'_{\Omega_r}(v_n) = o(1) \quad \text{strongly.}$$

COROLLARY 3. *Let* $\{u_n\} \subset M_{\Omega_r}$ *satisfy* $u_n \geqslant 0$, $f_{\Omega_r}(u_n) = c + o(1)$ *and* $\alpha < c < 2^{(p-2)/p}\alpha$. *Then* $\{u_n\}$ *contains a strongly convergent subsequence.*

PROOF: Set $v_n = c^{1/(p-2)}u_n$ for $n = 1, 2, \cdots$. Then

$$F_{\Omega_r}(v_n) = \left(\frac{1}{2} - \frac{1}{p}\right) c^{p/(p-2)} + o(1) \tag{2-2}$$
$$F'_{\Omega_r}(v_n) = o(1) \quad \text{strongly.}$$

By applying Theorem 1 we obtain solutions v^0 of equation (1_{Ω_r}) and positive solutions, $v^1, \cdots, v^k$ of equation $(1_{\mathbf{R}^N})$ and $\{x_n^i\}_{n=1}^{\infty}$ of the form $(x'_n, m_n + 1/2)$, m_n integers, $i = 1, \cdots, n$ such that

$$v_n(x) = v^0(x) + v^1(x - x_n^1) + \cdots + v^k(x - x_n^k) + o(1) \quad \text{strongly}$$
$$\|v_n\|_{\Omega_r}^2 = \|v^0\|_{\Omega_r}^2 + \|v^1\|^2 + \cdots + \|v^k\|^2 + o(1) \tag{2-3}$$
$$F_{\Omega_r}(v_n) = F_{\Omega_r}(v^0) + F(v^1) + \cdots + F(v^k) + o(1).$$

Note that if $v^i \geqslant 0$, $v^i \not\equiv 0$, $i = 1, 2, \cdots, k$, then we can take $v^i > 0$, v^i is unique up to a translation and $F(v^i) = (1/2 - 1/p)\alpha^{p/(p-2)}$ for $i = 1, 2, \cdots, k$. Therefore, by (2-2) and (2-3),

$$\left(\frac{1}{2} - \frac{1}{p}\right) c^{p/(p-2)} = F_{\Omega_r}(v^0) + k\left(\frac{1}{2} - \frac{1}{p}\right)\alpha^{p/(p-2)} + o(1).$$

If $v^0 \not\equiv 0$, then $v^0 > 0$ and $F_\Omega(v^0) > \left(1/2 - \frac{1}{p}\right)\alpha^{p/(p-1)}$ by Proposition 5 below. If $\alpha < c < 2^{(p-2)/p}\alpha$, then $k = 0, v^0 > 0$ and

$$v_n(x) = v^0(x) + o(1)$$

or

$$u_n(x) = u^0(x) + o(1)$$

where $u^0 = c^{-1/(p-2)}v^0$. Therefore $\{u_n\}$ contains a strongly convergent subsequence.

Take $\xi \in C^\infty(\mathbf{R}^+, \mathbf{R})$, $\eta \in C^\infty(\mathbf{R}, \mathbf{R})$ such that

$$\xi(t) = \begin{cases} 0 & 0 \leqslant t \leqslant \rho \\ 1 & t \geqslant 2\rho \end{cases}$$
$$\eta(t) = \begin{cases} 0 & t \leqslant 0 \\ 1 & t \geqslant 1 \end{cases}$$
$$0 \leqslant \xi \leqslant 1, \quad 0 \leqslant \eta \leqslant 1$$
$$f_y(x) = \xi(|x - a_r|)\eta(x_N)\overline{u}(x - y)$$
$$\varphi_y(x) = \frac{f_y(x)}{\|f_y\|_{L^p(\mathbf{R}^N)}} = c_y f_y(x) \quad \text{where } c_y = \frac{1}{\|f_y\|_{L^p(\mathbf{R}^N)}}.$$

Then $\varphi_y \in H_0^1(\Omega_r)$ and $\int_{\Omega_r} |\varphi_r|^p = 1$. Furthermore we have ∎

