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Reciprocity Law for Compatible Systems of Abelian mod p Galois Representations

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Abstract. The main result of the paper is a *reciprocity law* which proves that compatible systems of semisimple, abelian mod p representations (of arbitrary dimension) of absolute Galois groups of number fields, arise from Hecke characters. In the last section analogs for Galois groups of function fields of these results are explored, and a question is raised whose answer seems to require developments in transcendence theory in characteristic p.

1 Introduction

Motives defined over number fields give rise to compatible systems of p-adic and mod p representations of absolute Galois groups of number fields. Compatible systems of p-adic representations have been extensively studied. Although compatible systems of mod p representations have received less attention, they were considered by Serre in his work in the 1960's and 1970's studying adelic images of Galois groups acting on products of p-adic Tate modules of an elliptic curve for varying p.

In [Kh] we showed how studying compatible systems of mod p representations can be useful in proving results about p-adic representations. Specifically, using this point of view, we rederived in a simple way the result (see [He]) that strictly compatible systems of 1-dimensional p-adic representations (see [Se, I-11]) arise from Hecke characters. Compatible systems of mod p representations make quite apparent how to use the fact that we have a compatible system rather than just one representation at hand, while in the case of compatible p-adic systems this is not so apparent. The difference between these two types of compatible systems is mainly accounted for by the fact that given a p-adic representation it can be made part of at most one (semisimple) compatible system, while this is far from being true for a mod p representation.

In [Kh] we proved that 1-dimensional compatible mod p systems arise from Hecke characters. In this paper we would like to generalise results of [Kh] to the case of higher dimensional, but still abelian, compatible mod p systems. We recall the definition of compatible systems of n-dimensional, mod p representations of the absolute Galois group of a number field K.

Definition 1 Let K and L be number fields and S, T finite sets of places of K and L respectively. An L-rational strictly compatible system $\{\rho_\wp\}$ of n-dimensional mod $\wp$ representations of $G_K := \mathrm{Gal}(\overline{K}/K)$ with defect set T and ramification set S, consists of giving for each finite place $\wp$ of L not in T a continuous, semisimple representation

$$\rho_\wp : G_K \to GL_n(\mathbf{F}_\wp),$$

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for $\mathbf{F}_\wp$ the residue field of $\mathcal{O}_L$ at $\wp$ of characteristic p, such that:

- $\rho_\wp$ is unramified at the places outside $S \cup \{\text{places of } K \text{ above } p\}$;
- for each place r of K not in S there is a monic polynomial $f_r(X) \in L[X]$ such that for all places $\wp$ of L not in T, coprime to the residue characteristic of r, and such that $f_r(X)$ has coefficients that are integral at $\wp$, the characteristic polynomial of $\rho_\wp(\mathrm{Frob}_r)$ is the reduction of $f_r(X)$ mod $\wp$, where Frob_r is the conjugacy class of the Frobenius at r in the Galois group of the extension of K that is the fixed field of the kernel of $\rho_\wp$.

The following theorem was proved in [Kh].

Theorem 1 *An L-rational strictly compatible system $\{\rho_\wp\}$ of 1-dimensional mod $\wp$ representations of $\mathrm{Gal}(\overline{K}/K)$ arises from a Hecke character.*

In [Kh, §4], it is explained what one means by saying that a compatible system of 1-dimensional mod $\wp$ representations arises from a Hecke character.

The theorem below is the main result of this paper. It generalises the result of [Kh] to *abelian* compatible systems $(\rho_\wp)$: by this we mean that $(\rho_\wp)$ is a strictly compatible system as in the definition above and each $\rho_\wp$ has an abelian image. By saying that a compatible system $(\rho_\wp)$, as in the definition above, is the sum of compatible systems $(\rho_{i,\wp})$ $(i = 1, \ldots, n)$, we mean that $\rho_\wp \simeq \bigoplus_{i=1}^n \rho_{i,\wp}$ for all $\wp$ outside the finite defect set.

Theorem 2 *An L-rational strictly compatible system $\{\rho_\wp\}$ of abelian, mod $\wp$ representations of $\mathrm{Gal}(\overline{K}/K)$ is a direct sum of 1-dimensional compatible systems, each of which arises from a Hecke character.*

This theorem is but a small step in studying compatible systems of mod p Galois representations of arbitrary dimensions. Because of it, such compatible systems which are abelian are completely understood. Theorem 2 also proves Conjectures 1 and 2 of [Kh] for all abelian compatible systems of mod $\wp$ Galois representations, which might be summarised by saying that abelian compatible mod p systems of Galois representations are motivic (and all that this entails!).

