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TRANSLATION-INVARIANT OPERATORS ON $L^p(G)$, $0 < p < 1$ (II)

DANIEL M. OBERLIN

For a locally compact group G, let $L^p(G)$ be the usual Lebesgue space with respect to left Haar measure m on G. For $x \in G$ define the left and right translation operators L_x and R_x by $L_x f(y) = f(xy)$, $R_x f(y) = f(yx)$ $(f \in L^p(G), y \in G)$. The purpose of this paper is to prove the following theorem.

THEOREM. *Let G be a locally compact group and fix p with $0 < p < 1$. The bounded linear operators on $L^p(G)$ which commute with each $R_x (x \in G)$ are precisely those operators of the form*

$$(1) \qquad \sum_{i=1}^{\infty} a_i L_{x_i}, \quad x_i \in G, \sum_{i=1}^{\infty} |a_i|^p < \infty.$$

For compact abelian G this was proved in [**4**]. Here we give the details for the (somewhat more complicated) proof of the general case.

One half of the proof is trivial: for $0 < p < 1$ and complex numbers z and w we have $|z + w|^p \leqq |z|^p + |w|^p$. Thus it is obvious that (1) defines a bounded translation-invariant linear operator on $L^p(G)$. So assume that T is such an operator and we shall show, in two steps, that T has the form (1). First, though, we record some notation: the symbol $\int \cdots dx$ always stands for $\int_G \cdots dm(x)$, while for $0 < q \leqq 1$ and $f \in L^q(G)$, the symbol $||f||_q$ stands for the number $(\int |f(x)|^q dx)^{1/q}$.

Step 1. We shall prove:

(2) There exists a complex-valued regular Borel measure λ on G such that $Tf = \lambda * f$ for $f \in L^p(G) \cap L^1(G)$.

LEMMA 1. *Let S be a bounded real-linear operator on $L^p(G)$ which sends real-valued functions into real-valued functions. Let $||S||$ denote the number*

$$\sup \{||Sf||_p/||f||_p; f \in L^p(G), f \neq 0\}.$$

For any q with $0 < p < q \leqq 2$ and any real-valued continuous compactly-supported f on $G \times G$, we have

$$\int \left(\int |Sf(\cdot, y)(x)|^q dy \right)^{p/q} dx \leqq ||S||^p \int \left(\int |f(x, y)|^q dy \right)^{p/q} dx.$$

Proof of Lemma 1: For each $n = 1, 2, \ldots$ there exist $m (= m(n))$, pairwise disjoint Borel sets $E_1, \ldots, E_m \subseteq G$, and continuous compactly-supported

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real-valued functions $g_1, \ldots, g_m$ on G such that if χ_i is the characteristic function of E_i and if $f_n(x, y) = \sum_{i=1}^m g_i(x)\chi_i(y)$, then

(3) support $(f_n) \subseteq K$ for some compact $K \subseteq G \times G$ and
$$\sup \{|f_n(x, y) - f(x, y)|: (x, y) \in G \times G\} \to 0.$$

It follows that $\int \int |f_n(x, y) - f(x, y)|^p dx\, dy \to 0$, and so $\int \int |Sf_n(\cdot, y)(x) - Sf(\cdot, y)(x)|^p dx\, dy \to 0$. By passing to a subsequence we may assume that for almost every $x \in G$ we have $Sf_n(\cdot, y)(x) \to Sf(\cdot, y)(x)$ for almost every $y \in G$. Then Fatou's lemma yields

$$\int |Sf(\cdot, y)(x)|^q dy \leqq \liminf \int |Sf_n(\cdot, y)(x)|^q dx$$

for almost every x, and so

$$\int \left(\int |Sf(\cdot, y)(x)|^q dy \right)^{p/q} dx \leqq \liminf \int \left(\int |Sf_n(\cdot, y)(x)|^q dy \right)^{p/q} dx.$$

We will be done with the proof of Lemma 1 when we establish

(4) $$\liminf \int \left(\int |Sf_n(\cdot, y)(x)|^q dy \right)^{p/q} dx \leqq ||S||^p \int \left(\int |f(x, y)|^q dy \right)^{p/q} dx.$$

Fix n and recall that $f_n(x, y) = \sum_{i=1}^m g_i(x)\chi_i(y)$. For $i = 1, \ldots, m$, let $h_i(x) = g_i(x) m(E_i)^{1/q}$. A theorem of Marcinkiewicz and Zygmund [**3**, Théorème 2], implies that

$$\int \left(\sum_{i=1}^m |Sh_i(x)|^q \right)^{p/q} dx \leqq ||S||^p \int \left(\sum_{i=1}^m |h_i(x)|^q \right)^{p/q} dx,$$

and so

$$\int \left(\int |Sf_n(\cdot, y)(x)|^q dy \right)^{p/q} dx \leqq ||S||^p \int \left(\int |f_n(x, y)|^q dy \right)^{p/q} dx.$$

But

$$\int \left(\int |f_n(x, y)|^q dy \right)^{p/q} dx \to \int \left(\int |f(x, y)|^q dy \right)^{p/q} dx$$

by (3), so (4) is established.