Lemma 4. *Let $y = (y', y_N)$, then*

(1) $\|f_y - \overline{u}(\cdot - y)\|_{L^p(\mathbf{R}^N)} = o(1)$ *as* $|y - a_r| \to \infty$ *and* $y_N \to \infty$, *or* $\rho \to 0$ *and* $y_N \to \infty$

(2) $\|f_y - \overline{u}(\cdot - y)\| = o(1)$ *as* $|y - a_r| \to \infty$ *and* $y_N \to \infty$ *or* $\rho \to 0$ *and* $y_N \to \infty$

Proof: (1)

$$\begin{aligned}
&\|f_y(x) - \overline{u}(x - y)\|^p_{L^p(\mathbf{R}^N)} \\
&\quad = \int_{\mathbf{R}^N} |\xi(|x - a_r|)\eta(x_N) - 1|^p \, |\overline{u}(x - y)|^p \, dx \\
&\quad \leqslant 2^p \int_{B_{2\rho}(a_r)\cup\{x_N \leqslant 1\}} |\overline{u}(x - y)|^p \, dx \\
&\quad = o(1) \quad \text{as} \quad |y - a_r| \to \infty \quad \text{and} \quad y_N \to \infty, \quad \text{or} \quad \rho \to 0 \quad \text{and} \quad y_N \to \infty.
\end{aligned}$$

(2)

$$\begin{aligned}
&\|f_y(x) - \overline{u}(x - y)\|^2 \\
&\quad = \|(\xi(|x - x_r|)\eta(x_N) - 1)\overline{u}(x - y)\|^2 \\
&\quad \leqslant \frac{c}{\rho} \int_{B_{2\rho}(a_r)\cup\{x_N \leqslant 1\}} \left(|\nabla \overline{u}(x - y)|^2 + |\overline{u}(x - y)|^2\right) \\
&\quad = o(1) \quad \text{as} \quad |y - a_r| \to \infty \quad \text{and} \quad y_N \to \infty, \quad \text{or} \quad \rho \to 0 \quad \text{and} \quad y_N \to \infty.
\end{aligned}$$ ∎

Proposition 5. *Equation* (1_{Ω_r}) *does not have any ground state solution.*

Proof: Note that $\alpha_{\Omega_r} \geqslant \alpha$ since each function in $H_0^1(\Omega)$ can be extended by 0 outside Ω_r. Take a sequence $\{y^n\}$ in Ω_r such that

$$|y^n - a_r| \to \infty \quad \text{and} \quad y_N^n \to \infty \quad \text{as} \quad n \to \infty.$$

Then, by Lemma 4,

$$\begin{aligned}
\|f_{y^n} - \overline{u}(\cdot - y^n)\|_{L^p(\mathbf{R}^N)} &= o(1) \quad \text{as} \quad n \to \infty \\
\|f_{y^n} - \overline{u}(\cdot - y^n)\| &= o(1) \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Thus $\{\varphi_{y^n}\} \subset H_0^1(\Omega)$ is such that

$$\int_{\Omega_r} |\varphi_{y^n}|^p = 1 \quad \text{for} \quad n = 1, 2, \cdots$$

$$\|\varphi_{y^n}\|^2 \longrightarrow \alpha,$$

or $\alpha_{\Omega_r} \leqslant \alpha$. We then conclude that $\alpha_{\Omega_r} = \alpha$. By the maximum principle, there does not exist any ground state solution of equation (1_{Ω_r}). In other words, if u is a solution of equation (1_{Ω_r}) satisfying $\int_{\Omega_r} |u|^p = 1$, then $\|u\|^2_{\Omega_r} > \alpha$. ∎

REMARK 6. By Lemma 4(1), there is $r_1 > 0$ such that

$$\frac{1}{2} \leqslant \|f_y\|_{L^p(\Omega_r)} \leqslant \frac{3}{2} \tag{2-4}$$

where $r \geqslant r_1$ and $|y - a_r| \geqslant r/2$ and $y_N \geqslant r/2$.