The proof of Theorem 2 uses the ideas of [Kh]. The main problem that arises when generalising the arguments in [Kh] is that, although using arguments of [Kh] one can easily get that the characteristic polynomials $f_r(X)$ each have a shape that is consistent with the compatible system arising from Hecke characters, we cannot use the arguments in [Kh] directly to show that the nature of the roots of $f_r(X)$ for varying r is consistent with the compatible system arising from Hecke characters. It is this difficulty that is overcome in the lemma and corollary below and the ensuing arguments. We focus mainly on this difficulty as otherwise the arguments are similar to [Kh], where the theorem above was announced. Theorem 2 would directly follow from [Kh, Theorem 1] if we could prove that the compatible system in the theorem is the sum of 1-dimensional compatible systems. But this we *cannot* prove *a priori*. It looks like a hard problem to show that a compatible system of mod $\wp$ or $\wp$-adic representations is "the sum of two compatible systems" if each mod $\wp$ or $\wp$-adic representation in the compatible system is decomposable.

As in [Kh], Theorem 2 has the following corollary.

Corollary 1

(1) *An L-rational strictly compatible system* $\{\rho_\wp\}$ *of abelian (semisimple) mod* $\wp$ *representations of* $\mathrm{Gal}(\overline{K}/K)$ *lifts to a compatible system of n-dimensional* $\wp$*-adic representations* $\{\rho_{\wp,\infty}\}$ *as in* [Se, I-11].

(2) *An L-rational strictly compatible system of abelian (semisimple) n-dimensional* $\wp$*-adic representations* $\{\rho_{\wp,\infty}\}$ *as in* [Se, I-11] *arises from Hecke characters.*

The second part of the corollary is implied by difficult results of Waldschmidt in transcendental number theory, while the proof we offer is more "elementary".

In the last section of the paper we indicate what the analogs of our theorems for function fields should be. We do not prove these proposed analogs. We hope that someone will carry out the proofs of these analogs, and better still, answer the question at the end that we do know how to answer. The question asks for the natural analog to a result of [He] in the setting of function fields: to answer it will probably need transcendence results in characteristic p.

2 Proof of Theorem

Let $(\rho_\wp)$ be an abelian, compatible system of L-rational representations of G_K with K a number field that has finite defect set T and finite ramification set S. Using the arguments in [Kh, Lemma 1] which generalise easily to higher dimensional abelian compatible systems of mod $\wp$ representations, we can reduce to the case when K is a Galois extension of $\mathbf{Q}$ that contains $\sqrt{-1}$, L contains K and is again Galois over $\mathbf{Q}$.

Using class field theory as in the proof of Theorem 1 of [K], we are reduced to looking at a system of homomorphisms $\rho_\wp\colon \mathrm{Cl}_{\mathfrak{m}_\wp p} \to GL_n(\mathbf{F}_\wp)$. Here $\mathrm{Cl}_{\mathfrak{m}_\wp p}$ is the strict ray class group of conductor $\mathfrak{m}_\wp p$, with $\mathfrak{m}_\wp$ the prime to p part of the Artin conductor of $\rho_\wp$ which is divisible only by the primes in S. By using the fact that the subgroup of $\mathrm{Cl}_{\mathfrak{m}_\wp p}$ which is the image of principal ideals prime to $\mathfrak{m}_\wp p$ is of index that is bounded independently of $\wp$, we are further reduced to considering the induced system of homomorphisms $(\mathcal{O}_K/\mathfrak{m}_\wp p\mathcal{O}_K)^* \to GL_n(\mathbf{F}_\wp)$, which by abuse of notation we denote by the same symbol $\rho_\wp$. We shall now almost exclusively study only these homomorphisms. (For this reduction we are using the easy fact that abelian compatible systems of mod $\wp$ representations which all factor through the Galois group of a *fixed* finite extension of K arise from Dirichlet characters.) Thus what we have to show is how these homomorphisms *arise* from algebraic characters of K^*, where K^* is considered as the $\mathbf{Q}$-valued points of the algebraic group $\mathbf{Res}_{K/\mathbf{Q}}(\mathbf{G}_m)$ over $\mathbf{Q}$, that have a subgroup of finite index of the units $\mathcal{O}_K^*$ in their kernel (see [Kh, §4.1]).

