





# Symmetric tensors on the intersection of two quadrics and Lagrangian fibration

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## ABSTRACT

Let  $X$  be an  $n$ -dimensional (smooth) intersection of two quadrics, and let  $T^*X$  be its cotangent bundle. We show that the algebra of symmetric tensors on  $X$  is a polynomial algebra in  $n$  variables. The corresponding map  $\Phi : T^*X \rightarrow \mathbb{C}^n$  is a Lagrangian fibration, which admits an explicit geometric description; its general fiber is a Zariski open subset of an abelian variety, which is a quotient of a hyperelliptic Jacobian by a 2-torsion subgroup. In dimension 3,  $\Phi$  is the Hitchin fibration of the moduli space of rank 2 bundles with fixed determinant on a curve of genus 2.

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## 1. Introduction

Let  $X \subset \mathbb{P}_{\mathbb{C}}^{n+2}$  be a smooth  $n$ -dimensional complete intersection of two quadrics, with  $n \geq 2$ , and let  $T^*X$  be its cotangent bundle. The  $\mathbb{C}$ -algebra  $H^0(T^*X, \mathcal{O}_{T^*X})$  is canonically isomorphic to the algebra of symmetric tensors  $H^0(X, \mathbf{S}^*T_X)$ . Recall that  $T^*X$  carries a canonical symplectic structure. Our main result is the following theorem:

THEOREM 1.1.

- (a) *The vector space  $W := H^0(X, \mathbf{S}^2T_X)$  has dimension  $n$ , and the natural map  $\mathbf{S}^*W \rightarrow H^0(X, \mathbf{S}^*T_X)$  is an isomorphism.*
- (b) *The corresponding map,  $\Phi : T^*X \rightarrow W^* \cong \mathbb{C}^n$ , is a Lagrangian fibration.*
- (c) *When  $X$  is general, the general fiber of  $\Phi$  is of the form  $A \setminus Z$ , where  $A$  is an abelian variety and  $\text{codim } Z \geq 2$ .*

We will give a precise geometric description of the map  $\Phi$  and of the abelian variety  $A$  in Sections 4 and 5.

### 1.1 Comments

(1) For  $n = 2$ , (a) follows from Theorem 5.1 in [DOL19], while (b) and (c) are proved in [KL22]. The proof is based on the isomorphism  $T_X \cong \Omega_X^1(1)$ . The theorem also follows from the fact that  $X$  is a moduli space for parabolic rank 2 bundles on  $\mathbb{P}^1$  [Cas15], so  $\Phi : T^*X \rightarrow \mathbb{C}^2$  is identified to the *Hitchin fibration* (see [BHK10]).

For  $n = 3$ ,  $X$  is isomorphic to the moduli space of vector bundles of rank 2 and fixed determinant of odd degree [New68]; again, the theorem follows from the properties of the Hitchin fibration (see Section 2). It would be interesting to have a modular interpretation of  $\Phi$  for  $n \geq 4$ . Note that the Hitchin map for  $G$ -bundles is homogeneous quadratic only when  $G$  is  $\text{SL}(2)$  or a product of copies of  $\text{SL}(2)$ , so this limits the possibilities of using it.

(2) The map  $\Phi$  is an example of an algebraically completely integrable system; see Remark 5.1. There is an abundant literature on such systems; see, for instance, [A96].

A classical example, the geodesic flow on an ellipsoid, is discussed in detail in [K80]. The corresponding Lagrangian fibration takes place on the cotangent bundle of *one* quadric; it is not related to our  $\Phi$ . However, some of the tools we use in Sections 4 and 5, in particular the variety  $\mathcal{X}$  and the family of planes  $\mathcal{F}$ , appear already in [K80] (with a different purpose).

(3) Such a situation is rather exceptional: Most varieties do not admit nonzero symmetric tensors (for instance, hypersurfaces of degree  $\geq 3$  [HLS22]); when they do, even for varieties as simple

as quadrics, the algebra of symmetric tensors is fairly complicated (see, for instance, [BLi24]). We do not have a conceptual explanation for the particularly simple behavior in our case.

(4) For  $n = 2$  or  $3$ , the generality assumption on  $X$  in (c) is unnecessary. It seems likely that this is the case for all  $n$ , but our method does not allow us to make that conclusion.

## 1.2 Strategy

We will first treat the case  $n = 3$ , which is independent of the rest of this article (Section 2). For the general case, we will develop two different approaches. In the first one we exhibit a natural  $n$ -dimensional subspace  $W \subset H^0(X, S^2 T_X)$ , from which we deduce a map  $\Phi: T^*X \rightarrow W^* \cong \mathbb{C}^n$  (Section 3). We then show that  $\Phi$  has the required properties, which implies (a), (b) and (c) for general  $X$  (5.1). In the second approach (Section 7), we directly prove (a) for all smooth  $X$ , by realizing  $X$  as a double covering of a quadric.

## 1.3 Notations

Throughout this article,  $X$  will be a smooth complete intersection of two quadrics in  $\mathbb{P}^{n+2}$ , with  $n \geq 2$ . We denote by  $T^*X$  its cotangent bundle and by  $\mathbb{P}T^*X$  its projectivisation in the geometric sense (not in the Grothendieck sense). If  $V$  is a vector space, we denote by  $\mathbb{P}(V)$  the associated projective space  $V \setminus \{0\}/\mathbb{C}^*$  parametrising 1-dimensional subspaces of  $V$ .

## 2. The case $n=3$

In this section we show how our general results can be obtained in the case  $n = 3$  by interpreting  $X$  as a moduli space.

As in Section 4.1 below, we associate to  $X$  a genus 2 curve  $C$  such that the variety of lines in  $X$  is isomorphic to  $JC$ . Let us fix a line bundle  $N$  on  $C$  of degree 1; then  $X$  is isomorphic to the moduli space  $\mathcal{M}$  of rank 2 stable vector bundles on  $C$  with determinant  $N$  [New68]. The cotangent bundle  $T^*\mathcal{M}$  is naturally identified with the moduli space of *Higgs bundles*; that is, pairs  $(E, u)$  with  $E \in \mathcal{M}$  and  $u: E \rightarrow E \otimes K_C$  a homomorphism with  $\text{Tr} u = 0$ . The *Hitchin map*  $\Phi: T^*\mathcal{M} \rightarrow H^0(K_C^2)$  associates to a pair  $(E, u)$  the section  $\det u$  of  $K_C^2$ . It is a Lagrangian fibration [Hit87].

Let  $\omega \in H^0(K_C^2)$ . We assume in what follows that  $\omega$  vanishes at 4 distinct points. Let  $C_\omega$  be the curve in the cotangent bundle  $T^*C$  defined by  $z^2 = \omega$ . The projection  $\pi: C_\omega \rightarrow C$  is a double covering branched along  $\text{div}(\omega)$ , and  $C_\omega$  is a smooth curve of genus 5. Let  $P$  be the Prym variety associated to  $\pi$ , that is, the kernel of the norm map  $\text{Nm}: JC_\omega \rightarrow JC$ ; it is a 3-dimensional abelian variety.

**PROPOSITION 2.1.** *The fibre  $\Phi^{-1}(\omega)$  is isomorphic to the complement of a curve in  $P$ .*

*Proof.* Recall that the map  $L \mapsto \pi_* L$  establishes a bijective correspondence between line bundles on  $C_\omega$  and rank 2 vector bundles  $E$  on  $C$  endowed with a homomorphism  $u: E \rightarrow E \otimes K_C$  such that  $u^2 = \omega \cdot \text{Id}_E$  or, equivalently,  $\text{Tr} u = 0$  and  $\det u = \omega$  (see, for instance, [BNR89]). To get  $(E, u)$  in  $\Phi^{-1}(\omega)$ , we have to impose  $\det E = N$  and  $E$  stable. Since  $\det \pi_* L = \text{Nm}(L) \otimes K_C^{-1}$ , the first condition means that  $L$  belongs to the translate  $P_N := \text{Nm}^{-1}(K_C \otimes N)$  of  $P$ .

Then the vector bundle  $\pi_* L$  is unstable if and only if it contains an invertible subsheaf  $M$  of degree 1; this is equivalent to saying that there is a nonzero map  $\pi^* M \rightarrow L$ ; that is,  $L = \pi^* M(p)$  for some point  $p \in C_\omega$ . The condition  $L \in P_N$  means that  $M^2(\pi(p)) = K_C \otimes N$ , so

$M$  is determined by  $p$  up to the 2-torsion of  $J\mathcal{C}$ . Thus the locus of line bundles  $L \in P_N$  such that  $\pi_*L$  is unstable is a curve.  $\square$

Let  $\rho: C \rightarrow \mathbb{P}^1$  be the canonical double covering, with  $B \subset \mathbb{P}^1$  its branch locus. Since the homomorphism  $S^2 H^0(K_C) \rightarrow H^0(K_C^2)$  is surjective, the divisor of  $\omega$  is of the form  $\rho^*(p+q)$ , for some  $p, q \in \mathbb{P}^1$ ; by assumption, we have  $p \neq q$  and  $p, q \notin B$ .

**PROPOSITION 2.2.** *Let  $\Gamma$  be the double covering of  $\mathbb{P}^1$  branched along  $B \cup \{p, q\}$ . There is an exact sequence*

$$0 \rightarrow \mathbb{Z}/2 \rightarrow J\Gamma \rightarrow P \rightarrow 0.$$

*Proof.* Let  $\chi: \mathbb{P}^1 \rightarrow \mathbb{P}^1$  be the double covering branched along  $\{p, q\}$ . Since  $\text{div}(\omega) = \rho^*(p+q)$ , there is a cartesian diagram of double coverings

$$\begin{array}{ccc} C_\omega & \xrightarrow{\xi} & \mathbb{P}^1 \\ \pi \downarrow & & \downarrow \chi \\ C & \xrightarrow{\rho} & \mathbb{P}^1 \end{array}$$

which gives rise to two commuting involutions  $\sigma, \tau$  of  $C_\omega$ , exchanging the two sheets of  $\pi$  and  $\xi$ , respectively. The field of rational functions on  $C_\omega$  is

$$\mathbb{C}(x, y, z) \quad \text{with} \quad y^2 = f(x), z^2 = g(x),$$

where  $f$  and  $g$  are polynomials with  $\text{div} f = B$  and  $\text{div} g = \{p, q\}$ . Then  $\sigma$  and  $\tau$  change the sign of  $y$  and  $z$ , respectively.