Set

$$\chi(t) = \begin{cases} 1 & \text{if } 0 \leqslant t \leqslant 1 \\ \dfrac{1}{t} & \text{if } 1 \leqslant t < \infty \end{cases}$$

and define $\beta : H^1(\mathbf{R}^N) \to \mathbf{R}^N$ by

$$\beta(u) = \int_{\mathbf{R}^N} u^2(x)\chi(|x|)x\,dx.$$

For $r \geqslant r_1$, let

$$V_r = \left\{ u \in H_0^1(\Omega_r) \,\middle|\, \int_{\Omega_r} |u|^p = 1,\ \beta(u) = a_r \right\},$$
$$c_r = \inf_{u \in V_r} \|u\|_{\Omega_r}^2 .$$

Then we have:

LEMMA 7. $c_r > \alpha$.

PROOF: It is easy to see that $c_r \geqslant \alpha$. Suppose $c_r = \alpha$. Take a sequence $\{v_m\} \subset H_0^1(\Omega_r)$ such that

$$\|v_m\|_{L^p(\Omega_r)} = 1,\ \beta(v_m) = a_r \quad \text{for} \quad m = 1, 2, \cdots,$$
$$\|v_m\|_{\Omega_r}^2 = \alpha + o(1).$$

Let $u_m = \alpha^{1/(p-2)} v_m$ for $m = 1, 2, \cdots$. Then

$$F_{\Omega_r}(u_m) = \left(\frac{1}{2} - \frac{1}{p}\right) \alpha^{p/(p-2)} + o(1)$$
$$F'_{\Omega_r}(u_m) = o(1) \quad \text{strongly}.$$

By the maximum principle, $\{u_m\}$ does not contain any convergent subsequence. By Theorem 1, there is a sequence $\{x_m\}$ of the form $\left(x'_m, m + \frac{1}{2}\right)$ for integers m such that

$$|x_m| \longrightarrow \infty$$
$$u_m(x) = \overline{u}(x - x_m) + o(1) \quad \text{strongly}.$$

Since $\overline{u}$ is radially symmetric, we may take m to be positive. We may assume that $|x_m| \geqslant 4$ from $m = 1, 2, \cdots$. Now

$$\begin{aligned}\langle \beta(\overline{u}(x - x_m)), x_m \rangle &= \int_{\mathbf{R}^N} \overline{u}^2 (x - x_m) \chi(|x|) \langle x, x_m \rangle dx \\ &= \int_{\mathbf{R}^N_+} \overline{u}^2 (x - x_m) \chi(|x|) \langle x, x_m \rangle dx \\ &\quad + \int_{(\mathbf{R}^N_-)} \overline{u}^2 (x - x_m) \chi(|x|) \langle x, x_m \rangle dx \\ &\geqslant \int_{B_1(x_m)} \overline{u}^2 (x - x_m) \chi(|x|) \langle x, x_m \rangle dx \\ &\quad + \int_{\mathbf{R}^N_-} \overline{u}^2 (x - x_m) \chi(|x|) \langle x, x_m \rangle dx.\end{aligned}$$

Note that there are $c_1 > 0$, $c_2 > 0$ such that for $x \in B_1(x_m)$, we have

$$\begin{aligned}&\overline{u}^2 (x - x_m) \geqslant c_1, \\ &\langle x, x_m \rangle \geqslant c_2 |x| |x_m| \quad \text{for} \quad m = 1, 2, \cdots.\end{aligned}$$

Thus

$$\begin{aligned}\int_{B_1(x_m)} \overline{u}^2 (x - x_m) \chi(|x|) \langle x, x_m \rangle dx &\geqslant c_1 c_2 \int_{B_1(x_m)} \chi(|x|) |x| |x_m| dx \\ &\geqslant c_3 |x_m|^{N+1}, \qquad c_3 > 0 \quad \text{a constant.}\end{aligned}$$

Next, for $0 \leqslant s < \infty$, by (1-1),

$$\overline{u}(s) s^{(N-1)/2} e^{\sqrt{\lambda} s} \leqslant c_4 \quad \text{for} \quad c_4 > 0.$$