These homomorphisms all factor through the quotient by the image of the global units E of K. The compatible data translate into saying that for any principal prime ideal generated by an element r of K (we denote the ideal (r), generated by r, again by r), that is not in S, there is a monic polynomial $f_r(X) \in L[X]$ such that for all $\wp$ not in T and whose residue characteristic is different from that of r and at which $f_r(X)$ is integral, the characteristic polynomial of $\rho_\wp(\mathrm{Frob}_r)$ is the reduction mod $\wp$ of $f_r(X)$. Let the roots of $f_r(X)$ be $\gamma_{1,r}, \ldots, \gamma_{n,r}$. Note that when r varies, all these are algebraic numbers which lie in extensions of L that are of degree over L bounded by n. By using [Kh, Proposition 3], which generalises a result of [CS], and using the compatible data, it is easy to see (see [Kh, §4.2]) that there is an integer $t_{i,r}$ such that $\gamma_{i,r}^{t_{i,r}} = \Pi_\sigma \sigma(r)^{m'_{i,r,\sigma}}$, $i = 1, \ldots, n$, with the exponents some integers and with σ running through the distinct embeddings of K in $\overline{\mathbf{Q}}$. Note that the ramification indices of all primes in any field M of degree over $\mathbf{Q}$ bounded by some integer T, is bounded independently of M, and the number of roots of unity in such M is also bounded independently of M. From this we claim that there is a positive integer m independent of i and r such that $\gamma_{i,r}^m = \Pi_\sigma \sigma(r)^{m_{i,r,\sigma}}$, $i = 1, \ldots, n$, with the exponents $m_{i,r,\sigma}$ some integers. To see this we need the following two observations: First, from the fact of boundedness of the ramification indices mentioned above, the denominators of the reduced forms of $m'_{i,r,\sigma}/t_{i,r}$ are bounded independently of i, r, and then the fact that K is Galois, and the boundedness of the number of roots of unity in number fields of bounded degree recalled above, yields the claim. (Note that at this point we do not know that the exponents $m_{i,r,\sigma}$ are bounded independently of choice of i, r, σ.)

We would like to prove that the $m_{i,r,\sigma}$'s are independent of choice of the principal prime ideal (r) that is coprime to the primes in S. To prove this we cannot use the argument in [Kh] that relied on the fact that in the 1-dimensional situation the characteristic polynomial of $\rho_\wp(rr')$ (with rr' regarded as the image of rr' mod $\mathrm{m}_\wp p$) is the reduction mod $\wp$ of a fixed polynomial in $L[X]$ for almost all primes $\wp$ and where r and r' generate principal prime ideals, not in S. In the higher dimensional case this is not *a priori* the case, although *a posteriori* we will see that this is true. Thus we need to modify the arguments of [Kh] to take into account this complication, and indeed this is the main contribution of the present paper.

Consider the homomorphisms $\rho_\wp^m \colon (\mathcal{O}_K/\mathrm{m}_\wp p\mathcal{O}_K)^* \to GL_n(\overline{\mathbf{F}_\wp})$ (where by m-th power we just mean taking m-th powers of the n homomorphisms to $\overline{\mathbf{F}}_\wp^*$ that constitute $\rho_\wp$). Choose any principal prime ideals (r) and (r') of K that are not in S. Fix i between 1 and n. Then we see that for almost all primes $\wp$ there is an $i(\wp)$ that lies between 1 and n such that the reduction of $\Pi_\sigma \sigma(r)^{m_{i,r,\sigma}} \Pi_\sigma \sigma(r')^{m_{i(\wp),r',\sigma}}$ mod $\wp$ is a root of the characteristic polynomial of $\rho_\wp^m(rr')$.

We have the following lemma:

Lemma 1 *Let (r) and (r') be principal prime ideals of K not in S. Fix an integer i between 1 and n. Consider a prime ℓ of $\mathbf{Q}$ that is prime to the residue characteristics of the primes in S and is prime to the cardinalities of the multiplicative groups of the residue fields at primes of S. Then if a prime of K splits completely in $K(\zeta_\ell, (\sigma(rr'))^{1/\ell})$ with σ running through* Gal$(K/\mathbf{Q})$, *it also splits completely in one of the fields*

$$K\Big(\zeta_\ell, \big(\Pi_\sigma \sigma(r)^{m_{i,r,\sigma}} \Pi_\sigma \sigma(r')^{m_{j,r',\sigma}}\big)^{1/\ell}\Big)$$

for some j between 1 *and n.*

Proof Fix i between 1 and n. Then we see that for almost all primes $\wp$ (in particular we choose $\wp$ prime to the residue characteristics of the primes in S), there is an $i(\wp)$ that lies between 1 and n such that the reduction of $\Pi_\sigma \sigma(r)^{m_{i,r,\sigma}} \Pi_\sigma \sigma(r')^{m_{i(\wp),r',\sigma}}$ mod $\wp$ is a root of the characteristic polynomial of $\rho_\wp^m(rr')$. For almost all primes

s of K that split completely in $K(\zeta_\ell, (\sigma(rr'))^{1/\ell})$ for all $\sigma \in \mathrm{Gal}(K/\mathbf{Q})$, the cardinality of the residue field at s is 1 mod ℓ and further $\sigma(rr')$ is an ℓ-th power modulo s for all $\sigma \in \mathrm{Gal}(K/\mathbf{Q})$. Then by choice of ℓ, and using that $\rho_\wp^m$ is a homomorphism with abelian image, the reduction of $\Pi_\sigma \sigma(r)^{m_{i,r,\sigma}} \Pi_\sigma \sigma(r')^{m_{i(\wp),r',\sigma}}$ mod s is an ℓ-th power. (Here we are using that by choice of ℓ, the ℓ-Sylow subgroup of $(\mathcal{O}_K/\mathrm{m}_\wp p\mathcal{O}_K)^*$ maps isomorphically to that of $(\mathcal{O}_K/p\mathcal{O}_K)^*$, and the general fact that as $\rho_\wp$ is a semisimple representaion with abelian image, the orders of the roots of the characteristic polynomials of $\rho_\wp^m(g)$ all divide the order of g.) Thus s also splits in $K\big(\zeta_\ell, \big(\Pi_\sigma \sigma(r)^{m_{i,r,\sigma}} \Pi_\sigma \sigma(r')^{m_{i(\wp),r',\sigma}}\big)^{1/\ell}\big)$, which proves the lemma. ∎

Corollary 2 *Fix an integer i between 1 and n. For all sufficiently large primes ℓ of $\mathbf{Q}$, the subgroup generated by $\tau(rr')$ of $K^*/(K^*)^\ell$ where τ runs through $\mathrm{Gal}(K/\mathbf{Q})$, contains the image of $\Pi_\sigma \sigma(r)^{m_{i,r,\sigma}} \Pi_\sigma \sigma(r')^{m_{j,r',\sigma}}$ in $K^*/(K^*)^\ell$ for some j between 1 and n.*

Proof Consider $K^*/(K^*)^\ell$ as a $\mathbf{F}_\ell$ vector-space, and denote the $\mathbf{F}_\ell$ vector-space generated by $\sigma(rr')$ of $K^*/(K^*)^\ell$ for $\sigma \in \mathrm{Gal}(K/\mathbf{Q})$ by V. Let W_j be the 1-dimensional vector space of $K^*/(K^*)^\ell$ generated by $\Pi_\sigma \sigma(r)^{m_{i,r,\sigma}} \Pi_\sigma \sigma(r')^{m_{j,r',\sigma}}$ for each j between 1 and n. Let $\mathbf{V}$ be the span of the V's and the W_j's. Then using Kummer theory, the Cebotarev density theorem, the above lemma and the injectivity of the map $K^*/(K^*)^\ell \to K(\zeta_\ell)^*/(K(\zeta_\ell)^*)^\ell$ (note that $\sqrt{-1} \in K$: for this injectivity see [CS, Lemma 2.1]) we conclude that if an element of the dual $\mathbf{V}^*$ has a kernel that contains V, then its kernel also contains W_j for some j between 1 and n. But if ℓ is large enough, this forces one of the W_j's to be contained in V by the following easy claim.

Claim If a prime ℓ is bigger than a given positive integer k, a (non-zero) finite dimensional vector space over $\mathbf{F}_\ell$ cannot be written as the union of k *proper* subspaces.