The involution  $\sigma\tau$  is fixed-point free, so the quotient  $\Gamma := C_\omega / \langle \sigma\tau \rangle$  has genus 3; its field of functions is  $\mathbb{C}(x, w)$ , with  $w = yz$  and  $w^2 = f(x)g(x)$ . We have again a cartesian square

$$\begin{array}{ccc} C_\omega & \xrightarrow{\varphi} & \Gamma \\ \pi \downarrow & & \downarrow \psi \\ C & \xrightarrow{\rho} & \mathbb{P}^1. \end{array}$$

Let  $\alpha \in J\Gamma$ . We have  $\text{Nm}_\pi \varphi^* \alpha = \rho^* \text{Nm}_\psi \alpha = 0$ ; hence,  $\varphi^*$  maps  $J\Gamma$  into  $P \subset JC_\omega$ . Since  $\varphi$  is étale, we have  $\text{Ker} \varphi^* = \mathbb{Z}/2$ ; since  $\dim J\Gamma = \dim P = 3$ ,  $\varphi^*$  is surjective.  $\square$

### 3. Definition of $\Phi$

Let  $Y$  be a smooth degree  $d$  hypersurface in  $\mathbb{P}^N$ , defined by an equation  $f=0$ . Recall that one associates to  $f$  a section  $h_f$  of  $S^2 \Omega_Y^1(d)$ , the *hessian* or *second fundamental form* of  $f$  [GH79]: at a point  $y$  of  $Y$ , the intersection of  $Y$  with the tangent hyperplane  $H$  to  $Y$  at  $y$  is a hypersurface in  $H$  singular at  $y$ , and  $h_f(y)$  is the degree 2 term in the Taylor expansion of  $f|_H$  at  $y$ .

Now let  $X \subset \mathbb{P}^{n+r}$  be a smooth complete intersection of  $r$  hypersurfaces of degree  $d$ ; let

$$V \subset H^0(\mathbb{P}^{n+r}, \mathcal{O}_{\mathbb{P}}(d))$$

be the  $r$ -dimensional subspace of degree  $d$  polynomials vanishing on  $X$ . By restricting  $h_f$ , for  $f \in V$ , to  $X$ , we get a linear map

$$V \otimes \mathcal{O}_X \longrightarrow \mathbb{S}^2 \Omega_X^1(d),$$

which gives at each point  $x \in X$  a linear space of quadratic forms on the tangent space  $T_x(X)$ . Note that when  $d=2$ , the corresponding quadrics in  $\mathbb{P}(T_x(X))$  can be viewed geometrically as follows: The projective space  $\mathbb{P}(T_x(X))$  can be identified with the space of lines in  $\mathbb{P}^{n+r}$  passing through  $x$  and tangent to  $X$ ; then for each  $q \in V$ , the quadric defined by  $h_q(x)$  parameterises the lines passing through  $x$  and contained in the quadric  $\{q=0\}$ .

Now we want to consider the ‘inverse’ of the quadratic form  $h_f(x)$  on  $T_x(X)$ ; that is, the form on  $T_x^*(X)$  given in coordinates by the cofactor matrix. Intrinsically, each  $f \in V$  gives a twisted symmetric morphism

$$h_f : T_X \longrightarrow \Omega_X^1(d),$$

which induces a twisted symmetric morphism on  $(n-1)$ -th exterior powers, namely,

$$\wedge^{n-1} h_f : \bigwedge^{n-1} T_X \longrightarrow \bigwedge^{n-1} \Omega_X^1((n-1)d).$$

We now observe that  $K_X = \mathcal{O}_X(-n-1-r+dr)$ ; hence

$$\bigwedge^{n-1} T_X \cong \Omega_X^1(n+1-r(d-1)) \text{ and } \bigwedge^{n-1} \Omega_X^1 \cong T_X(-n-1+r(d-1)),$$

so  $\wedge^{n-1} h_f$  induces a symmetric morphism from  $\Omega_X^1(n+1-r(d-1))$  to  $T_X((n-1)d-n-1+r(d-1))$ , hence provides a section

$$\wedge^{n-1} h_f \in H^0(X, \mathbb{S}^2 T_X(d(n+2r-1)-2(n+r+1))).$$

Being locally given by the cofactor matrix,  $\wedge^{n-1} h_f$  is homogeneous of degree  $n-1$  in  $f$ . Hence, we have constructed a linear map

$$\alpha : \mathbb{S}^{n-1} V \longrightarrow H^0(X, \mathbb{S}^2 T_X(d(n+2r-1)-2(n+r+1))) \text{ such that } \alpha(f^{n-1}) = \wedge^{n-1} h_f.$$

From now on, we restrict to the case  $d=2$ ,  $r=2$ , so  $X$  is the complete intersection of two quadrics, and the previous construction gives a linear map

$$\alpha : \mathbb{S}^{n-1} V \longrightarrow H^0(X, \mathbb{S}^2 T_X).$$

Using the canonical isomorphism  $H^0(T^*X, \mathcal{O}_{T^*X}) = H^0(X, \mathbb{S}^\bullet T_X)$ , we deduce from  $\alpha$  a morphism

$$\Phi : T^*X \longrightarrow \mathbb{S}^{n-1} V^* \cong \mathbb{C}^n.$$

We have  $\Phi(\lambda v) = \lambda^2 \Phi(v)$  for  $v \in T^*X$ ,  $\lambda \in \mathbb{C}$ , so  $\Phi$  induces a rational map

$$\varphi : \mathbb{P} T^*X \dashrightarrow \mathbb{P}^{n-1},$$

whose indeterminacy locus  $Z$  is the image of  $\Phi^{-1}(0)$ .

**PROPOSITION 3.1.**

- (1)  $\alpha$  is injective.
- (2)  $\Phi$  is surjective.
- (3) The image of  $Z$  by the structure map  $p : \mathbb{P} T^*X \rightarrow X$  is a proper subvariety of  $X$ .

*Proof.* Let  $x$  be a general point of  $X$ . We claim that the base locus in  $\mathbb{P}(T_x(X))$  of the pencil of quadratic forms  $\{h_q(x)\}_{q \in V}$  is smooth. Indeed, this locus can be viewed as the variety  $F_x$  of lines in  $X$  passing through  $x$ . Let  $F$  be the Fano variety of lines contained in  $X$ , and let

$$G \subset F \times X = \{(\ell, y) \mid y \in \ell\}.$$

Then  $F$  and therefore  $G$  are smooth [Reid72, Theorem 2.6], hence  $F_x$ , which is the fibre above  $x$  of the projection  $G \rightarrow X$ , is smooth since  $x$  is general. It follows that in an appropriate system of coordinates  $(k_1, \dots, k_n)$  of  $T_x(X)$ , the forms  $\{h_q(x)\}$  can be written as

$$t \sum k_i^2 + \sum \alpha_i k_i^2 \quad \text{with } \alpha_i \text{ distinct in } \mathbb{C}, t \in \mathbb{C}.$$

Then  $\wedge^{n-1} h_q(x)$  is given by the diagonal matrix with entries  $\beta_i := \prod_{j \neq i} (t + \alpha_j)$  ( $i = 1, \dots, n$ ).

These polynomials in  $t$  are linearly independent; hence, they generate the space of quadratic forms on  $T_x^*(X)$ , which are diagonal in the basis  $(k_i)$ . This linear system has dimension  $n$ , so  $\alpha$  is injective; it has no base point, so  $\varphi$  induces a finite, surjective morphism  $\mathbb{P}(T_x^*(X)) \rightarrow \mathbb{P}^{n-1}$ . Thus,  $\Phi$  is surjective, and  $Z \cap \mathbb{P}(T_x^*(X)) = \emptyset$ , which gives (2) and (3).  $\square$

We want to give a geometric construction of the rational map  $\varphi: \mathbb{P}T^*X \dashrightarrow \mathbb{P}^{n-1}$ . A point of  $\mathbb{P}T^*X$  is a pair  $(x, H)$ , where  $x \in X$  and  $H$  is a hyperplane in  $T_x(X)$ . Restricting the pencil  $\{h_q(x)\}_{q \in V}$  to  $H$  gives a pencil of quadrics on  $H$ , which for general  $(x, H)$  contains  $n-1$  singular quadrics  $q_1, \dots, q_{n-1}$ . The subset  $\{q_1, \dots, q_{n-1}\}$  of  $\mathbb{P}(V)$  corresponds to a point  $\varphi_{x,H}$  of  $\mathbb{P}(\mathbb{S}^{n-1}V^*)$ ; namely, the hyperplane in  $\mathbb{S}^{n-1}V$  spanned by  $q_1^{n-1}, \dots, q_{n-1}^{n-1}$ .

**PROPOSITION 3.2.**  $\varphi(x, H) = \varphi_{x,H}$ .

*Proof.* We can assume that  $x$  is general. We have seen that the restriction of  $\varphi$  to  $\mathbb{P}(T_x^*X)$  is the morphism given by the linear system of quadratic forms  $W \cong \mathbb{S}^{n-1}V$  spanned by the forms  $\wedge^{n-1} h_q(x)$ , for  $q \in V$ ; in other words,  $\varphi$  maps the point  $H$  of  $\mathbb{P}(T_x^*(X))$  to the hyperplane of forms in  $W$  vanishing at  $H$ .