Now

$$\begin{aligned}\int_{\mathbf{R}^N_-} \overline{u}^2 (x - x_m) \chi(|x|) \langle x, x_m \rangle dx &\leqslant c_4^2 \int_{\mathbf{R}^N_-} \frac{\chi(|x|) |x| |x_m|}{|x - x_m|^{(N-1)} e^{2\sqrt{\lambda}|x - x_m|}} \\ &\leqslant \frac{c_5}{e^{\sqrt{\lambda}|x_m|}}, \quad c_5 > 0 \quad \text{a constant.}\end{aligned}$$

Therefore

$$\langle \beta(\overline{u}(x - x_m)), x_m \rangle \geqslant c_3 |x_m|^{N+1} - \frac{c_5}{e^{\sqrt{\lambda}|x_m|}},$$

or

$$\langle \beta(\overline{u}(x - x_m)), \frac{x_m}{|x_m|} \rangle \geqslant c_3 |x_m|^N - \frac{c_5}{|x_m| e^{\sqrt{\lambda}|x_m|}}.$$

We conclude that

$$\begin{aligned}\alpha^{1/(p-2)}\,|a_r| &\geqslant \langle\beta(u_m),\frac{x_m}{|x_m|}\rangle\\ &= \langle\beta\left(\overline{u}(x-x_m)\right),\frac{x_m}{|x_m|}\rangle + o(1)\\ &\geqslant c_3\,|x_m|^N + o(1),\end{aligned}$$

a contradiction. Thus $c_r > \alpha$. ∎

REMARK 8. By Lemma 4 (2), there is $r_2 \geqslant r_1$ such that

$$\alpha < \|\varphi_y\|^2 < \frac{c_r+\alpha}{2} \tag{2-5}$$

where $r \geqslant r_2$ and $|y - a_r| \geqslant r/2$ and $y_N \geqslant r/2$.

LEMMA 9. *There is $r_3 \geqslant r_2$ such that if $r \geqslant r_3$, then*

$$\langle\beta(\varphi_y),y\rangle > 0 \quad \text{for} \quad y \in \partial\left(B_{r/2}\left(a_r\right)\right).$$

PROOF: By (2-4), $2/3 \leqslant c_y \leqslant 2$. For $r \geqslant r_2$, let

$$\begin{aligned}A_{((3/8)r,(5/8)r)} &= \left\{x \in \mathbf{R}^N \;\middle|\; \frac{3}{8}r \leqslant |x-a_r| \leqslant \frac{5}{8}r\right\},\\ \mathbf{R}^N_+(y) &= \left\{x \in \mathbf{R}^N \mid \langle x,y\rangle > 0\right\},\\ \mathbf{R}^N_-(y) &= \left\{x \in \mathbf{R}^N \mid \langle x,y\rangle < 0\right\}.\end{aligned}$$

$$\begin{aligned}\langle\beta(\varphi_y),y\rangle &= c_y\left[\int_{\mathbf{R}^N_+(y)} \xi^2\left(|x-a_r|\right)\eta^2(x_N)\overline{u}^2(x-y)\chi(|x|)\langle x,y\rangle dx\right.\\ &\qquad\left. + \int_{\mathbf{R}^N_-(y)} \xi^2(|x-a_r|)\eta^2(x_N)\overline{u}^2(x-y)\chi(|x|)\langle x,y\rangle dx\right]\\ &\geqslant \frac{2}{3}\left[\int_{A((3/8)r,(5/8)r)} \overline{u}^2(x-y)\chi(|x|)\langle x,y\rangle\right.\\ &\qquad + \int_{\mathbf{R}^N_-(y)} \overline{u}^2(x-y)\chi(|x|)\langle x,y\rangle dx\ .\end{aligned}$$

Now

$$\begin{aligned}\int_{A((3/8)r,(5/8)r)} \overline{u}^2(x-y)\chi(|x|)\langle x,y\rangle dx &\geqslant c_6 \int_{A((3/8)r,(5/8)r)} \chi(|x|)\,|x|\,|y|\,dx \quad \text{for } c_6>0\\ &\geqslant c_6\,|y|\left[\left(\frac{5}{8}r\right)^N-\left(\frac{3}{8}r\right)^N\right]\\ &\geqslant c_7 r^{N+1} \quad \text{for} \quad c_7>0.\\ \int_{\mathbf{R}^N_-(y)} \overline{u}^2(x-y)\chi(|x|)\langle x,y\rangle dx &\leqslant c_8 \int_{\mathbf{R}^N_-(y)} \frac{|y|}{|x-y|^{(N-1)}e^{2\sqrt{\lambda}|x-y|}}dx \quad \text{for } c_8>0\\ &\leqslant c_9\frac{1}{e^{\sqrt{\lambda}r}} \quad \text{for} \quad c_9>0.\end{aligned}$$