We apply this claim to the vector space $X = (\mathbf{V}/V)^*$ and the subspaces $(V/W_j)^* \cap X$ where the intersection is taking place in $\mathbf{V}^*$. From the claim we see that W_j is contained in V for some j and thus the corollary follows. ∎

We now claim that fixing a prime (r) not in S and which lies above a prime of $\mathbf{Q}$ that splits completely in K (we call such a prime a split prime), for all split primes (r') of K of different residue characteristic from that of r, the sets $\{(m_{i,r,\sigma})_\sigma | i = 1,\dots,n\}$ and $\{(m_{i,r',\sigma})_\sigma | i = 1,\dots,n\}$ are the same. Namely, fixing an i between 1 and n, it follows from the corollary that there is a fixed $j(i)$ between 1 and n such that for infinitely many primes ℓ, the image of $\Pi_\sigma \sigma(r)^{m_{i,r,\sigma}} \Pi_\sigma \sigma(r')^{m_{j(i),r',\sigma}}$ in $K^*/(K^*)^\ell$ is contained in the subgroup generated by the images of $\sigma(rr')$ as σ runs through $\mathrm{Gal}(K/\mathbf{Q})$. From this, as in [CS, Theorem 1(3)], using the unit theorem, we see that some power of $\Pi_\sigma \sigma(r)^{m_{i,r,\sigma}} \Pi_\sigma \sigma(r')^{m_{j(i),r',\sigma}}$ is contained in the subgroup of K^* generated by $\sigma(rr')$ as σ runs through $\mathrm{Gal}(K/\mathbf{Q})$. From this, as r and r' generate *split* primes of different residue characteristic, we see that $m_{i,r,\sigma} = m_{j(i),r',\sigma}$. Thus we get an injection from the the set $\{(m_{i,r,\sigma})_\sigma \mid i = 1,\dots,n\}$ to the set $\{(m_{i,r',\sigma})_\sigma \mid i = 1,\dots,n\}$. Now as r, r' play symmetric roles, the claim follows. Repeating the argument by fixing a split principal prime of residue charateristic prime to r, we in fact conclude that the set $\{(m_{i,r,\sigma})_\sigma \mid i = 1, \dots, n\}$ is independent of r with r any split principal ideal prime to S.

We now need a small argument to prove in fact that even the multiplicities with which distinct tuples occur in the collection $\langle (m_{i,r,\sigma})_\sigma \mid i = 1, \dots, n\rangle$ are independent of r. The difficulty here is related to the fact that if we have a linear representation ρ of a group G such that for every $g \in G$, $\rho(g)$ has 1 as an eigenvalue, it does not follow that the identity representation occurs in the semisimplification of ρ.

We know by our work that there are only finitely many possibilities for the $m_{i,r,\sigma}$'s as i, r, σ vary (r varies over split principal prime ideals not in S). Let N be a natural number greater than all the finitely many integers $|2m_{i,r,\sigma}|$'s, and let (α) be a split prime ideal of K not in S. Observe that for almost all rational prime p' (we consider only those that are coprime to the places in S and T and α), whenever $1 - \Pi_\sigma \sigma(\alpha)^{m_\sigma}$ is not coprime to p', with m_σ integers and $|m_\sigma| \leq N$, then all the m_σ's are 0. This is true as the $\sigma(\alpha)$'s for $\sigma \in \mathrm{Gal}(K/\mathbf{Q})$ are multiplicatively independent. Fix such a p', choose a prime $\wp'$ above p' in L, and consider $\rho_{\wp'}$ and the corresponding prime to p' Artin conductor $\mathfrak{m}_{\wp'}$ of $\rho_{\wp'}$. Consider integral elements β in K such that β is congruent to α mod $\mathfrak{m}_{\wp'} p'$, and β generates a split prime ideal not in S. We claim that the unordered collection of n, $[K:\mathbf{Q}]$-tuples $\langle (m_{i,\alpha,\sigma})_\sigma \mid i = 1, \dots, n\rangle$, is the same as $\langle (m_{i,\beta,\sigma})_\sigma \mid i = 1, \dots, n\rangle$. This is because the numbers $\Pi_{\sigma \in \mathrm{Gal}(K/\mathbf{Q})} \sigma(\alpha)^{m_{i,\alpha,\sigma}}$ and $\Pi_{\sigma \in \mathrm{Gal}(K/\mathbf{Q})} \sigma(\beta)^{m_{i,\beta,\sigma}}$, with the latter congruent to $\Pi_{\sigma \in \mathrm{Gal}(K/\mathbf{Q})} \sigma(\alpha)^{m_{i,\beta,\sigma}}$ mod p' by choice of β, are congruent mod $\wp'$ under some ordering because they map to the same element of $(\mathcal{O}_K/\mathfrak{m}_{\wp'} p' \mathcal{O}_K)^*$, and thus must have the same image under the homomorphism $\rho^m_{\wp'}$. From this and the fact that p' was chosen so that, whenever $1 - \Pi_\sigma \sigma(\alpha)^{m_\sigma}$ is not coprime to p' and $|m_\sigma| \leq N$, then all the m_σ's are 0, the claim follows. Then for almost all primes p, if we consider the images of such β's that are futher prime to p in $(\mathcal{O}_K/m_\wp p\mathcal{O}_K)^*$ and the subgroup generated by these images, then this is a subgroup whose index is bounded independently of p. This follows from the Cebotarev density theorem as the only condition on the β's is that they generate split prime ideals not in S and that for a fixed prime p' and ideal $\mathfrak{m}_{\wp'}$ of $\mathbf{Z}$ and $\mathcal{O}_K$ respectively, they be congruent to a fixed number α mod p'. We denote the common value of $m_{i,\beta,\sigma}$ for all such β's by $m_{i,\sigma}$. (At this point we know that the multiplicities with which distinct tuples occur in $\langle (m_{i,r,\sigma})_\sigma \rangle$ is independent of r.)