The preceding lemma is essentially Lemma 2 in [**1**], where it is stated without proof. We have included the details for the sake of completeness. We note that part of the argument which follows was inspired by the proof of the theorem in [**1**].

We return to the proof of (2). Let S be either of the real-linear operators $f \to \operatorname{Re}(Tf), f \to \operatorname{Im}(Tf)$. If we show:

(5) There exists a real-valued regular Borel measure μ on G such that $Sf = \mu * f$ for real-valued $f \in L^p(G) \cap L^1(G)$,

then (2) will follow. We note that S is a real-linear operator on $L^p(G)$ which commutes with each $R_x (x \in G)$ and satisfies the hypothesis of Lemma 1.

Let U and V be neighborhoods of the identity e in G with U relatively compact, V symmetric, and $V^2 \subseteq U$. Let u and h be continuous real-valued compactly-supported functions on G with $u(x) = 1$ for $x \in U$ and h supported in V. Taking $q = 1$ and $f(x, y) = u(x)h(xy)$ in Lemma 1, we get

$$||S|| \cdot ||u||_p \cdot ||h||_1 \geqq \left(\int \left(\int |S(u(\cdot)h(\cdot y))(x)|dy\right)^p dx\right)^{1/p}$$
$$\geqq \left(\int \left(\int_V |S(u(\cdot)h(\cdot y))(x)|dy\right)^p dx\right)^{1/p},$$

and so, since S commutes with each R_x,

$$(6) \qquad ||S|| \cdot ||u||_p \cdot ||h||_1 \geqq \left(\int \left(\int_V |S(u(\cdot y^{-1})h(\cdot))(xy)|dy\right)^p dx\right)^{1/p}.$$

Since $V^2 \subseteq U$, V is symmetric, and support $(h) \subseteq V$, it follows that $u(\cdot y^{-1})$ is equal to 1 on the support of h as long as $y \in V$. Thus, if χ_V denotes the characteristic function of V,

$$\int_V |S(u(\cdot y^{-1})h(\cdot))(xy)|dy = \int_V |Sh(xy)|dy = \int |Sh(y)\chi_V(x^{-1}y)|dy$$
$$= \int |Sh(y)\chi_V(y^{-1}x)|dy$$

since V is symmetric. Now (6) yields

$$||S|| \cdot ||u||_p \cdot ||h||_1 \geqq \left(\int \left(\int |Sh(y)\chi_V(y^{-1}x)|dy\right)^p dx\right)^{1/p}$$
$$\geqq \int \left(\int |\chi_V(y^{-1}x)|^p dm(x)\right)^{1/p} |Sh(y)|dy = (m(V))^{1/p}||Sh||_1,$$

where the last inequality follows from an application of Minkowski's integral inequality. Thus we have established

$$(7) \qquad ||S|| \cdot ||u||_p (m(V))^{-1/p}||h||_1 \geqq ||Sh||_1$$

for any real-valued continuous h supported in V. It is easy to check that (7) continues to hold for an arbitrary real-valued $h \in L^1(G)$ so long as h is supported in V. But any compactly-supported $h \in L^1(G)$ can be written as a finite sum of right translates of L^1 functions supported in V—say $h = \sum_{i=1}^n R_i h_i$ where R_i is right translation by some $x_i \in G$—and we can arrange to have the sets $\{x \in G: R_i h_i \neq 0\}$ pairwise disjoint. With Δ denoting the modular function of G we then have

$$||Sh||_1 \leqq \sum_{i=1}^{n} ||SR_ih_i||_1 = \sum_{i=1}^{n} ||R_iSh_i||_1 = \sum_{i=1}^{n} \Delta(x_i^{-1})||Sh_i||_1$$
$$\leqq ||S||\cdot||u||_p\cdot(m(V))^{-1/p} \sum_{i=1}^{n} \Delta(x_i^{-1})||h_i||_1.$$

Since

$$\sum_{i=1}^{n} \Delta(x_i^{-1})||h_i||_1 = \sum_{i=1}^{n} ||R_ih_i||_1 = ||h||_1,$$

it follows that (7) holds for any compactly-supported real-valued function h in $L^1(G)$. Now (5) follows from Wendel's theorem [**2**, Theorem 35.5]. This completes Step 1.

Step 2. We will show that the measure λ of (2) is of the form $\sum_{i=1}^{\infty} a_i\delta_i$ where δ_i is the unit mass at some $x_i \in G$ and $\sum_{i=1}^{\infty}|a_i|^p < \infty$. This will complete the proof of the theorem. We begin by showing that λ is a discrete measure. We will need the following lemma.

LEMMA 2 (Lemma 1 of [**4**]). *Let K be a compact Hausdorff space and let λ be a complex-valued regular Borel measure on K. If for some $p(0 < p < 1)$ and for some finite positive number M we have*

$$(8) \qquad \sum_{j=1}^{m} |\lambda(E_j)|^p \leqq M$$

for each m and each finite Borel partition $\{E_j\}_{j=1}^{m}$ of K, then λ is of the form $\sum_{i=1}^{\infty} a_i\delta_i$, where δ_i is the unit mass at some point $x_i \in K$ and $\sum_{i=1}^{\infty}|a_i|^p \leqq M$.