On the other hand,  $\varphi_{x,H}$  is the hyperplane of  $\mathbb{S}^{n-1}V$  spanned by the  $q^{n-1}$  for those  $q \in V$  such that  $h_q(x)|_H$  is singular; this condition is equivalent to saying that the form  $\wedge^{n-1} h_q(x)$  on  $T_x^*X$  vanishes at  $H$ . Therefore,  $\varphi_{x,H}$  is spanned by quadratic forms vanishing at  $H$ , hence coincides with  $\varphi(x, H)$ .  $\square$

**COROLLARY 3.3.**  $\text{codim} Z \geq 2$ .

*Proof.* Suppose that  $Z$  contains a component  $Z_0$  of codimension 1; since  $p(Z) \neq X$ , we have  $Z_0 = p^{-1}(p(Z_0))$ . We claim that this is impossible; in fact,  $Z$  cannot contain a fibre  $p^{-1}(x)$ . Indeed, its doing this would mean that for  $q \in V$ , the form  $h_q(x)$  is singular along all hyperplanes  $H \subset T_x(X)$ ; that is,  $h_q(x)$  has rank  $\leq n-2$ . But the rank of  $h_q(x)$  is the rank of the restriction of  $q$  to the projective tangent subspace to  $X$  at  $x$ . Restricting a quadratic form to a hyperplane lowers its rank by up to two. Since a general  $q$  in  $V$  has rank  $n+3$ , its restriction to a codimension 2 subspace has rank  $\geq n-1$ .  $\square$

#### 4. Fibers of $\varphi$

In an appropriate system of coordinates  $(x_0, \dots, x_{n+2})$ , our variety  $X$  is defined by the equations  $q_1 = q_2 = 0$ , with

$$q_1 = \sum x_i^2 \quad q_2 = \sum \mu_i x_i^2 \quad , \text{ with } \mu_i \in \mathbb{C} \text{ distinct.}$$

Let  $\Pi = \mathbb{P}(V)$  ( $\cong \mathbb{P}^1$ ) be the pencil of quadrics containing  $X$ . We choose a coordinate  $t$  on  $\Pi$  so that the quadrics of  $\Pi$  are given by  $tq_1 - q_2 = 0$ . Then the singular quadrics of  $\Pi$  correspond to the points  $\mu_0, \dots, \mu_{n+2}$ .

The goal of this section is to describe the general fibre of the rational map  $\varphi: \mathbb{P}T^*X \dashrightarrow \mathbb{S}^{n-1}\Pi$  ( $\cong \mathbb{P}^{n-1}$ ). For  $\lambda = (\lambda_1, \dots, \lambda_{n-1}) \in \mathbb{S}^{n-1}\Pi$ , let  $C_{\mu,\lambda}$  denote the hyperelliptic curve  $y^2 = \prod(t - \mu_i) \prod(t - \lambda_j)$ , of genus  $n$ . We will then prove the following:

**PROPOSITION 4.1.** *For  $\lambda$  general in  $\mathbb{S}^{n-1}\Pi$ , the fibre  $\varphi^{-1}(\lambda)$  is birational to the quotient of the Jacobian  $JC_{\mu,\lambda}$  by the group  $\Gamma := \{\pm 1_{JC}\} \times \Gamma^+$ , where  $\Gamma^+ \cong (\mathbb{Z}/2\mathbb{Z})^{n-2}$  is a group of translations by 2-torsion elements.*

## 4.1 Odd-dimensional intersection of 2 quadrics

We briefly recall here the results of Reid's thesis ([Reid72]; see also [DR76]). Let  $Y \subset \mathbb{P}^{2g+1}$  be a smooth intersection of 2 quadrics, and let  $\Xi$  ( $\cong \mathbb{P}^1$ ) be the pencil of quadrics containing  $Y$ . Let  $\Sigma \subset \Xi$  be the subset of  $2g+2$  points corresponding to singular quadrics, and let  $C$  be the double covering of  $\Xi$  branched along  $\Sigma$ ; this is a hyperelliptic curve of genus  $g$ . The intermediate Jacobian  $JY$  of  $Y$  is isomorphic to  $JC$  (as principally polarized abelian varieties). The variety  $F$  of  $(g-1)$ -planes contained in  $Y$  is also isomorphic to  $JC$ , but this isomorphism is not canonical.

In an appropriate system of coordinates, the equations of  $Y$  are of the form

$$\sum x_i^2 = \sum \alpha_i x_i^2 = 0, \quad \text{with } \alpha_i \in \mathbb{C} \text{ distinct};$$

then  $\Sigma = \{\alpha_1, \dots, \alpha_{2g+2}\}$ . The group  $\Gamma := (\mathbb{Z}/2\mathbb{Z})^{2g+1}$  acts on  $Y$  (hence also on  $F$ ) by changing the signs of the coordinates. Let  $\Gamma^+ \subset \Gamma$  be the subgroup of elements that change an even number of coordinates. Choose an element  $\gamma \in \Gamma \setminus \Gamma^+$ ; there is an isomorphism  $F \xrightarrow{\sim} JC$  such that  $\gamma$  corresponds to  $(-1_{JC})$ . Then the image of  $\Gamma^+$  in  $\text{Aut}(JC)$  is the group  $T_2$  of translations by 2-torsion elements of  $JC$ , and the image of  $\Gamma$  is  $T_2 \times \{\pm 1_{JC}\}$  [DR76, Lemma 4.5].

## 4.2 An auxiliary construction

We consider the projective space  $\mathbb{P}^{2n+1}$  equipped with the system of homogeneous coordinates

$$x_0, \dots, x_{n+2}; y_1, \dots, y_{n-1}$$

and the affine space  $\mathbb{A}^{n-1}$  equipped with the affine coordinates  $\lambda_1, \dots, \lambda_{n-1}$ . Let

$$\mathcal{X} \subset \mathbb{P}^{2n+1} \times \mathbb{A}^{n-1}$$

be the complete intersection of the two quadrics with equations

$$Q_1 = Q_2 = 0 \quad \text{with} \quad Q_1 = \sum_{i=0}^{n+2} x_i^2 + \sum_{j=1}^{n-1} y_j^2, \quad Q_2 = \sum_{i=0}^{n+2} \mu_i x_i^2 + \sum_{j=1}^{n-1} \lambda_j y_j^2.$$

The second projection,  $\mathcal{X} \rightarrow \mathbb{A}^{n-1}$ , gives a family of complete intersections of two quadrics  $\mathcal{X}_\lambda$  of dimension  $2n-1$  parameterised by  $\mathbb{A}^{n-1}$ . Note that  $X$  is the intersection of  $\mathcal{X}$  with the subspace  $\mathbb{P}^{n+2} \subset \mathbb{P}^{2n+1}$  defined by  $y_1 = \dots = y_{n-1} = 0$ .

Let  $p: \mathcal{F} \rightarrow \mathbb{A}^{n-1}$  be the family of  $(n-1)$ -planes contained in the  $\mathcal{X}_\lambda$ ; that is

$$\mathcal{F} = \{(P, \lambda) \mid \lambda \in \mathbb{A}^{n-1}, P \text{ } (n-1)\text{-plane} \subset \mathcal{X}_\lambda\}.$$

For  $\lambda$  general, the fibre  $\mathcal{F}_\lambda$  is isomorphic to the Jacobian of the hyperelliptic curve  $C_{\mu,\lambda}$  (4.1).

Let  $(P, \lambda)$  be a general point of  $\mathcal{F}$ . Then  $P \cap \mathbb{P}^{n+2}$  is a point  $x$  of  $X$ . Let  $\pi: \mathbb{P}^{2n+1} \dashrightarrow \mathbb{P}^{n+2}$  be the projection  $(x_i, y_j) \mapsto (x_i)$ . Since the  $\pi_*$  differentials of  $Q_i$  and  $q_i$  coincide at  $x$ , the differential  $\pi_*$  maps  $T_x(P) \subset T_x(\mathcal{X})$  into  $T_x(X)$ . Since  $P$  is general,  $\pi_* T_x(P)$  is a hyperplane in  $T_x(X)$ ; this



will follow from the proof of Proposition 4.2, (1) below, where we explicitly construct pairs  $(P, \lambda)$  with this property.

Therefore, we have a rational map

$$\psi : \mathcal{F} \dashrightarrow \mathbb{P}T^*X \quad (P, \lambda) \mapsto (x = P \cap \mathbb{P}^{n+2}, \pi_*T_x(P)).$$

The symmetric group  $\mathfrak{S}_{n-1}$  acts on  $\mathbb{P}^{2n+1}$  by permuting the  $y_j$  and acts on the group  $(\mathbb{Z}/2\mathbb{Z})^{n-1}$  by changing their signs; this gives an action of the semi-direct product  $G := (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes \mathfrak{S}_{n-1}$ . We make  $G$  act on  $\mathbb{A}^{n-1}$  through its quotient  $\mathfrak{S}_{n-1}$ , by permutation of the  $\lambda_i$ . This induces an action of  $G$  on  $\mathcal{X}$  and therefore on  $\mathcal{F}$ , which is compatible via  $p$  with the action on the base. The map  $\psi$  is invariant under this action; hence, it factors through the quotient  $\mathcal{F}/G$ . By passing to the quotient, we get a map  $p^\sharp : \mathcal{F}/G \rightarrow \mathbb{A}^{n-1}/\mathfrak{S}_{n-1}$ .

PROPOSITION 4.2. (1)  $\psi$  induces a birational map  $\psi^\sharp : \mathcal{F}/G \dashrightarrow \mathbb{P}T^*X$ .

(2) There is a commutative diagram

$$\begin{array}{ccc} \mathcal{F}/G & \xrightarrow{\psi^\sharp} & \mathbb{P}T^*X \\ p^\sharp \downarrow & & \downarrow \varphi \\ \mathbb{A}^{n-1}/\mathfrak{S}_{n-1} & \xrightarrow[\sigma]{\sim} & \mathbb{A}^{n-1} \subset \mathbb{P}^{n-1} \end{array}$$

where  $p^\sharp$  is deduced from  $p$  and where  $\sigma$  is the isomorphism given by symmetric functions.

Proof. (1) Let  $(x, H) \in \mathbb{P}T^*X$ ; we want to describe the pairs  $(P, \lambda)$  such that  $P \cap \mathbb{P}^{n+2} = \{x\}$  and  $\pi_*T_x(P) = H$ . The latter condition says that via the decomposition

$$T_x(\mathbb{P}^{2n+1}) = T_x(\mathbb{P}^{n+2}) \oplus \text{Ker } \pi_*,$$

$T_x(P)$  identifies with the graph of a linear map

$$\alpha : H \rightarrow \text{Ker } \pi_*.$$

Using the basis  $(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_{n-1}})$  of  $\text{Ker } \pi_*$ , we have  $\alpha = (\alpha_1, \dots, \alpha_{n-1})$ , where the  $\alpha_i$  are linear forms on  $H$ . The condition  $P \subset \mathcal{X}_\lambda$  implies that the Hessians  $h_{Q_1}(x)$  and  $h_{Q_2}(x)$  vanish on  $T_x(P)$ , which gives

$$h_{q_1}(x)|_H = - \sum_i \alpha_i^2 \quad h_{q_2}(x)|_H = - \sum_i \lambda_i \alpha_i^2. \quad (1)$$

This is a simultaneous diagonalisation of the quadratic forms  $h_{q_1}(x)|_H$  and  $h_{q_2}(x)|_H$ ; when they are in general position, this determines the  $\lambda_i$  up to permutation and the  $\alpha_i$  up to sign and permutation, which proves (1).