From this we conclude (see [Kh, §4.2] for more details) that the homomorphisms $\rho^m_\wp \colon (\mathcal{O}_K/\mathfrak{m}_\wp p\mathcal{O}_K)^* \to GL_n(\mathbf{F}_\wp)$ for almost all primes $\wp$, when restricted to a subgroup whose index is bounded indpendently of $\wp$, are the direct sums of the homomorphisms that arise from reducing the homomorphisms $x \to \Pi_{\sigma \in \mathrm{Gal}(K/\mathbf{Q})} \sigma(x)^{m_{i,\sigma}}$ mod $\wp$. Thus, remembering that the $\mathfrak{m}_\wp$'s were divisible by only finitely many primes, we conclude that $\rho_\wp$ factors through $(\mathcal{O}_K/\mathfrak{m} p\mathcal{O}_K)^*$ for a non-zero ideal $\mathfrak{m}$ independent of p. Then again as in proof of Theorem 1 [Kh, §4.2], we repeat the argument above using principal split prime ideals that are congruent to 1 mod $\mathfrak{m}$. Namely, we see that for such prime ideals r, the roots of the characteristic polynomial $f_r(X)$ are $\Pi_\sigma \sigma(r)^{m''_{i,r,\sigma}}$ with integers $m''_{i,r,\sigma}$. Using the argument above, we conclude that the $m''_{i,r,\sigma}$'s are independent of r, and thus we can set $m''_{i,r,\sigma} = m''_{i,\sigma}$. From this we deduce that the system of homomorphisms $\rho_\wp \colon (\mathcal{O}_K/\mathfrak{m} p\mathcal{O}_K)^* \to GL_n(\mathbf{F}_\wp)$, when restricted to subgroups whose index (in $(\mathcal{O}_K/\mathfrak{m} p\mathcal{O}_K)^*$) is bounded independently of $\wp$, *arise* from algebraic characters of K^*, where K^* is considered as the $\mathbf{Q}$-valued points of the algebraic group $\mathbf{Res}_{K/\mathbf{Q}}(\mathbf{G}_m)$ over $\mathbf{Q}$, that have a subgroup of finite index of the units $\mathcal{O}_K^*$ in their kernel. From this we conclude that the compatible system $(\rho_\wp)$ of Theorem 2 is the direct sum of the 1-dimensional compatible systems that arise from Hecke characters χ_i for $i = 1, \dots, n$ such that χ_i has infinity type $(m''_{i,\sigma})_\sigma$. This finishes the proof of the theorem. ■

Remark There is a simpler *dévissage* argument, in which one takes determinants of the compatible system and deduces Theorem 2 from [Kh, Theorem 1], to prove Theorem 2 in the case when we know that the compatible system we are dealing with is *integral* (see [Kh, Definition 1]).

3 Analogs for Function Fields

While both here and in [Kh] we have been concerned with representations of absolute Galois groups of number fields, it looks plausible that the methods of [Kh] and this paper should carry over to this setting and lead to the classification of abelian compatible mod $\wp$ systems of absolute Galois groups of function fields.