To show that λ is discrete it is enough to show that the restriction of λ to K satisfies the hypothesis of Lemma 2 for each compact $K \subseteq G$. So fix such a K and a relatively compact neighborhood E of e in G. We will show that (8) holds for any Borel partition $\{E_j\}_{j=1}^{m}$ of K with $M = ||T||^p m(K^{-1}E)/m(E)$.

Fix $\epsilon > 0$, compact sets $K_j \subseteq E_j$, and pairwise disjoint open subsets U_j of G such that

$$(9) \qquad \sum_{j=1}^{m} |\lambda(E_j) - \lambda(F_j)|^p < \epsilon \quad \text{if each } F_j \text{ satisfies } K_j \subseteq F_j \subseteq U_j.$$

Let U be a symmetric neighborhood of e in G with $K_jU^2 \subseteq U_j$ for each j, and let $\{S_k\}_{k=1}^{n}$ be a partition of $K^{-1}E$ such that each S_k is contained in some right translate of U. Then if χ_k is the characteristic function of $S_k(k = 1, \ldots, n)$, we have

$$||T||^p m(K^{-1}E) = ||T||^p \sum_{k=1}^{n} ||\chi_k||_p^p \geqq \int \sum_{k=1}^{n} |T\chi_k(x)|^p dx$$
$$= \int \sum_{k=1}^{n} |\lambda(xS_k^{-1})|^p dx \geqq \int_E \sum_{k=1}^{n} |\lambda(xS_k^{-1})|^p dx.$$

Thus there exists some $x \in E$ (which we now fix) with

$$(10)\quad \sum_{k=1}^{n} |\lambda(xS_k^{-1})|^p \leqq ||T||^p m(K^{-1}E)/m(E).$$

For $j = 1, \ldots, m$, let $F_j = \bigcup xS_k^{-1}$, where the union is over all k such that $xS_k^{-1} \cap K_j \neq \emptyset$. Since $\{xS_k^{-1}\}_{k=1}^n$ partitions $xE^{-1}K \supseteq K$, it follows that $K_j \subseteq F_j$. Since each S_k is contained in a right translate of U and since $K_jU^2 \subseteq U_j$ for each j, it follows that $F_j \subseteq U_j$. Now (9), (10), and the definition of the sets F_j yield

$$\sum_{j=1}^{m} |\lambda(E_j)|^p \leqq \epsilon + \sum_{j=1}^{m} |\lambda(F_j)|^p \leqq \epsilon + \sum_{k=1}^{n} |\lambda(xS_k^{-1})|^p \leqq \epsilon + ||T||^p m(K^{-1}E)/m(E).$$

But for an arbitrary $\epsilon > 0$, this is (8) with $M = ||T||^p\, m(K^{-1}E)/m(E)$. It follows that λ is discrete, say $\lambda = \sum_{i=1}^{\infty} a_i\delta_i$ with δ_i the unit mass at some $x_i \in G$ and with $\sum_{i=1}^{\infty}|a_i| < \infty$. To complete the proof we need only show that $\sum_{i=1}^{\infty}|a_i|^p < \infty$.

With no loss of generality we may suppose that no $a_i = 0$ and that the x_i are distinct. Let N_1 be an arbitrary positive integer and let $N_2 > N_1$ be so large that $\sum_{i=N_2+1}^{\infty}|a_i| \leqq |a_j|/2$ if $1 \leqq j \leqq N_1$. Let U be a neighborhood of e such that $x_iU \cap x_jU = \emptyset$ if $1 \leqq i < j \leqq N_2$. Now

$$(11)\quad \begin{aligned} ||T||^p m(U) = ||T||^p ||\chi_U||_p^p &\geqq \int |\lambda(xU^{-1})|^p dx \\ &\geqq \sum_{j=1}^{N_1} \int_{x_jU} |\lambda(xU^{-1})|^p dx = \sum_{j=1}^{N_1} \int_{x_jU} \left| \sum_{x_i \in xU^{-1}} a_i \right|^p dx. \end{aligned}$$

Fix j with $1 \leqq j \leqq N_1$ and fix $x \in x_jU$. Then $x_j \in xU^{-1}$ and if $1 \leqq i \leqq N_2$, $i \neq j$, then $x_i \notin xU^{-1}$. Thus

$$\int_{x_jU} \left| \sum_{x_i \in xU^{-1}} a_i \right|^p dx \geqq \int_{x_jU} \left(|a_j| - \sum_{i=N_2+1} |a_i| \right)^p dx \geqq \int_{x_jU} (|a_j|/2)^p dx.$$

This and (11) yield

$$2^p ||T||^p \geqq \sum_{j=1}^{N_1} |a_j|^p.$$

Since N_1 was arbitrary, $\sum_{j=1}^{\infty}|a_j|^p < \infty$ as desired.

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Florida State University,
Tallahassee, Florida