Therefore, there is $r_3 \geqslant r_2$, such that if $r \geqslant r_3$, $|y-a_r| = r/2$

$$\langle \beta(\varphi_y), y\rangle \geqslant c_7 r^{N+1} - c_8\frac{1}{e\sqrt{\lambda}r} > 0.$$

This completes the proof. □

By Lemma 4 and Lemma 9, fix $\rho_0>0$, $r_0 \geqslant r_3$ such that if $0<\rho\leqslant\rho_0$, $r\geqslant r_0$, then $\|\varphi_y\|^2_{\Omega_r} < 2^{(p-2)/p}\alpha$ for $y \in \overline{B_{r/2}}(a_r)$. From now on, fix ρ_0, r_0, for $r \geqslant r_0$. Let

$$B = \left\{\varphi_y \;\middle|\; |y-a_r| \leqslant \frac{r}{2}\right\},$$

$$\Gamma = \left\{h \in C(V_r,V_r) \;\middle|\; h(u)=u \quad \text{if} \quad \|u\|^2_{\Omega_r} < \frac{c_r+\alpha}{2}\right\}.$$

Lemma 10. *$h(B)\cap V_r \neq \emptyset$ for each $h\in\Gamma$.*

Proof: Let $h\in\Gamma$ and $H(x) = \beta\circ h\circ\varphi_x : \mathbf{R}^N \to \mathbf{R}^N$. Consider the homotopy, for $0\leqslant t\leqslant 1$,

$$F(t,x) = (1-t)H(x) + tI(x) \quad \text{for} \quad x\in\mathbf{R}^N.$$

If $x\in\partial\big(B_{r/2}(a_r)\big)$, then, by Remark 8 and Lemma 9,

$$\langle \beta(\varphi_x), x\rangle > 0,$$
$$\alpha < \|\varphi_x\|^2 < \frac{c_r+\alpha}{2}.$$

Then

$$\begin{aligned}\langle F(t,x),x\rangle &= \langle (1-t)H(x),x\rangle + \langle tx,x\rangle\\ &= (1-t)\langle\beta(\varphi_x),x\rangle + t\langle x,x\rangle\\ &> 0.\end{aligned}$$

https://doi.org/10.1017/S0004972700016956 Published online by Cambridge University Press

Thus $F(t,x) \neq 0$ for $x \in \partial(B_{r/2}(a_r))$. By the homotopic invariance of the degree

$$d(H(x), B_{r/2}(a_r), a_r) = d(I, B_{r/2}(a_r), a_r) = 1.$$

There is $x \in B_{r/2}(a_r)$ such that

$$a_r = H(x) = \beta(h \circ \varphi_x).$$

Thus $h(B) \cap V_r \neq \emptyset$ for each $h \in \Gamma$.

Now we are in the position to prove Theorem A: Consider the class of mappings

$$F = \left\{ h \in C\left(\overline{B_{r/2}(a_r)}\right), H^1(R_N) : \ h|_{\partial B_{r/2}(a_r)} = \varphi_y \right\}$$

and set

$$c = \inf_{h \in F} \sup_{y \in \overline{B_{r/2}(a_r)}} \|h(y)\|^2_{\Omega_r}$$

It follows from Lemmas 4-10, with the appropriate choice of r that

$$\alpha < c_r = \inf_{u \in V_\gamma} \|u\|^2_{\Omega_r} \leqslant c < 2^{(p-2)/p}\alpha$$

and

$$\max_{\partial B_{r/2}(a_r)} \|h(y)\|^2_{\Omega_r} < \max_{B_{r/2}(a_r)} \|h(y)\|^2_{\Omega_r}.$$

Theorem A then follows by applying the version of the mountain pass theorem from Brezis-Nirenberg [6]. ∎

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