(2) Let  $(P, \lambda) \in \mathcal{F}$ , and let  $(x, H) := \psi(P, \lambda)$ . According to Proposition 3.2,  $\varphi(x, H)$  is given by the  $(n-1)$ -uple of quadrics  $q \in \Pi$  such that the form  $h_q(x)|_H$  is singular. Using  $(\alpha_1, \dots, \alpha_{n-1})$  as coordinates on  $H$ , we see from (1) that this  $(n-1)$ -uple is given by  $(\lambda_1, \dots, \lambda_{n-1})$ , which proves (2).  $\square$

### 4.3 Proof of Proposition 4.1.

Let  $\lambda$  be a general element of  $\mathbb{A}^{n-1}$ . Let us denote by  $\Gamma$  the subgroup  $(\mathbb{Z}/2\mathbb{Z})^{n-1}$  of  $G$ . From Proposition 4.2 and the cartesian diagram



$$\begin{array}{ccc} \mathcal{F}/\Gamma & \longrightarrow & \mathcal{F}/G \\ \downarrow p & & \downarrow p^i \\ \mathbb{A}^{n-1} & \longrightarrow & \mathbb{A}^{n-1}/\mathfrak{S}_{n-1} \end{array}$$

we see that the fibre  $\varphi^{-1}(\lambda)$  is birational to the quotient  $\mathcal{F}_\lambda/\Gamma$ . By (4.1) there is an isomorphism  $\mathcal{F}_\lambda \xrightarrow{\sim} JC_{\mu,\lambda}$  such that  $\Gamma$  acts on  $JC_{\mu,\lambda}$  as  $\{\pm 1_J\} \times \Gamma^+$ , where  $\Gamma^+$  is a group of translations by 2-torsion elements. This proves the proposition.

## 5. Fibres of $\Phi$

### 5.1 Results

We keep the settings of the previous section. Recall that our parameter  $\lambda$  lives in  $\mathbb{A}^{n-1} \subset \mathbb{S}^{n-1}\Pi \cong \mathbb{P}^{n-1}$ . For  $\lambda$  in  $\mathbb{A}^{n-1}$ , we denote by  $\tilde{\lambda}$  a lift of  $\lambda$  in  $\mathbb{C}^n$  for the quotient map  $\mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{P}^{n-1}$ .

**THEOREM 5.1.** *Assume that  $X$  is general. For  $\lambda \in \mathbb{A}^{n-1}$  general, the fibre  $\Phi^{-1}(\tilde{\lambda})$  is isomorphic to  $A \setminus Z$ , where:*

- $A$  is the abelian variety quotient of  $JC_{\mu,\lambda}$  by a 2-torsion subgroup, isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^{n-2}$ ;
- $Z$  is a closed subvariety of codimension  $\geq 2$  in  $A$ .

**COROLLARY 5.2.** *For every smooth complete intersection of two quadrics  $X \subset \mathbb{P}^{n+2}$ , the fibration  $\Phi: T^*X \rightarrow \mathbb{C}^n$  is Lagrangian.*

*Proof.* Assume first that  $X$  is general. The symplectic form on  $T^*X$  is  $d\eta$ , where  $\eta$  is the Liouville form. By Theorem 5.1 and Hartogs' principle, the pull-back of  $\eta$  to a general fibre of  $\Phi$  is the restriction of a 1-form on an abelian variety, hence is closed. This implies the result.

Let  $p: \mathcal{X} \rightarrow B$  be a complete family of smooth intersection of two quadrics in  $\mathbb{P}^{n+2}$ . The constructions of §3 can be globalised over  $B$ : we have a rank 2 vector bundle  $\mathcal{V}$  over  $B$  whose fibre at a point  $b \in B$  is the space of quadratic forms vanishing on  $\mathcal{X}_b$ . We get a homomorphism  $\mathbb{S}^{n-1}\mathcal{V} \rightarrow p_*T_{\mathcal{X}/B}$ , which thus gives rise to a morphism  $\Phi: T^*(\mathcal{X}/B) \rightarrow \mathbb{S}^{n-1}\mathcal{V}^*$  over  $B$  which induces over each point  $b \in B$  our map  $\Phi$ . There is a natural Liouville form  $\eta$  on  $T^*(\mathcal{X}/B)$ : Since  $d\eta$  vanishes on a general fibre of  $\Phi$ , it vanishes on all fibres.  $\square$

**COROLLARY 5.3.** *Assume that  $X$  is general. The multiplication map  $\mathbb{S} \cdot H^0(X, \mathbb{S}^2 T_X) \rightarrow H^0(X, \mathbb{S} \cdot T_X)$  is an isomorphism.*

(We will give in Section 7 a proof that is valid with no generality assumption.)

*Proof.* Theorem 5.1 implies that every function on a general fibre of  $\Phi$  is constant; hence, the pull-back  $\Phi^*: H^0(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) \rightarrow H^0(T^*X, \mathcal{O}_{T^*X})$  is an isomorphism. The right-hand space is canonically isomorphic to  $H^0(X, \mathbb{S} \cdot T_X)$ ; hence, we get an algebra isomorphism  $\mathbb{C}[t_1, \dots, t_n] \xrightarrow{\sim} H^0(X, \mathbb{S} \cdot T_X)$ . By construction, the  $t_i$  are mapped to elements of  $H^0(X, \mathbb{S}^2 T_X)$ , so the Corollary follows.  $\square$

**Remark 5.4.** Let  $V_1, \dots, V_n$  be the Hamiltonian vector fields on  $T^*X$  that are associated to the components of  $\Phi$ . For  $\lambda$  general in  $\mathbb{C}^n$ , let us identify  $\Phi^{-1}(\lambda)$  to  $A \setminus Z$  as in the theorem. Then by Hartogs' principle the  $V_i$  linearise on  $A$ ; that is, they extend to a basis of  $H^0(A, T_A)$ . In principle, this allows to write explicit solutions of the Hamilton equations for  $\Phi_i$  in terms of theta functions.

## 5.2 Proof of the theorem: lemmas

We fix a general point  $\lambda \in \mathbb{A}^{n-1}$ . We denote by  $\mathcal{F}^\circ$  the open subset of  $\mathcal{F}$  where the rational map  $\psi$  is well-defined and denote by  $\mathcal{F}_\lambda^\circ$  its intersection with the fibre  $\mathcal{F}_\lambda$ . Since  $\lambda$  is general, the complement of  $\mathcal{F}_\lambda^\circ$  in  $\mathcal{F}_\lambda$  has codimension  $\geq 2$ . The rational map  $\psi$  induces a morphism  $\psi^\circ: \mathcal{F}^\circ \rightarrow \mathbb{P}T^*X$ ; we denote by  $\psi_\lambda^\circ$  its restriction to  $\mathcal{F}_\lambda^\circ$ . Let  $Z \subset \mathbb{P}T^*X$  be the indeterminacy locus of  $\varphi$  (§ 3), and let  $\mathcal{F}_\lambda^{\text{bad}} := (\psi_\lambda^\circ)^{-1}(Z) \subset \mathcal{F}_\lambda$ .

PROPOSITION 5.5.  $\mathcal{F}_\lambda^{\text{bad}}$  has codimension  $\geq 2$  in  $\mathcal{F}_\lambda$ .

We postpone the proof of Proposition 5.5 to the next section; here we show how it implies Theorem 5.1.

Let  $0_X \subset T^*X$  be the zero section, and let  $q: T^*X \setminus 0_X \rightarrow \mathbb{P}T^*X$  be the quotient map. Let  $\varphi^\circ: \mathbb{P}T^*X \setminus Z \rightarrow \mathbb{P}^{n-1}$  be the morphism induced by  $\varphi$ . We thus have  $q(\Phi^{-1}(\tilde{\lambda})) = (\varphi^\circ)^{-1}(\lambda)$ , and the restriction

$$q_\lambda: \Phi^{-1}(\tilde{\lambda}) \rightarrow (\varphi^\circ)^{-1}(\lambda)$$

is an étale double cover, with Galois involution  $\iota$  induced by  $(-1_{T^*X})$ .

We put  $\mathcal{F}_\lambda^{\circ\circ} := \mathcal{F}_\lambda^\circ \setminus \mathcal{F}_\lambda^{\text{bad}}$  and consider the restriction

$$\psi_\lambda^\circ: \mathcal{F}_\lambda^{\circ\circ} \rightarrow (\varphi^\circ)^{-1}(\lambda) \quad \text{of } \psi^\circ.$$

LEMMA 5.6. The fibre  $\Phi^{-1}(\tilde{\lambda})$  is Lagrangian, and has a trivial tangent bundle.

Proof. The étale double cover  $q_\lambda$  induces by fibred product an étale double cover

$$\pi: \tilde{\mathcal{F}}_\lambda^{\circ\circ} \rightarrow \mathcal{F}_\lambda^{\circ\circ}$$

such that  $\psi_\lambda^\circ$  lifts to a morphism  $\tilde{\psi}_\lambda^\circ: \tilde{\mathcal{F}}_\lambda^{\circ\circ} \rightarrow \Phi^{-1}(\tilde{\lambda})$ .

By Proposition 5.5, the complement of  $\mathcal{F}_\lambda^{\circ\circ}$  in  $\mathcal{F}_\lambda$  has codimension  $\geq 2$ , so  $\pi$  extends to an étale double cover  $\tilde{\mathcal{F}}_\lambda \rightarrow \mathcal{F}_\lambda$ , where  $\tilde{\mathcal{F}}_\lambda$  is an abelian variety or the disjoint union of two abelian varieties. The morphism  $\tilde{\psi}_\lambda^\circ: \tilde{\mathcal{F}}_\lambda^{\circ\circ} \rightarrow \Phi^{-1}(\tilde{\lambda})$  is generically of maximal rank. Again by Proposition 5.5, the holomorphic 1-forms on  $\tilde{\mathcal{F}}_\lambda^{\circ\circ}$  are closed; hence by pull-back, the same holds for the holomorphic 1-forms on  $\Phi^{-1}(\tilde{\lambda})$ . As in the proof of Corollary 5.2, this implies that  $\Phi^{-1}(\tilde{\lambda})$  is Lagrangian. The second assertion is a basic property of Lagrangian fibres.  $\square$

LEMMA 5.7 The morphism  $\psi_\lambda^\circ$  lifts to a morphism  $\tilde{\psi}_\lambda^\circ: \mathcal{F}_\lambda^{\circ\circ} \rightarrow \Phi^{-1}(\tilde{\lambda})$ .