Let $\mathbf{F}$ be a finite field of characteristic p and consider the ring of polynomials $\mathbf{F}[t]$ and rational functions $\mathbf{F}(t)$. These will serve as our analogs of $\mathbf{Z}$ and $\mathbf{Q}$, respectively. Any function field we consider, say K, will be a finite separable extension of $\mathbf{F}(t)$, and the ring of integers $\mathcal{O}_K$ will be the integral closure of $\mathbf{F}[t]$ in K. Thus by our choices, if K corresponds to the function field of a smooth projective curve X that is geometrically irreducible over a finite extension of $\mathbf{F}$, then X comes equipped with a finite set of infinite places. We denote by G_K the Galois group of the separable closure K^s of K over K.

To get the right analog of compatible systems in this setting one would need to consider the field of rationality L, which is part of the definition of compatible systems, to be a finite extension of the function field K. In [Go] there is an account of compatible systems of $\wp$-adic representations in this setting. Here is the definition of compatible mod$\wp$ systems in this setting.

Definition 2 Let K and L be function fields as above, with a choice of infinite places of K, L as above, and S, T finite sets of places of K and L, respectively, that contain the infinite places. An L-rational strictly compatible system $\{\rho_\wp\}$ of n-dimensional mod $\wp$ representations of G_K with defect set T and ramification set S, consists of giving for each finite place $\wp$ of L not in T a continuous, semisimple representation

$$\rho_\wp : G_K \to GL_n(\mathbf{F}_\wp),$$

for $\mathbf{F}_\wp$ the residue field of $\mathcal{O}_L$ at $\wp$ of characteristic p, such that

- $\rho_\wp$ is unramified at the places outside

$$S \cup \{\text{ all places of } K \text{ above the place of } \mathbf{F}_q[T] \text{ below } \wp\};$$

- $\rho_\wp(G_{\infty_i})$ is trivial where G_{∞_i} is a decomposition group at the infinite places ∞_i;

- for each place r of K not in S there is a monic polynomial $f_r(X) \in L[X]$ such that for all places $\wp$ of L not in T, and that do not lie above the prime of $\mathbf{F}[T]$ below r, and such that $f_r(X)$ has coefficients that are integral at $\wp$, the characteristic polynomial of $\rho_\wp(\mathrm{Frob}_r)$ is the reduction of $f_r(X)$ mod $\wp$, where Frob_r is the conjugacy class of the Frobenius at r in the Galois group of the extension of K that is the fixed field of the kernel of $\rho_\wp$.

The condition of being split, or at least potentially split, via a base change independent of $\wp$, at the infinite places is natural in this context as pointed out to us by Gebhard Böckle. In [Gr], Gross' definition of Hecke characters depends on the choice of infinite places. There he indicates how to attach a compatible system of $\wp$-adic representations to Hecke characters where they are (potentially) split at the infinite places. Then we expect that the methods here should be able to prove that an L-rational strictly compatible system $\{\rho_\wp\}$ of abelian, semisimple mod $\wp$ representations of G_K is a direct sum of 1-dimensional compatible systems each of which arises from a Hecke character. From this will follow the following two statements:

(1) An L-rational strictly compatible system $\{\rho_\wp\}$ of abelian, semisimple mod $\wp$ representations of G_K lifts to a compatible system of n-dimensional $\wp$-adic representations $\{\rho_{\wp,\infty}\}$ (split at all infinite places).
(2) An L-rational strictly compatible system of abelian, semisimple n-dimensional $\wp$-adic representations $\{\rho_{\wp,\infty}\}$ of G_K arises from Hecke characters.

This will lead to a proof that the compatible system of characteristic p Galois representations attached by G. Böckle, R. Pink and others to Drinfeld modular forms arise from Hecke characters. It is also seems plausible that the second statement above can be strengthened considerably:

Question If we have a (continuous) abelian semisimple representation $\rho\colon G_K \to GL_n(L_\wp)$ that is (potentially) split at the infinite places, with $\wp$ a finite place of a function field L and $L_\wp$ the completion of L at $\wp$, unramified outside a finite set of places S, and for places v not in S, the characteristic polynomial of $\rho(\mathrm{Frob}_v)$ has coefficients in L, then is ρ the sum of 1-dimensional compatible systems that arise from Hecke characters in the sense of Gross [Gr]?

The analog of this question for number fields is answered affirmatively by G. Henniart in [He] using results of Waldschmidt in transcendental number theory. Of course, a question like the one above cannot be answered using only the elementary algebraic techniques of the present paper.

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