Proof. It suffices to show that the double covering  $\pi: \tilde{\mathcal{F}}_\lambda^{\circ\circ} \rightarrow \mathcal{F}_\lambda^{\circ\circ}$  splits.

Assuming the contrary,  $\tilde{\mathcal{F}}_\lambda$  is an abelian variety. By Lemma 5.6  $H^0(\Phi^{-1}(\tilde{\lambda}), \Omega^1)$  has dimension  $n$ . It follows that the pull-back  $(\tilde{\psi}_\lambda^\circ)^*: H^0(\Phi^{-1}(\tilde{\lambda}), \Omega^1) \rightarrow H^0(\tilde{\mathcal{F}}_\lambda^{\circ\circ}, \Omega^1)$  is bijective. Since the Galois involution of the double covering  $\pi$  acts trivially on holomorphic 1-forms, the same holds for the Galois involution  $\iota$  of the double covering  $q_\lambda: \Phi^{-1}(\tilde{\lambda}) \rightarrow (\varphi^\circ)^{-1}(\lambda)$ .

Now we observe that the 1-forms on  $\Phi^{-1}(\tilde{\lambda})$  are ‘pure’; that is, they extend to any smooth projective compactification of  $\Phi^{-1}(\tilde{\lambda})$ . This follows from the fact that this holds after pull-back to  $\tilde{\mathcal{F}}_\lambda^{\circ\circ}$ . But the quotient  $\Phi^{-1}(\tilde{\lambda})/\iota$  is isomorphic to a Zariski open subset of  $(\varphi^\circ)^{-1}(\lambda)$ , which, by Proposition 4.1, has no nonzero holomorphic 1-forms, so any Zariski open subset has no nonzero closed pure holomorphic 1-forms. This contradiction proves the lemma.  $\square$

### 5.3 Proof of Theorem 5.1

Lemma 5.7 gives a factorisation,

$$\psi_\lambda^\circ: \mathcal{F}_\lambda^{\circ\circ} \xrightarrow{\tilde{\psi}_\lambda^\circ} \Phi^{-1}(\tilde{\lambda}) \xrightarrow{q_\lambda} (\varphi^\circ)^{-1}(\lambda).$$

By Proposition 4.1,  $\psi_\lambda^\circ$  induces a birational morphism,

$$\psi_{\lambda,\Gamma}^\circ: \mathcal{F}_\lambda^{\circ\circ}/\Gamma \longrightarrow (\varphi^\circ)^{-1}(\lambda)$$

it follows that for some subgroup  $\Gamma' \subset \Gamma$  of index 2, the morphism  $\tilde{\psi}_\lambda^\circ: \mathcal{F}_\lambda^{\circ\circ} \rightarrow \Phi^{-1}(\tilde{\lambda})$  factors through a birational morphism,

$$\tilde{\psi}_{\lambda,\Gamma'}^\circ: \mathcal{F}_\lambda^{\circ\circ}/\Gamma' \longrightarrow \Phi^{-1}(\tilde{\lambda}).$$

By Lemma 5.6, the cotangent bundle of  $\Phi^{-1}(\tilde{\lambda})$  is trivial. Therefore, the cotangent bundle of  $\mathcal{F}_\lambda^{\circ\circ}/\Gamma'$  is generically generated by its global sections. This implies that  $\Gamma'$  acts trivially on holomorphic 1-forms and, hence, is the subgroup  $\Gamma^+$  of  $\Gamma$  generated by translations, isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^{n-2}$ ; thus  $\mathcal{F}_\lambda/\Gamma'$  is an abelian variety  $A$ .

To simplify notation, we write  $A^\circ := \mathcal{F}_\lambda^{\circ\circ}/\Gamma'$  and  $u := \tilde{\psi}_{\lambda,\Gamma'}^\circ$ . The rational map  $u^{-1}: \Phi^{-1}(\tilde{\lambda}) \dashrightarrow A$  is everywhere defined (e.g. [BL92, Theorem 4.9.4]), so we have two morphisms

$$A^\circ \xrightarrow{u} \Phi^{-1}(\tilde{\lambda}) \xrightarrow{u^{-1}} A$$

whose composition is the inclusion  $A^\circ \hookrightarrow A$ . Since the tangent bundles of  $A$  and  $\Phi^{-1}(\tilde{\lambda})$  are trivial, the determinant of  $Tu: T_{A^\circ} \rightarrow u^*T_{\Phi^{-1}(\tilde{\lambda})}$  is a function on  $A^\circ$ , hence constant by Proposition 5.5. Therefore,  $u$  is étale and birational, hence an open embedding. This implies that every function on  $\Phi^{-1}(\tilde{\lambda})$  is constant (because its restriction to  $A^\circ$  is constant). Then the previous argument shows that  $u^{-1}$  is also an open embedding, hence  $\Phi^{-1}(\tilde{\lambda})$  is isomorphic to an open subset of  $A$  containing  $A^\circ$ . This proves the theorem.

## 6. Proof of Proposition 5.5

We keep the notations of Section 4.2. Recall that we have coordinates  $(x_0, \dots, x_{n+2}; y_1, \dots, y_{n-1})$  on  $\mathbb{P}^{2n+1}$  and subspaces  $\mathbb{P}^{n+2}$  and  $\mathbb{P}^{n-2}$  in  $\mathbb{P}^{2n+1}$  defined by  $y = 0$  and  $x = 0$ .

Let  $q_1(x) = q_2(x) = 0$  be the equations defining  $X$  in  $\mathbb{P}^{n+2}$ , and let  $R$  be the vector space of quadratic forms in  $y = (y_1, \dots, y_{n-1})$ . We define an extended family  $\mathcal{X}^e \subset \mathbb{P}^{2n+1} \times R^2$  as

$$\mathcal{X}^e = \{((x, y); (r_1, r_2)) \in \mathbb{P}^{2n+1} \times R^2 \mid q_1(x) + r_1(y) = q_2(x) + r_2(y) = 0\}.$$

The fibre  $\mathcal{X}_r^e$  at a point  $r = (r_1, r_2)$  of  $R^2$  is the intersection in  $\mathbb{P}^{2n+1}$  of the two quadrics  $q_1(x) + r_1(y) = q_2(x) + r_2(y) = 0$ . Let  $\mathbb{G}$  be the Grassmannian of  $(n-1)$ -planes in  $\mathbb{P}^{2n+1}$ ; we define as before

$$\mathcal{F}^e := \{(P, r) \in \mathbb{G} \times R^2 \mid P \subset \mathcal{X}_r^e\}$$

and the extended rational map  $\psi^e: \mathcal{F}^e \dashrightarrow \mathbb{P}T^*X$ , which maps a general  $P \subset \mathcal{X}_r^e$  to the pair  $(x, H)$ , with  $\{x\} = P \cap \mathbb{P}^{n+2}$  and  $H = \pi_*T_x(P)$ .

We observe that a general pair  $r = (r_1, r_2)$  of  $R^2$  is simultaneously diagonalisable, so the restriction of  $\psi^e$  to  $\mathcal{F}_r^e$  coincides, for an appropriate choice of the coordinates  $(y_i)$ , with the map  $\psi_\lambda$  that we want to study.

PROPOSITION 6.1. Assume that  $X$  is general.

1. Let  $\Gamma \subset \mathcal{F}^e$  be the locus of points  $(P, r)$  such that either  $\dim P \cap \mathbb{P}^{n+2} > 0$ , or  $P \cap \mathbb{P}^{n-2} \neq \emptyset$ . Then  $\Gamma$  has codimension  $\geq 2$  in  $\mathcal{F}^e$ .

2. There exists no divisor in  $\mathcal{F}^e \setminus \Gamma$  that dominates  $R^2$  and that is mapped to the base locus  $Z \subset \mathbb{P}T^*X$  by  $\psi_e$ .

We claim that this implies Proposition 5.5. Indeed, as just explained above, it suffices to prove the analogue of Proposition 5.5 for  $\psi^e$ . Next, it is clear that the indeterminacy locus of  $\psi^e$  is contained in  $\Gamma$ , so  $\psi^e$  is well-defined on  $\mathcal{F}^e \setminus \Gamma$ . By Proposition 6.1, (1), it now suffices to prove the analogue of Proposition 5.5 for the restriction of  $\psi^e$  to  $\mathcal{F}^e \setminus \Gamma$ . This is exactly the statement of Proposition 6.1, (2).

*Proof of Proposition 6.1:* (1) Let  $\mathcal{Q}$  be the vector space of quadratic forms on  $\mathbb{P}^{2n+1}$  of the form  $q(x) + r(y)$  for some quadratic forms  $q$  and  $r$ . For each pair of integers  $(k, l)$  with  $k \geq 0$ ,  $l \geq -1$ , let  $\mathbb{G}_{k,l}$  be the locally closed subvariety of  $(n-1)$ -planes  $P \in \mathbb{G}$  such that

$$\dim(P \cap \mathbb{P}^{n+2}) = k \quad \dim(P \cap \mathbb{P}^{n-2}) = l.$$

(We put, by convention,  $l = -1$  if  $P \cap \mathbb{P}^{n-2} = \emptyset$ .) Let

$$\mathcal{F}^{\mathcal{Q}} := \{(P, (Q_1, Q_2)) \in \mathbb{G} \times \mathcal{Q}^2 \mid Q_1|_P = Q_2|_P = 0\}$$

and

$$\mathcal{F}_{k,l}^{\mathcal{Q}} := \mathcal{F}^{\mathcal{Q}} \cap (\mathbb{G}_{k,l} \times \mathcal{Q}^2).$$

The general fibre of the projection  $\mathcal{F}^{\mathcal{Q}} \rightarrow \mathcal{Q}^2$  is an abelian variety, and we recover  $\mathcal{F}^e$  by restricting  $\mathcal{F}^{\mathcal{Q}}$  to pairs of quadratic forms of the form  $(q_1(x) + r_1(y), q_2(x) + r_2(y))$ , with  $(q_1(x), q_2(x))$  fixed. Because we assume  $X$  general, the pair  $(q_1(x), q_2(x))$  is general in  $\mathcal{Q}^2$ . It thus suffices to prove the result for the larger family  $\mathcal{F}^{\mathcal{Q}}$ ; that is, to show that  $\mathcal{F}_{k,l}^{\mathcal{Q}}$  has codimension  $\geq 2$  in  $\mathcal{F}^{\mathcal{Q}}$ .

This is done by a dimension count. For  $P \in \mathbb{G}$ , let  $\varphi_P$  be the restriction map  $\mathcal{Q} \rightarrow H^0(P, \mathcal{O}_P(2))$ . The fibre of the projection  $\mathcal{F}^{\mathcal{Q}} \rightarrow \mathbb{G}$  is the vector space  $(\text{Ker } \varphi_P)^{\oplus 2}$ . For  $P$  general,  $\varphi_P$  is surjective: This is the case, for instance, if  $P$  is contained in the  $(n+2)$ -plane in  $\mathbb{P}^{2n+1}$  defined by  $y_i = x_i$  ( $i = 1, \dots, n-1$ ). However,  $\varphi_P$  is not surjective for  $P \in \mathbb{G}_{k,l}$  because the forms  $r(y)|_P$  are singular along  $P \cap \mathbb{P}^{n+2}$  and the forms  $q(x)|_P$  are singular along  $P \cap \mathbb{P}^{n-2}$ ; this implies that the subspaces  $P \cap \mathbb{P}^{n+2}$  and  $P \cap \mathbb{P}^{n-2}$  are apolar for all forms in  $\text{Im } \varphi_P$ . Therefore, the corank of  $\varphi_P$  is  $\geq (k+1)(l+1)$ , and there is equality when  $P$  is contained in the subspace defined by  $x_0 = \dots = x_{n+1-k} = y_1 = \dots = y_{n-2-l} = 0$ , hence for  $P$  general in  $\mathbb{G}_{k,l}$ . Thus our assertion follows from:

$$\begin{aligned} \text{codim}(\mathcal{F}_{k,l}^{\mathcal{Q}}, \mathcal{F}^{\mathcal{Q}}) &= \text{codim}(\mathbb{G}_{k,l}, \mathbb{G}) - 2(k+1)(l+1) \\ &= k(k+1) + (l+1)(l+4) - 2(k+1)(l+1) \\ &= (k-l)(k-l-1) + 2(l+1) \\ &\geq 2 \quad \text{if } k \geq 1 \text{ or } l \geq 0. \end{aligned}$$

(2) The base locus  $Z \subset \mathbb{P}T^*X$  has codimension  $\geq 2$  (Corollary 3.3). Note that  $\psi^e$  is well-defined in  $\mathcal{F}^e \setminus \Gamma$ . If  $\mathcal{D}$  is a codimension 1 subvariety in  $\mathcal{F}^e \setminus \Gamma$ , with  $\psi^e(\mathcal{D}) \subset Z$ , the map  $\psi^e$  does not have maximal rank along  $\mathcal{D}$ . This contradicts the following lemma:

LEMMA 6.2.  $\psi^e$  has maximal rank on  $\mathcal{F}^e \setminus \Gamma$ .

*Proof.* Let  $(x, H)$  be a point of  $\mathbb{P}T^*X$ ; we view  $H$  as a hyperplane in the projective tangent space to  $x$  at  $X$ . The fibre of  $\psi^e: \mathcal{F}^e \setminus \Gamma \rightarrow \mathbb{P}T^*X$  at  $(x, H)$  is the locus

$$(\psi^e)^{-1}(x, H) = \{(P, r_1, r_2) \in \mathbb{G} \times R^2 \mid P \cap \mathbb{P}^{n+2} = \{x\}, P \cap \mathbb{P}^{n-2} = \emptyset, \pi(P) = H, \quad (2)$$

$$(q_i + r_i)|_P = 0 \quad (i = 1, 2)\}. \quad (3)$$

The equations (2) define a smooth, locally closed subvariety  $\mathbb{G}_{x,H}$  of  $\mathbb{G}$ . Let  $P \in \mathbb{G}_{x,H}$ , and let  $\chi_P: R \rightarrow H^0(P, \mathcal{O}_P(2))$  be the restriction map. We will show below that the image of  $\chi_P$  is the space of quadratic forms on  $P$  that are singular at  $x$ . Since the forms  $q_i|_P$  are singular at  $x$ , this implies that the solutions of (3) form an affine space over  $(\text{Ker} \chi_P)^{\oplus 2}$ . Therefore,  $(\psi^e)^{-1}(x, H)$  admits an affine fibration over  $\mathbb{G}_{x,H}$ , hence is smooth.

Clearly the quadrics in  $\text{Im} \chi_P$  are singular at  $x$ . To prove the opposite inclusion, choose the coordinates  $(x_i)$  so that  $x = (1, 0, \dots, 0)$ . Since  $P \cap \mathbb{P}^{n+2} = \{x\}$ , there exist linear forms  $\ell_1, \dots, \ell_{n+2}$  in the  $y_j$  so that  $P$  is defined by  $x_i = \ell_i(y)$  for  $i = 1, \dots, n+2$ . Then a quadratic form on  $\mathbb{P}^{2n+1}$  singular at  $x$  can be written as a form in  $x_1, \dots, x_{n+2}; y_1, \dots, y_{n-1}$ ; hence, its restriction to  $P$  is in  $\text{Im} \chi_P$ . This proves the lemma and, hence, also the proposition.  $\square$

## 7. Symmetric tensors: second approach

### 7.1 The cotangent bundle of a smooth quadric

We consider a smooth quadric  $Q \subset \mathbb{P}^{n+1}$  defined by an equation  $q = 0$ . Its cotangent bundle  $\mathbb{P}T^*Q$  parameterises pairs  $(x, P)$  with  $x \in Q$  and  $P$  a  $(n-1)$ -plane tangent to  $Q$  at  $x$ . Thus, we get a morphism  $\gamma$  from  $\mathbb{P}T^*Q$  to the Grassmannian  $\mathbb{G}$  of  $(n-1)$ -planes in  $\mathbb{P}^{n+1}$ , which is the morphism defined by the linear system  $|\mathcal{O}_{\mathbb{P}T^*Q}(1)|$ . It is birational onto its image, but contracts the subvariety  $\mathcal{C} \subset \mathbb{P}T^*Q$  that consists of pairs  $(x, P)$ , such that  $P$  is tangent to  $Q$  along a line  $\ell \subset Q$  with  $x \in \ell$ ; then  $\gamma^{-1}(P)$  consists of the pairs  $(x, P)$  with  $x \in \ell$ .

Let  $h_q \in H^0(Q, S^2\Omega_Q^1(2))$  be the hessian form of  $q$  (§3). Choosing coordinates  $(x_i)$  such that  $q(x) = \sum x_i^2$ , we have  $h_q = \sum (dx_i)^2$  (note that this is, up to a scalar, the unique element of  $H^0(Q, S^2\Omega_Q^1(2))$  invariant under  $\text{Aut}(Q)$ ). Then  $h_q(x)$  is non-degenerate at each point  $x$  of  $Q$ , so  $h_q$  induces an isomorphism  $\Omega_Q^1(1) \xrightarrow{\sim} T_Q(-1)$ , hence also  $S^2\Omega_Q^1(2) \xrightarrow{\sim} S^2T_Q(-2)$ . The image in  $H^0(Q, S^2T_Q(-2))$  of  $h_q$  by this isomorphism is  $h'_q = \sum \partial_j^2$ . We will view  $h'_q$  as an element of  $H^0(\mathbb{P}T^*Q, \mathcal{O}_{\mathbb{P}T^*Q}(2) \otimes p^*\mathcal{O}_Q(-2))$ , where  $p: \mathbb{P}T^*Q \rightarrow Q$  is the projection.

**PROPOSITION 7.1.** *The divisor of  $h'_q$  is  $\mathcal{C}$ . The projection  $p|_{\mathcal{C}}: \mathcal{C} \rightarrow Q$  is a smooth quadric fibration, and  $\mathcal{C}$  is a prime divisor for  $n \geq 3$ .*

*Proof.* Let  $x \in Q$ ; the hyperplane tangent to  $Q$  at  $x$  cuts out a cone over the smooth quadric  $Q_x \subset \mathbb{P}(T_x(Q))$  defined by  $h_q(x) = 0$  (Section 3). The isomorphism  $T_x(Q) \xrightarrow{\sim} T_x^*(Q)$  given by  $h_q(x)$  carries  $Q_x$  into the dual quadric  $Q_x^*$  in  $\mathbb{P}(T_x^*(Q))$ . On the other hand, a point  $y \in p^{-1}(x)$  corresponds to a hyperplane  $H_y \subset \mathbb{P}(T_x(Q))$ , and  $y$  belongs to  $\mathcal{C}$  if and only if  $H_y$  is tangent to  $Q_x$ ; that is, if  $y \in Q_x^*$ . This proves the equality  $\mathcal{C} = \text{div}(h'_q)$  and thus, that the fiber of  $p|_{\mathcal{C}}: \mathcal{C} \rightarrow Q$  at  $x$  is  $Q_x$ , which is smooth and connected if  $n \geq 3$ .  $\square$

**Remark 7.2** The variety  $\mathcal{C}$  is an example of a total dual VMRT [HLS22]. For the proof of the theorem, we will combine this tool with the birational transformation of  $\mathbb{P}T^*X$  defined by a double cover. (Compare with [AH23]).

We will have to consider the following situation: Let  $Q'$  be another quadric in  $\mathbb{P}^{n+1}$ , such that the intersection  $B := Q \cap Q'$  is a smooth hypersurface in  $Q$ . The surjection  $T_Q \rightarrow N_{B/Q}$  gives a section of  $\mathbb{P}T^*Q$  over  $B$ , hence an embedding  $s : B \hookrightarrow \mathbb{P}T^*Q$ .

LEMMA 7.3. *The image  $s(B)$  is not contained in  $\mathcal{C}$ .*

*Proof.* Let  $x \in B$ . The point  $s(x)$  in  $\mathbb{P}(T_x^*(Q))$  corresponds to the hyperplane image of  $T_x(B)$  in  $T_x(Q)$ ; we must therefore show that this hyperplane is not tangent to the quadric  $Q_x := h_q(x)$ . In terms of projective space, this means that the projective tangent space to  $Q'$  at  $x$  is not tangent, at a smooth point  $y$ , to the cone cut out on  $Q$  by the projective tangent space to  $Q'$  at  $x$ .

Suppose that this is the case, with  $y = (y_0, \dots, y_{n+1})$ . We can assume that  $Q'$  is defined by  $\sum \alpha_i x_i^2 = 0$ , with  $\alpha_i \in \mathbb{C}$  distinct. Then the (projective) tangent space to  $Q'$  at  $x$ , given by  $\sum (\alpha_i x_i) \xi_i = 0$ , must coincide with the tangent space to  $Q$  at  $y$ , given by  $\sum y_i \xi_i = 0$ . This implies  $y = (\alpha_0 x_0, \dots, \alpha_{n+1} x_{n+1})$ . Thus the point  $x$  must satisfy

$$\sum x_i^2 = \sum \alpha_i x_i^2 = \sum \alpha_i^2 x_i^2 = 0.$$

If these relations hold for all  $x$  in  $B$ , the quadric  $\sum \alpha_i^2 x_i^2 = 0$  must belong to the pencil spanned by  $Q$  and  $Q'$ . This means that there exist scalars  $\lambda, \mu, \nu$  such that

$$\lambda \alpha_i^2 + \mu \alpha_i + \nu = 0 \quad \text{for all } i,$$

which is impossible since the  $\alpha_i$  are distinct. Therefore, there exists  $x \in B$  such that  $s(x) \notin \mathcal{C}$ .  $\square$

## 7.2 Explicit description of symmetric tensors

We keep the notation of the previous sections:  $X \subset \mathbb{P} = \mathbb{P}^{n+2}$  is defined by  $q_1 = q_2 = 0$ , and with

$$q_1 = \sum_{i=0}^{n+2} x_i^2, \quad q_2 = \sum_{i=0}^{n+2} \mu_i x_i^2 \quad \text{with } \mu_i \in \mathbb{C} \text{ distinct.}$$

We put  $\partial_i := \frac{\partial}{\partial x_i}$ . We have an exact sequence

$$0 \rightarrow T_X \rightarrow T_{\mathbb{P}|X} \xrightarrow{(dq_1, dq_2)} \mathcal{O}_X(2)^2 \rightarrow 0,$$

where  $dq_i$  maps the restriction of a vector field  $V$  on  $\mathbb{P}$  to  $V \cdot q_i$ . This gives the exact sequence of symmetric tensors

$$0 \rightarrow S^2 T_X \rightarrow S^2 T_{\mathbb{P}|X} \xrightarrow{(dq_1, dq_2)} T_{\mathbb{P}|X}(2)^2, \quad (4)$$

where  $dq_i(V_1 V_2) = (V_1 \cdot q_i) V_2 + (V_2 \cdot q_i) V_1$  for  $V_1, V_2$  in  $H^0(X, T_{\mathbb{P}|X})$ .

PROPOSITION 7.4. *The quadratic vector fields  $s_i := \sum_{j \neq i} \frac{(x_i \partial_j - x_j \partial_i)^2}{\mu_j - \mu_i}$  in  $H^0(X, S^2 T_{\mathbb{P}|X})$  belong to the image of  $H^0(X, S^2 T_X)$ .*

*Proof.* According to the exact sequence (4), we have to prove  $dq_1(s_i) = dq_2(s_i) = 0$ .

We have  $(x_i \partial_j - x_j \partial_i) \cdot q_1 = 0$ ; hence,  $dq_1(s_i) = 0$  and  $dq_2(x_i \partial_j - x_j \partial_i, x_i \partial_j - x_j \partial_i) = 4(\mu_j - \mu_i)x_i x_j (x_i \partial_j - x_j \partial_i)$ . Hence, using  $\sum x_j \partial_j = 0$  and  $q_1|_X = 0$ ,

$$dq_2(s_i) = 4x_i^2 \sum_{j \neq i} x_j \partial_j - 4x_i \left( \sum_{j \neq i} x_j^2 \right) \partial_i = 0, \quad \text{which proves the proposition.}$$

□

In the rest of this article, we will consider the  $s_i$  to be elements of  $H^0(X, S^2 T_X)$ .

### 7.3 The double cover

Let  $p_0: \mathbb{P}^{n+2} \dashrightarrow \mathbb{P}^{n+1}$  be the projection  $(x_0, \dots, x_{n+2}) \mapsto (x_1, \dots, x_{n+2})$ . The image  $p_0(X)$  is the smooth quadric  $Q$  in  $\mathbb{P}^{n+1}$  defined by

$$\sum_{i=1}^{n+2} (\mu_i - \mu_0) x_i^2 = 0.$$

The restriction  $\pi: X \rightarrow Q$  of  $p_0$  is a double covering that is branched along the subvariety  $B \subset Q$  defined by

$$\sum_{i=1}^{n+2} x_i^2 = \sum_{i=1}^{n+2} \mu_i x_i^2 = 0.$$

It is a smooth complete intersection of 2 quadrics in  $\mathbb{P}^{n+1}$ . The ramification locus  $R \subset X$  of  $\pi$  (isomorphic to  $B$ ) is the hyperplane section  $x_0 = 0$  of  $X$ .

The tangent map of  $\pi: X \rightarrow Q$  gives a morphism,

$$\tau: T_X \rightarrow \pi^* T_Q,$$

which is an isomorphism outside of  $R$ . Consider the normal exact sequence

$$0 \rightarrow T_R \rightarrow T_{X|R} \rightarrow N_{R/X} \rightarrow 0.$$

The involution  $\iota: (x_0, \dots, x_{n+2}) \mapsto (-x_0, x_1, \dots, x_{n+2})$  acts on  $T_{X|R}$ ; this splits the exact sequence, giving a decomposition

$$T_{X|R} = T_R \oplus N_{R/X}$$

into eigenspaces for the eigenvalues  $+1$  and  $-1$ . Let  $\rho: T_{X|R} \rightarrow T_R$  be the projection on the first summand. We deduce from  $\rho$  a sequence of homomorphisms

$$h^k: H^0(X, S^k T_X) \longrightarrow H^0(X, S^k T_{X|R}) \xrightarrow{S^k \rho} H^0(R, S^k T_R).$$

Since  $\iota_* \partial_0 = -\partial_0$  and  $\iota_* \partial_j = \partial_j$  for  $j > 0$ , we have

$$h^2(s_0) = 0 \quad \text{and} \quad h^2(s_i) = \sum_{\substack{j > 0 \\ j \neq i}} \frac{(x_i \partial_j - x_j \partial_i)^2}{\mu_j - \mu_i} \quad \text{for } i > 0 \quad (5)$$

in other words,  $h^2$  maps  $s_1, \dots, s_{n+2}$  to the elements  $\hat{s}_1, \dots, \hat{s}_{n+2}$  of  $H^0(R, S^2 T_R)$  constructed in Proposition 7.2.1 applied to  $R$ .

Let  $\pi^* \mathbb{P} T^* Q$  be the pull-back under  $\pi$  of the projective bundle  $\mathbb{P} T^* Q \rightarrow Q$ . The homomorphism  $\tau: T_X \rightarrow \pi^* T_Q$  gives rise to the birational map  $g: \pi^* \mathbb{P} T^* Q \dashrightarrow \mathbb{P} T^* X$ . Following the geometric description of the tangent map as an elementary transformation of vector bundles in the sense of Maruyama in [Mar72] and [Mar73, Corollary 1.1.1], one has a commutative diagram



$$\begin{array}{ccc}
 & \Gamma & \\
 \mu \swarrow & & \searrow \nu \\
 \pi^* \mathbb{P}T^*Q & \overset{g}{\dashrightarrow} & \mathbb{P}T^*X \\
 p \searrow & & \swarrow q \\
 & X &
 \end{array} \tag{6}$$

where  $p$  and  $q$  are the canonical projections;  $\nu: \Gamma \rightarrow \mathbb{P}T^*X$  is the blow-up along the subspace  $\mathbb{P}T^*R \subset \mathbb{P}T^*X$  defined by the projection  $\rho$ ; and  $\mu: \Gamma \rightarrow \pi^*\mathbb{P}T^*Q$  is the blow-up of the image  $B'$  of the embedding  $B \hookrightarrow \pi^*\mathbb{P}T^*Q$  deduced from the surjective homomorphism  $\pi^*T_Q \rightarrow \pi^*N_{B/X}$ .

Let  $E_\mu$  be the exceptional divisor of  $\mu$ . By [Mar73, Theorem 1.1], there is an isomorphism

$$\mu^* \mathcal{O}_{\pi^*\mathbb{P}T^*Q}(1) \otimes \mathcal{O}_\Gamma(-E_\mu) \cong \nu^* \mathcal{O}_{\mathbb{P}T^*X}(1) \tag{7}$$

as well as the equality

$$\nu_* E_\mu = q^* R. \tag{8}$$

## 7.4 The divisor of $s_0$

We now consider the divisor  $\mathcal{C} \subset \mathbb{P}T^*Q$  defined in (7.1) and the cartesian diagram

$$\begin{array}{ccc}
 \pi^* \mathbb{P}T^*Q & \xrightarrow{\pi'} & \mathbb{P}T^*Q \\
 \downarrow & & \downarrow \\
 X & \xrightarrow{\pi} & Q.
 \end{array}$$

Set  $\mathcal{C}' := \pi'^{-1}(\mathcal{C})$ . The projection  $\mathcal{C}' \rightarrow X$  is again a smooth quadric fibration, so  $\mathcal{C}'$  is smooth and connected for  $n \geq 3$ .

Recall that we have defined the element  $s_0 := \sum_{j=1}^{n+2} \frac{(x_0 \partial_j - x_j \partial_0)^2}{\mu_j - \mu_0} \in H^0(X, S^2 T_X)$  (7.2). We

will now view  $s_0$  as an element of  $H^0(\mathbb{P}T^*X, \mathcal{O}(2))$ .

**PROPOSITION 7.5** *Assume  $n \geq 3$ . We have  $g_* \mathcal{C}' = \text{div}(s_0)$ .*

*Proof.* We first show that  $g_* \mathcal{C}' \in |\mathcal{O}_{\mathbb{P}T^*X}(2)|$ . By Proposition 7.1 we have  $\mathcal{C}' \in |\mathcal{O}_{\pi^*\mathbb{P}T^*Q}(2) \otimes p^* \mathcal{O}_X(-2)|$ . Using (7), (8) and the projection formula, we get the linear equivalences

$$\nu_* \mu^* \mathcal{C}' \sim 2\nu_* \mu^* (c_1(\mathcal{O}_{\pi^*\mathbb{P}T^*Q}(1) - p^* R)) \sim 2(c_1(\mathcal{O}_{\mathbb{P}T^*X}(1)) + q^* R) - 2q^* R = c_1(\mathcal{O}_{\mathbb{P}T^*X}(2)).$$

Thus, it is enough to prove that  $\nu_* \mu^* \mathcal{C}'$  is irreducible. Since  $\mathcal{C}'$  is irreducible and  $\mu$  is the blow-up along  $B' \subset \pi^*\mathbb{P}T^*Q$ , it suffices to show that  $B'$  is not contained in  $\mathcal{C}'$ . If this is the case, then we have  $\pi'(B') \subset \pi'(\mathcal{C}') = \mathcal{C}$ . But  $\pi'(B') = s(B)$ , where  $s: B \hookrightarrow \mathbb{P}T^*Q$  is the embedding defined by the surjective homomorphism  $T_Q \rightarrow N_{B/Q}$ . Then the result follows from Lemma 7.3.

Since  $g_* \mathcal{C}'$  and  $\text{div}(s_0)$  are linearly equivalent effective divisors and  $g_* \mathcal{C}'$  is irreducible, it suffices to show that their restrictions to  $\mathbb{P}T_x^*(X)$  coincide at a general point  $x \in X$ .

Fix a point  $x = (x_0, \dots, x_{n+2}) \in X \setminus R$  so that  $x_0 \neq 0$ . Then the tangent map  $T\pi(x): T_x(X) \rightarrow T_{\pi(x)}(Q)$  is an isomorphism; in diagram (6), the maps  $\mu, \nu$  and  $g$  restricted over the fibres at  $x$  are all isomorphisms. Let us show that  $\mathcal{C}'$  and  $T\pi(\text{div}(s_0))$  define the same quadric in  $\mathbb{P}(T_{\pi(x)}(Q))$ .

Note that  $\mathcal{C}' \cap \mathbb{P}(T_x^*(X)) = \mathcal{C} \cap \mathbb{P}(T_{\pi(x)}^*(Q))$  is the quadric defined by the element  $h'_q$  of (7.1).

In the coordinates  $(z_i)$  defined by  $z_i = (\mu_i - \mu_0)^{1/2} x_i$ , the equation of  $Q$  is  $\sum_{j=1}^{n+2} z_j^2 = 0$ , so

$$h'_q = \sum_{j=1}^{n+2} \left( \frac{\partial}{\partial z_j} \right)^2 = \sum_{j=1}^{n+2} \frac{\partial_j^2}{\mu_j - \mu_0}.$$

On the other hand, since  $\pi(x_0, \dots, x_{n+2}) = (x_1, \dots, x_{n+2})$ , we have  $T\pi(\partial_0) = 0$  and  $T\pi(\partial_j) = \partial_j$  for  $j > 0$ ; hence,

$$T\pi(s_0) = x_0^2 \sum_{j=1}^{n+2} \frac{\partial_j^2}{\mu_j - \mu_0}.$$

Since  $x_0 \neq 0$ , this proves the proposition.  $\square$

## 7.5 Proof of part (a) of the theorem

Suppose now that  $n \geq 3$ . Consider the double cover  $\pi : X \rightarrow Q$  and the ramification divisor  $R \subset X$ . The restriction maps  $h^k$  defined in Section 7.3 yield a homomorphism of graded  $\mathbb{C}$ -algebras

$$h : S(X) := H^0(X, S^\bullet T_X) \longrightarrow H^0(R, S^\bullet T_R) =: S(R).$$

PROPOSITION 7.6 *The kernel  $\mathcal{I}$  of  $h$  is the ideal generated by  $s_0$ .*

*Proof.* Since  $\mathcal{I}$  is a homogeneous ideal, it suffices to prove that every homogeneous element  $s \in \mathcal{I}$  can be written as  $s = s' s_0$  for some element  $s' \in S(X)$ .

Choose an element  $s \in \mathcal{I}$  of degree  $k$ . This element corresponds to an effective Cartier divisor  $G$  in the linear system  $|\mathcal{O}_{\mathbb{P}T^*X}(k)|$ . Recall the commutative diagram (6):

$$\begin{array}{ccc} & \Gamma & \\ \mu \swarrow & & \searrow \nu \\ \pi^* \mathbb{P}T^*Q & \xrightarrow{\quad g \quad} & \mathbb{P}T^*X \\ p \searrow & & \swarrow q \\ & X & \end{array}$$

Choose  $\hat{G} := \mu_* \nu^* G \subset \pi^* \mathbb{P}T^*Q$ . By (7),  $\hat{G}$  belongs to the linear system  $|\mathcal{O}_{\pi^* \mathbb{P}T^*Q}(k)|$ .

Here is the key observation: Since  $s \in \mathcal{I}$ , the divisor  $\hat{G} \subset \pi^* \mathbb{P}T^*Q$  contains  $p^* R$ . Indeed, since  $(\pi^* T_Q)|_R$  is invariant under  $\iota$ , the homomorphism  $\tau|_R$  factors as

$$\tau|_R : T_{X|R} \xrightarrow{\rho} T_R \longrightarrow (\pi^* T_Q)|_R.$$

Therefore, we have a commutative diagram,

$$\begin{array}{ccc} H^0(X, S^k T_X) & \xrightarrow{h^k} & H^0(R, S^k T_R) \\ S^k \tau \downarrow & & \downarrow \\ H^0(X, S^k \pi^* T_Q) & \longrightarrow & H^0(R, S^k (\pi^* T_Q)|_R) \end{array}$$

and  $S^k\tau(s)$  vanishes on  $R$ . But  $\hat{G}$  is the divisor of  $S^k\tau(s)$ , viewed as a section of  $\mathcal{O}_{\pi^*\mathbb{P}T^*Q}(k)$ ; hence,  $\hat{G}$  contains  $p^*R$ .

Now we want to show that the divisor  $\mathcal{C}' \subset \pi^*\mathbb{P}T^*Q$  is a component of  $\hat{G} - p^*R$ . Recall from (7.1) that  $\mathcal{C}$  is the union of the lines  $\ell$  that are contracted by the morphism  $\gamma: \mathbb{P}T^*Q \rightarrow \mathbb{G}$  and that  $c_1(\mathcal{O}_{\mathbb{P}T^*Q}(1)) \cdot \ell = 0$ . Thus the curves  $\ell' := \pi'^*\ell$  cover  $\mathcal{C}'$  and satisfy  $c_1(\mathcal{O}_{\pi^*\mathbb{P}T^*Q}(1)) \cdot \ell' = 0$ . On the other hand, the divisor  $R \subset X$  is a hyperplane section, so  $p^*R \cdot \ell' = R \cdot p_*\ell' > 0$ . Therefore,

$$(\hat{G} - p^*R) \cdot \ell' < 0,$$

so  $\mathcal{C}'$  is a component of  $\hat{G}$ . Thus,  $g_*\mathcal{C}'$  is a component of  $G$ . Since  $g_*\mathcal{C}' = \text{div}(s_0)$  by Proposition 7.5, this proves the proposition.  $\square$

The following proposition implies part (a) of our main theorem:

**PROPOSITION 7.7.** *Assume  $n \geq 2$ . For any choice of indices  $0 \leq i_1 < \dots < i_n \leq n+2$ , the homomorphism  $\mathbb{C}[t_1, \dots, t_n] \rightarrow S(X)$ , which maps  $t_j$  to  $s_{i_j}$ , with  $\deg(t_i) = 2$ , is an isomorphism of graded  $\mathbb{C}$ -algebras.*

*Proof.* We argue by induction on  $n$ . The statement for  $n = 2$  follows from [DOL19, Theorem 5.1], except for the fact that any two of the  $s_i$  generate  $H^0(X, S^2T_X)$ . Up to the permuting of the coordinates, it suffices to prove that  $s_0$  and  $s_1$  are linearly independent. But  $h^2: H^0(X, S^2T_X) \rightarrow H^0(R, S^2T_R)$  maps  $s_0$  to zero and maps  $s_i$  (for  $i > 0$ ) to the corresponding elements  $\hat{s}_i$  of  $H^0(R, S^2T_R)$ ; this implies our assertion.

Assume  $n \geq 3$ . By the induction hypothesis, the homomorphism  $\mathbb{C}[t_1, \dots, t_{n-1}] \rightarrow S(R)$ , which maps  $t_i$  to  $\hat{s}_i$ , is an isomorphism of graded  $\mathbb{C}$ -algebras (with  $\deg(t_i) = 2$ ). It follows that  $h$  is surjective and that  $(s_0, \dots, s_{n-1})$  form a basis of  $H^0(X, S^2T_X)$  and generate the  $\mathbb{C}$ -algebra  $S(X)$ . Thus we have a surjective homomorphism  $u: \mathbb{C}[t_0, \dots, t_{n-1}] \rightarrow S(X)$ , with  $u(t_i) = s_i$ .

In particular, the Krull dimension of  $S(X)$  is at most  $n$ . On the other hand, the ring  $S(X)$  is a domain, and  $s_0$  is neither zero nor a unit. Thus, by Krull's Hauptidealsatz, the Krull dimension of  $S(X)$  is equal to  $n$ ; hence,  $u$  is an isomorphism. By permutation of the coordinates, we get the same result for any choice of  $n$  elements in  $\{s_0, \dots, s_{n+2}\}$ . This proves the proposition.  $\square$

## CONFLICTS OF INTEREST

None.

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